Development of One-Equation Turbulence Models

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Governing Equations and the Closure Problem

Consider a simple case of turbulent flow, namely, incompressible flow without gravity. In the Navier-Stokes equations, the energy equation is uncoupled from the continuity and momentum equations. The mean flow equations reduce to:

$$\frac{\partial U_i}{\partial x_i} = 0 \quad (1), \text{ and } \rho\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial U_i}{\partial x_j} - \rho\overline{u_i u_j}\right) \quad (2).$$

This represents 4 equations and 4 primary unknowns (U_i and P), which are functions of (x_i and t). But, the Reynolds stress tensor, $\left(\tau_{ij}\right)_{\text{turbulent}} \equiv -\rho\overline{u_i u_j}$ in the momentum equation represents *six additional independent unknowns* which must be modeled to *close* the problem.

Boussinesq Eddy Viscosity and Mixing Length Approximation

The eddy viscosity, along with the concept of a mixing length, allows us to *approximate* the Reynolds stress tensor as follows:

$$\left(\tau_{ij}\right)_{\text{turbulent}} = -\rho\overline{u_i u_j} \approx -\tfrac{2}{3}\rho K \delta_{ij} + 2\mu_e E_{ij} \quad (3), \text{ where } E_{ij} \equiv \tfrac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \quad (4), \text{ and } \mu_e \approx \text{constant} \cdot \rho \cdot \ell_m u' \quad (5).$$

Prandtl's One-Equation Turbulence Model (High Reynolds number form)

In Prandtl's "mixing length - t.k.e." turbulence model, mixing length ℓ_m must be provided (typically from experimental data). u' in the above expression for eddy viscosity is written in terms of the turbulent kinetic energy per unit mass, $K = q^2 = \tfrac{1}{2}\overline{u_i u_i}$, simply as $u' = \sqrt{K}$. K is now an *additional unknown* in the problem, but K is solved by one *additional transport equation* – the turbulent kinetic energy (t.k.e.) equation:

***Exact* Turbulent Kinetic Energy Transport Equation**: (from a previous handout, with no gravity term) *Note*: $\varepsilon = 2\nu\overline{e_{ij} e_{ij}}$

$$\frac{DK}{Dt} \equiv \frac{\partial K}{\partial t} + U_j \frac{\partial K}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{p u_j} + \tfrac{1}{2}\overline{u_i u_i u_j} - 2\nu\overline{u_i e_{ij}}\right) - \overline{u_i u_j}\frac{\partial U_i}{\partial x_j} - \varepsilon \quad (6), \text{ where } e_{ij} \equiv \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

I II III IV V VII

***Modeled* Turbulent Kinetic Energy Transport Equation:**

- Term I is the total rate of change of t.k.e. There is no need to model the LHS of the t.k.e. equation since we solve for K.
- Terms II, III, and IV collectively represent diffusion or transport of t.k.e. Term IV is typically much smaller than the other two, which are usually modeled collectively as one term, $\frac{\partial}{\partial x_j}\left(\nu_k \frac{\partial K}{\partial x_j}\right)$, using the **gradient diffusion model**, where ν_K is the **turbulent transport coefficient for t.k.e.**, usually expressed as $\nu_K = \nu_e / \sigma_K$, where σ_K is a constant, but ν_K is not.
- Term V represents production of t.k.e., and needs no further modeling since Equations (3) have already provided a model for the Reynolds stress tensor. Substitution of (3) into term V reduces term V to $\nu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}$.
- Term VII is viscous dissipation of t.k.e., modeled as $\varepsilon \approx \text{constant} \cdot \frac{u'^3}{\ell} \approx C_D \frac{K^{3/2}}{\ell_m}$, where C_D = the ***dissipation coefficient***.

Finally, the equation set is closed, assuming ℓ_m is known. The ***modeled* t.k.e. equation** becomes

$$\frac{DK}{Dt} \equiv \frac{\partial K}{\partial t} + U_j \frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\nu_e}{\sigma_k}\frac{\partial K}{\partial x_j}\right) + \nu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \varepsilon \quad (7), \text{ with } \varepsilon = C_D \frac{K^{3/2}}{\ell_m}, \text{ and } \nu_e \approx C_P \ell_m \sqrt{K}.$$

Total rate of change = Diffusion (transport) + Production - Dissipation

The constants are determined by experiment. For example, Emmons (1954) got C_P = 1.0, C_D = 0.07 to 0.09, and σ_K = 1.0.

Alternate Version of Modeled t.k.e. Equation: *Note*: Wilcox (and some other authors) define both dissipation and molecular transport differently. With his definitions, the modeled t.k.e. equation (7) is given as

$$\frac{DK}{Dt} \equiv \frac{\partial K}{\partial t} + U_j \frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_e}{\sigma_k}\right)\frac{\partial K}{\partial x_j}\right) + \nu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \varepsilon \quad (8) \text{ with } \varepsilon \equiv \nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} \approx C_D \frac{K^{3/2}}{\ell_m}.$$