Mean Kinetic Energy and Turbulent Kinetic Energy Equations

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1. Mean Kinetic Energy (MKE) Equation
Consider nearly incompressible turbulent flow (Boussinesq approximation). Manipulation of the mean continuity and momentum equations yields an equation for the mean kinetic energy per unit mass $\frac{1}{2}U_i^2$:

$$\frac{D}{Dt}\left(\frac{1}{2}U_i^2\right) = \frac{\partial}{\partial x_j}\left[-\frac{PU_i}{\rho_0} + 2\nu U_i E_{ij} - u_iu_j \frac{\partial U_j}{\partial x_i}\right] + u_iu_j \frac{\partial U_j}{\partial x_i} + \frac{g}{\rho_0} \bar{p}U_j - 2\nu E_{ij} E_{ij}$$  \hspace{1cm} (1)

where the terms are labeled and defined as follows:

- **I** Total rate of change of mean kinetic energy per unit mass following a fluid particle.
- **II** Rate of spatial transport of mean kinetic energy per unit mass by pressure work.
- **III** Rate of spatial transport of mean kinetic energy per unit mass by mean viscous stresses.
- **IV** Rate of spatial transport of mean kinetic energy per unit mass by turbulent stresses.
- **V** Rate of destruction (or production) of mean kinetic energy per unit mass into turbulence.
- **VI** Rate of destruction (or production) of mean kinetic energy per unit mass into potential energy.
- **VII** Rate of viscous dissipation of mean kinetic energy per unit mass (turns mean kinetic energy into thermal energy, i.e., heat). *This term is always negative*, indicating a loss of mean kinetic energy per unit mass.

Note: In Equation (1), $E_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) = \text{mean strain rate tensor}$, sometimes called $S_{ij}$ (as in T&L).  \hspace{1cm} (2)

2. Turbulent Kinetic Energy (tke) Equation (also called the tke budget)
Manipulation of the continuity and momentum equations for the turbulent fluctuations yields an equation for the turbulent kinetic energy per unit mass $q^2 = \frac{1}{2}u_ku_k$ (Note: $q^2$ is sometimes given the notation $K$ or $k$ or tke):

$$\frac{D}{Dt}(q^2) = \frac{\partial}{\partial x_j}\left[-\frac{1}{\rho_0} pu_j - \frac{1}{2}u_k u_k + 2\nu u_{ij} - u_iu_j \frac{\partial u_k}{\partial x_l} + g\alpha wT - 2\nu e_{ij} e_{ij}\right]$$  \hspace{1cm} (3)

where the terms are labeled and defined as follows:

- **I** Total rate of change of turbulent kinetic energy per unit mass following a fluid particle.
- **II** Rate of spatial transport of turbulent kinetic energy per unit mass by pressure work.
- **III** Rate of spatial transport of turbulent kinetic energy per unit mass by turbulent velocity fluctuations (convective diffusion).
- **IV** Rate of spatial transport of turbulent kinetic energy per unit mass by turbulent viscous stresses.
- **V** Rate of production (or destruction) of turbulent kinetic energy per unit mass from mean shear.
- **VI** Rate of production (or destruction) of turbulent kinetic energy per unit mass from fluctuating potential energy (buoyancy).
- **VII** Rate of viscous dissipation of turbulent kinetic energy per unit mass (turns tke into thermal energy, i.e., heat). *This term is always negative*, indicating a loss of turbulent kinetic energy.

Note: In Equation (3), $e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \text{fluctuating strain rate tensor}$, sometimes called $s_{ij}$ (as in T&L).  \hspace{1cm} (4)

3. Rate of Viscous Dissipation of Turbulent Kinetic Energy per Unit Mass
The last term (VII) in Equation (3) is extremely important in analysis of turbulent flows. The negative of term VII is given a special name and symbol – the viscous dissipation rate of turbulent kinetic energy per unit mass, sometimes called the scalar dissipation rate or simply the dissipation rate,

$$\varepsilon = 2\nu e_y e_y$$  \hspace{1cm} (5)

Since $e_y e_y$ is positive definite, *the viscous dissipation rate is always a positive quantity* ($\varepsilon > 0$).