Reynolds Stress Turbulence Models (Second-Order Closure)

Author: John M. Cimbala, Penn State University
Latest revision: 14 April 2008

The *Exact* Reynolds Stress Transport Equation

Consider incompressible turbulent flow without gravity. The mean continuity and momentum equations reduce to

$\frac{\partial U_i}{\partial x_i} = 0$ (1) , and $\rho\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial U_i}{\partial x_j} - \rho\overline{u_i u_j}\right)$ (2). In addition to these 4 equations and 4 primary unknowns U_i and P, which are functions of x_i and t, The Reynolds stress tensor in the momentum equation adds *six additional independent unknowns*. *Exact* transport equations for the Reynolds stress tensor can be derived:

$$\underbrace{\frac{\partial \tau_{ij_\text{turb}}}{\partial t} + U_k \frac{\partial \tau_{ij_\text{turb}}}{\partial x_k}}_{\text{I}} = \underbrace{-\left(\tau_{ik_\text{turb}} \frac{\partial U_j}{\partial x_k} + \tau_{jk_\text{turb}} \frac{\partial U_i}{\partial x_k}\right)}_{\text{II}} \underbrace{-\Pi_{ij}}_{\text{III}} + \frac{\partial}{\partial x_k}\Big[\underbrace{\nu \frac{\partial \tau_{ij_\text{turb}}}{\partial x_k}}_{\text{IV}} + \underbrace{C_{ijk}}_{\text{V}}\Big] + \underbrace{\varepsilon_{ij}}_{\text{VI}} \quad \text{where} \quad \tau_{ij_\text{turb}} \equiv -\rho\overline{u_i u_j} \qquad (3)$$

I = Total rate of change of Reynolds stress following a fluid particle (unsteady plus advective terms).

II = Production of Reynolds stress, often given the symbol $\overline{P}_{ij}$.

III = Pressure-strain correlation tensor (a redistribution term), often called the "pressure-strain redistribution term," and sometimes given the symbol π_{ij} or ϕ_{ij} instead. Here, $\Pi_{ij} \equiv \overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$.

IV = Spatial transport (diffusion) of Reynolds stress by molecular (viscous) effects.

V = Spatial transport (diffusion) of Reynolds stress by turbulent fluctuations. Here, C_{ijk}, sometimes called D_{ijk} instead, is the "diffusion correlation," defined as $C_{ijk} \equiv \rho\overline{u_i u_j u_k} + \overline{p\left(u_i \delta_{jk} + u_j \delta_{ik}\right)}$.

VI = Rate of viscous dissipation of Reynolds stress, $\varepsilon_{ij} \equiv 2\mu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$.

The *Modeled* Reynolds Stress Transport Equation

Terms I, II, and IV are exact. However, terms III, V, and VI need to be modeled. Let's consider each of these separately:

- Term III. This is the hardest term to model, and has received the most attention. It is typically split into "slow" and "rapid" parts, $\Pi_{ij} = A_{ij} + M_{ijkl} \frac{\partial U_k}{\partial x_l}$, plus a "wall-reflection effect" or "pressure-echo effect" near solid walls. The ***slow pressure-strain term*** is modeled as $A_{ij} \approx C_1 \frac{\varepsilon}{K}(\tau_{ij_\text{turb}} + \tfrac{2}{3}\rho K \delta_{ij})$. There are many models of the rapid and wall-reflection terms, the most popular of which is that of Launder, Reece, and Rodi (LRR), given below. (See Wilcox (1998) for others.)
- Term V. This term is modeled by the *gradient diffusion approximation*, $C_{ijk} \approx \tfrac{2}{3} C_s \frac{K^2}{\varepsilon}\left(\frac{\partial \tau_{jk_\text{turb}}}{\partial x_i} + \frac{\partial \tau_{ik_\text{turb}}}{\partial x_j} + \frac{\partial \tau_{ij_\text{turb}}}{\partial x_k}\right)$.
- Term VI. Since most of the viscous dissipation occurs at the smallest scales, which are nearly isotropic, most modelers assume $\varepsilon_{ij} \approx \frac{2}{3}\rho\varepsilon\delta_{ij}$, where ε is the ***isotropic dissipation rate***, $\varepsilon \equiv \nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}}$. Since ε is another unknown in the problem, a seventh additional transport equation (for ε) must be solved – generally similar to that used in the K-ε model.

A complete example, the ***Launder-Reece-Rodi model*** is provided below (n is the normal distance from the nearest wall):

$$\frac{\partial \tau_{ij_{turb}}}{\partial t} + U_k \frac{\partial \tau_{ij_{turb}}}{\partial x_k} = -\overline{P}_{ij} - \Pi_{ij} + \tfrac{2}{3} C_s \frac{\partial}{\partial x_k}\left[\frac{K^2}{\varepsilon}\left(\frac{\partial \tau_{jk_{turb}}}{\partial x_i} + \frac{\partial \tau_{ik_{turb}}}{\partial x_j} + \frac{\partial \tau_{ij_{turb}}}{\partial x_k}\right)\right] + \frac{2}{3}\rho\varepsilon\delta_{ij}, \quad \overline{P}_{ij} = \tau_{ik_\text{turb}} \frac{\partial U_j}{\partial x_k} + \tau_{jk_\text{turb}} \frac{\partial U_i}{\partial x_k}, \quad \overline{P} = \tfrac{1}{2} P_{kk},$$

$$\Pi_{ij} \approx C_1 \frac{\varepsilon}{K}(\tau_{ij_\text{turb}} + \tfrac{2}{3}\rho K \delta_{ij}) - \hat{\alpha}(\overline{P}_{ij} - \tfrac{2}{3}\overline{P}\delta_{ij}) - \hat{\beta}(D_{ij} - \tfrac{2}{3}\overline{P}\delta_{ij}) - \hat{\gamma}\rho K(E_{ij} - \tfrac{1}{3}E_{kk}\delta_{ij}) + \left[0.125\frac{\varepsilon}{K}(\tau_{ij_\text{turb}} + \tfrac{2}{3}\rho K \delta_{ij}) - 0.015(\overline{P}_{ij} - D_{ij})\right]\frac{K^{\frac{3}{2}}}{\varepsilon n}$$

with $D_{ij} = \tau_{ik_\text{turb}} \frac{\partial U_k}{\partial x_j} + \tau_{jk_\text{turb}} \frac{\partial U_k}{\partial x_i}$, $\hat{\alpha}$ = $(8+C_2)/11$, $\hat{\beta}$ = $(8C_2-2)/11$, $\hat{\gamma}$ = $(60C_2-4)/55$, C_1 =1.8, C_2 =0.60, C_s =0.11. We also need an equation for ε: $\rho\frac{\partial \varepsilon}{\partial t} + \rho U_k \frac{\partial \varepsilon}{\partial x_k} = C_{\varepsilon_1}\frac{\varepsilon}{K}\tau_{ik_\text{turb}}\frac{\partial U_i}{\partial x_k} - C_\varepsilon \frac{\partial}{\partial x_j}\left(\frac{K}{\varepsilon}\tau_{ij_\text{turb}}\frac{\partial \varepsilon}{\partial x_i}\right) - C_{\varepsilon_2}\rho\frac{\varepsilon^2}{K}$, with C_ε =0.18, C_{ε_1} =1.44, and C_{ε_2} =1.92.