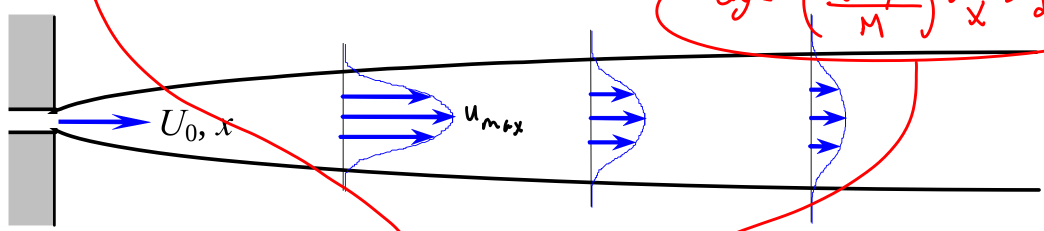
Today, we will:

- Continue to analyze the steady, laminar, 2-D, thin (boundary layer approx.) jet similarity solution
- Do an example problem – 2-D jet entrainment
- Start talking about axisymmetric boundary layers

Recall, $\boxed{u = u_{\max}\operatorname{sech}^2\eta}$ where $\boxed{u_{\max} = \left[\frac{3}{32}\frac{M^2}{\rho^2\nu}\right]^{1/3} x^{-1/3}}$ and $\boxed{\eta = \frac{y}{bx^n} = \left[\frac{M}{48\nu^2\rho}\right]^{1/3} yx^{-2/3}}$

$$dy = \left(\frac{48\nu^2\rho}{M}\right)^{1/3} x^{2/3}\, d\eta$$



M = momentum flux = constant @ any x

$\dot{m}$ = mass flux

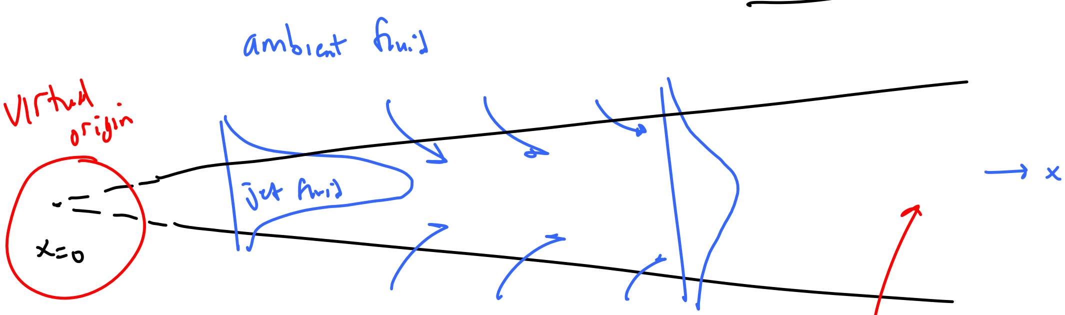
$$\dot{m} = \rho\int_{-\infty}^{\infty} u\, dy$$ = mass flux per unit depth @ any x location

get

$$\boxed{\dot{m} = \underbrace{3.302\,(\rho\mu M)^{1/3}}_{\text{const.}}\, x^{1/3}}$$

$\dot{m}$ increases with x, like $x^{1/3}$

This is called <u>entrainment</u>



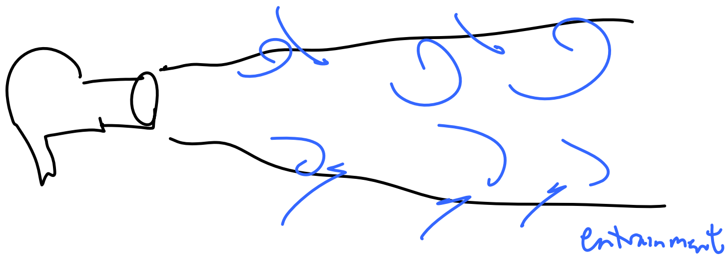
@ $x=0$, $\dot{m}=0$, but $M \neq 0$

Here, only a very small fraction of the jet fluid is original jet fluid

(Mostly ambient fluid)

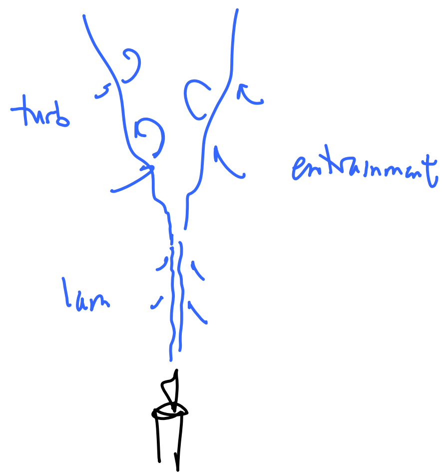
Related examples:

1) Hair dryer (axisymm., turbulent)



2) Candle or cigarette smoke

Buoyant plume



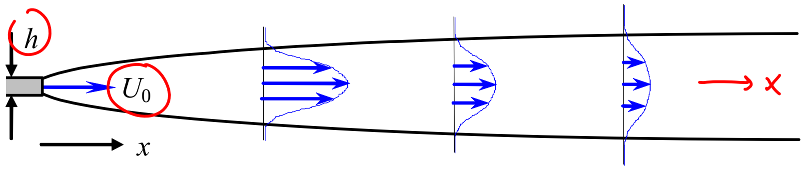
EXAMPLE

↓

Example Problem: 2-D Incompressible Laminar Jet

Author: John M. Cimbala, Penn State University
Latest revision: 14 January 2008

Given: A fully submerged 2-dimensional channel of height h is dumping polluted water into a large quiescent lake. Approximate the initial jet as a uniform plug flow (constant velocity U_0 over the entire height h). For simplicity, assume the properties ρ, μ, etc. of the effluent (the polluted water from the drain), are the same as those of the ambient water in the lake.



To do:
(a) Calculate the momentum flux of the jet (per unit width into the page).

Solution:

We integrate at $x = 0$, the origin of the jet, $M = \int_{-\infty}^{\infty} \rho u^2 dy = \rho \int_{-h/2}^{h/2} U_0^2 dy = \rho U_0^2 h$. In other words, since the jet is initially uniform, the momentum flux per unit width into the page is simply $\boxed{M = \rho U_0^2 h}$. ***This momentum flux is constant at any x location***, as discussed in class.

b) Calculate the initial mass flux of the jet (per unit width into the page).

Solution:

Similarly, we integrate at $x = 0$, the origin of the jet, $\dot{m}_0 = \int_{-\infty}^{\infty} \rho u dy = \rho \int_{-h/2}^{h/2} U_0 dy = \rho U_0 h$. Again, since the jet is initially uniform, the mass flux per unit width into the page is simply $\boxed{\dot{m}_0 = \rho U_0 h}$. ***This mass flux is not constant with x location***, as discussed in class – in fact, the mass flux increases downstream due to entrainment of the surrounding ambient fluid.

& thin

c) If the jet remains steady, laminar, and 2-D, the boundary layer solution discussed in class is applicable. Namely, as x increases, the jet entrains ambient fluid like $x^{1/3}$, and the polluted water becomes diluted. Assuming that at any x location the fluid inside the jet is fully mixed (well-mixed), generate an expression for how far downstream one must go in order that the jet contains only 1% of the initial effluent, and 99% ambient water.

Solution:

At any x-location, the mass flux is given by $\dot{m} = 3.302(\rho\mu M)^{1/3} x^{1/3}$. So we need to find x where $\dot{m}_0 / \dot{m} = 0.01$, or

$3.302(\rho\mu M)^{1/3} x^{1/3} = 100\dot{m}_0$, or $3.302(\rho^2 \mu U_0^2 hx)^{1/3} = 100\rho U_0 h$. Solving for x gives $\boxed{x = \left(\frac{100}{3.302}\right)^3 \frac{h^2 U_0}{\nu}}$, where $\nu \equiv \mu/\rho$ is the kinematic viscosity. (*Notice that the answer is independent of density*.)

d) For $h = 5.00$ cm, $U_0 = 10.0$ cm/s, and $\nu = 0.0100$ cm^2/s, what downstream distance is required for the conditions of Part (c)?

Solution:

Plugging in the numbers, we get $\boxed{x = 6.9 \times 10^8 \text{ cm, or approximately 4300 miles}}$!

[We are also assuming that the Schmidt # is 1 so that species & momentum diffuse at the same rate.]

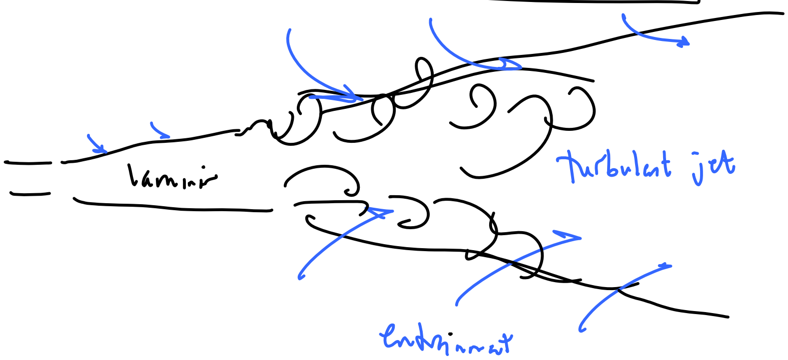
Discussion: Clearly this is *not* a realistic answer!
Why not? What have we assumed incorrectly? What would happen in real life?

Unrealistic because the jet will go turbulent (will not remain laminar!)

Define local Reynolds # (at a given x location)

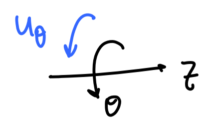
$$Re = \frac{u_{max}\,\delta_c}{\nu} \sim \frac{x^{-1/3}\,x^{2/3}}{\nu} \sim x^{1/3}$$

$\boxed{Re \text{ increases with } x}$



jet goes turbulent @ Re ~ 30 or 40

G. Axisymmetric BLs

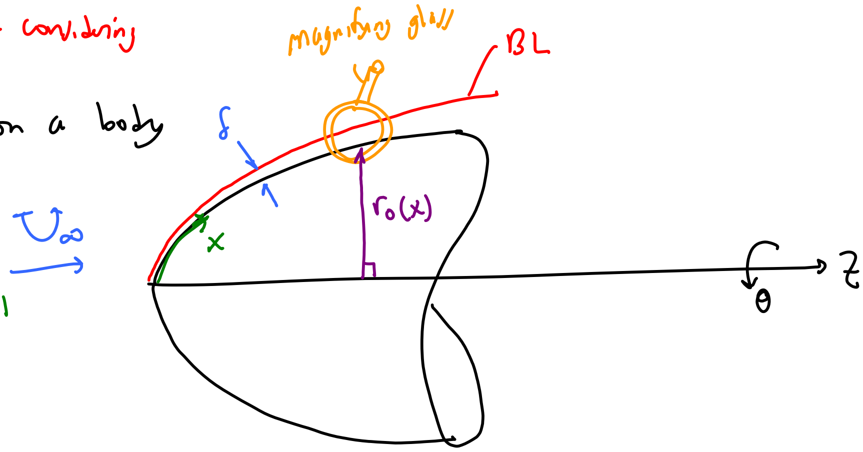


1. Intro   rotationally symmetric $\Rightarrow \frac{\partial}{\partial\theta} = 0$ but u_θ can be non-zero (there can be swirl)

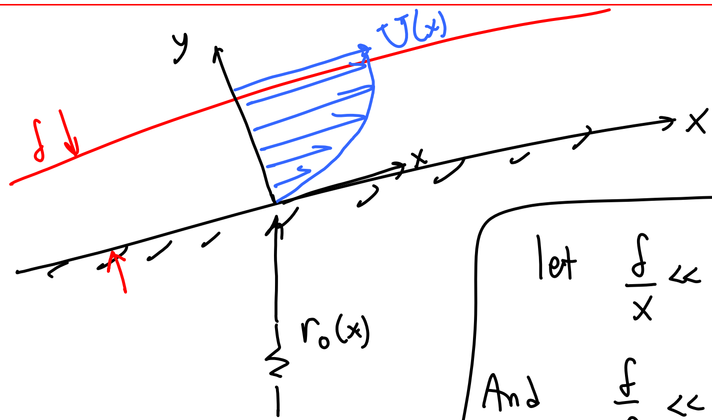
☆ axisymmetric $\rightarrow \frac{\partial}{\partial\theta} = 0$ and $u_\theta = 0$ (no swirl)

We are considering

Axisymmetric BL on a body



X = distance from stag. pt. along the wall



let $\frac{\delta}{x} \ll 1$ (just like in 2-D)

And $\frac{\delta}{r_0} \ll 1$

$\Rightarrow$ ☆ THIN BL APPROX. FOR AXISYM. BL

We are tempted to use the 2-D BL eqs. for this case

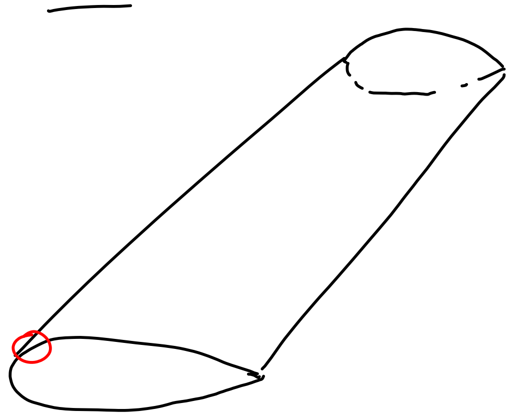
$\swarrow$

<u>This is not correct</u>

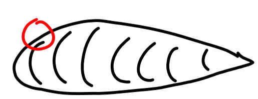
Cons. of mass requires that $r_0(x)$ appear in the continuity eq. even when $\delta \ll r_0$

Consider a 2-D & an axisym. body of the same cross-sectional shape:

<u>2-D</u>



<u>axi</u>

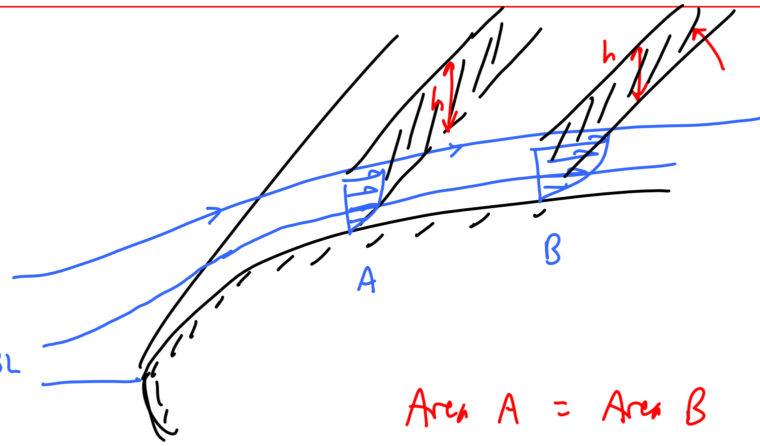


Look at a region where the streamlines outside the BL are parallel to the body surface

2-D

U(x) is increasing
BL is growing

Combined effect is that streamlines are // outside the BL

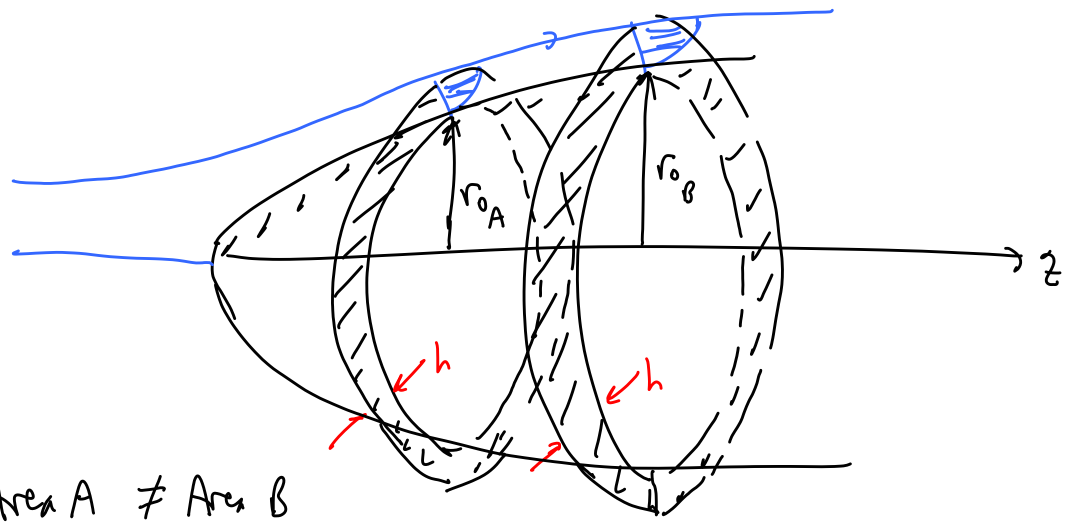


Area = h·b

b = depth into the page

Area A = Area B for 2-D case

Axisym:

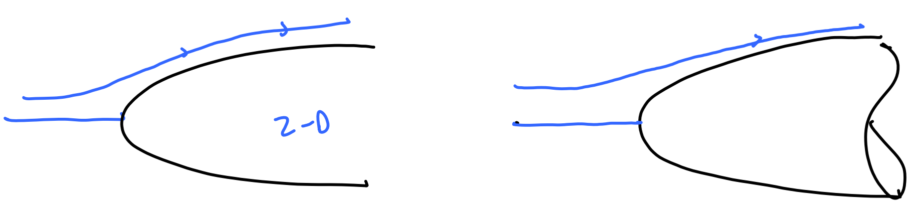


Area A ≠ Area B

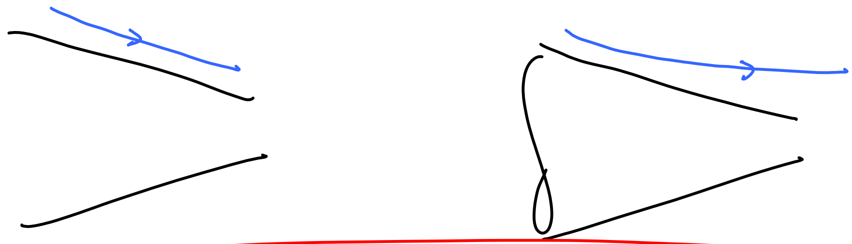
Area B > Area A even in this case where h is the same

The BL thickness in the axisym. case must be smaller than that in the 2-D case for the same cross-sectional area (streamlines must converge towards the wall in axi. case)

Front of body



Back of body



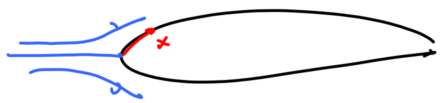
Bottom line:

- The axisymm. BL eqs. must contain r_0 even when $\delta/r_0 \ll 1$
- ☆ You cannot simply apply the 2-D BL eqs to an axisymm. body even if $\delta/r_0 \ll 1$

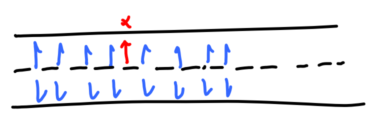
Another case — Stagnation point flow

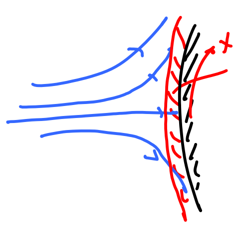
2-D

side view



front view



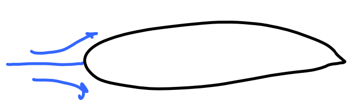


BL thickness turns out to be constant

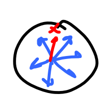
$f \approx$ constant near stag. pt.

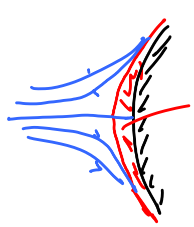
Axisymm

side view



front view





BL thickness must decrease with x