Reynolds Decomposition

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1. Introduction

- Start with the Boussinesq equations (Navier-Stokes equations for buoyant flows) for *total* flow variables ($\tilde{q}$):

$\boxed{\frac{\partial \tilde{u}_i}{\partial x_i} = 0}$ (1), $\boxed{\left(\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}\right) = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x_i} - g\delta_{i3}\left[1-\alpha\left(\tilde{T}-T_0\right)\right] + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}}$ (2), and $\boxed{\frac{\partial \tilde{T}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{T}}{\partial x_j} = \kappa \frac{\partial^2 \tilde{T}}{\partial x_j \partial x_j}}$ (3).

- This represents 5 equations and 5 unknowns ($\tilde{u}_i$, $\tilde{p}$, $\tilde{T}$), functions of (x_i, t). Note that these equations are *exact* (within the limits of the Boussinesq approximation of nearly incompressible flow, of course).

2. Procedure

- **Step 1.** Substitute Reynolds decomposition $\boxed{\tilde{q} = Q + q}$ into (1), (2), and (3): This generates ***instantaneous equations***, (1i), (2i), and (3i).
- **Step 2.** Take the ensemble average of the instantaneous equations: This generates the ***Reynolds averaged equations***, or the ***equations for the mean flow***, (1m), (2m), and (3m).

Summary of rules for ensemble averaging (considering two generic flow variables, $\tilde{q} = Q + q$ and $\tilde{p} = P + p$):

$\overline{\tilde{q}} = Q$	Ensemble average of a total flow quantity equals the mean flow quantity
$\overline{C_1 \tilde{q}} = C_1 \overline{\tilde{q}} = C_1 Q$	A constant (C_1) is not affected by an ensemble average
$\overline{q} = 0$	Ensemble average of a fluctuating quantity is identically zero (by definition)
$\overline{Q} = Q$	Ensemble average of a mean quantity does not change that mean quantity
$\overline{\tilde{q} P} = QP$	Ensemble average of a total flow quantity times a mean quantity is the product of the two mean quantities (P is already ensemble averaged, so it doesn't change further)
$\overline{C_1 q} = C_1 \overline{q} = C_1 \cdot 0 = 0$	Ensemble average of a constant (C_1) times a fluctuating quantity is identically zero
$\overline{qP} = \overline{q} P = 0 \cdot P = 0$	Ensemble average of a fluctuating quantity times a mean quantity is identically zero
$\overline{\tilde{q}\tilde{p}} = QP + \overline{qp}$	Ensemble average of the product of two total flow quantities yields two terms
$\overline{\tilde{q} + \tilde{p}} = \overline{\tilde{q}} + \overline{\tilde{p}} = Q + P$	The order (sequence) of addition or ensemble average does not matter
$\overline{\frac{\partial \tilde{q}}{\partial x_i}} = \frac{\partial \overline{\tilde{q}}}{\partial x_i} = \frac{\partial Q}{\partial x_i}$	The order (sequence) of spatial derivation or ensemble average does not matter. *Note*: Same thing holds with *time* derivatives.
$\overline{\int \tilde{q}\,dx} = \int \overline{\tilde{q}}\,dx$	The order (sequence) of spatial integration or ensemble average does not matter. *Note*: Same thing holds with *time* integration.

- **Step 3.** Subtract the mean flow equations from the instantaneous equations to generate the ***equations for the turbulent fluctuations***, (1f), (2f), and (3f).

Example

Consider the continuity equation, Equation (1). Let's work through the three steps above:

- **Step 1.** $\frac{\partial\left(U_i + u_i\right)}{\partial x_i} = \frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0$, or $\boxed{\frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0}$ (1i)

  Equation (1i) is the ***instantaneous continuity equation***.

- **Step 2.** $\overline{\frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}} = \overline{\frac{\partial U_i}{\partial x_i}} + \overline{\frac{\partial u_i}{\partial x_i}} = \frac{\partial U_i}{\partial x_i} + \cancel{\frac{\partial \overline{u_i}}{\partial x_i}} = \frac{\partial U_i}{\partial x_i} = 0$, or $\boxed{\frac{\partial U_i}{\partial x_i} = 0}$ (1m)

  Equation (1m) is the ***mean continuity equation***, or the ***Reynolds averaged continuity equation***.

- **Step 3.** Subtracting (1m) from (1i) yields $\boxed{\frac{\partial u_i}{\partial x_i} = 0}$ (1f)

  Equation (1f) is the ***continuity equation for the turbulent fluctuations***.

Note: All three equations above are still *exact*, since we haven't made any further simplifications, like linearization, etc. The same procedure must be done on the momentum and energy equations, (2) and (3) respectively.