Today, we will:

- Continue to analyze the steady, laminar, 2-D, thin (boundary layer approx.) jet similarity solution
- Do an example problem – 2-D jet entrainment
- Start talking about axisymmetric boundary layers

Recall, $u = u_{\text{max}} \operatorname{sech}^2 \eta$, where $u_{\text{max}} = \left[ \frac{3}{32} \frac{M^2}{\rho^2 \nu} \right]^{1/3} x^{-1/3}$ and $\eta = \frac{y}{bx^n} = \left[ \frac{M}{48 \nu^2 \rho} \right]^{1/3} y x^{-2/3}$.

$$M = \text{momentum flux} = \text{constant at any } x$$

$$\dot{m} = \text{mass flux} \quad \dot{m} = \rho \int_0^\infty u \, dy = \text{mass flux per unit depth at any } x \text{ location}$$

Get: $\dot{m} = 3.302 (\rho \mu M)^{1/2} y^{1/2}$

This is called entrainment.

Here, only a very small fraction of the jet fluid is entrained.
Related examples: 1) Hair dryer (axiymmetric, turbulent)

2) Candle or cigarette smoke

Example↓
Given: A fully submerged 2-dimensional channel of height $h$ is dumping polluted water into a large quiescent lake. Approximate the initial jet as a uniform plug flow (constant velocity $U_0$ over the entire height $h$). For simplicity, assume the properties $\rho$, $\mu$, etc. of the effluent (the polluted water from the drain), are the same as those of the ambient water in the lake.

To do:
(a) Calculate the momentum flux of the jet (per unit width into the page).

Solution:
We integrate at $x = 0$, the origin of the jet, $M = \int_{-\infty}^{\infty} \rho u^2 dy = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} U_0^2 dy = \rho U_0^2 h$. In other words, since the jet is initially uniform, the momentum flux per unit width into the page is simply $M = \rho U_0^2 h$. This momentum flux is constant at any $x$ location, as discussed in class.

(b) Calculate the initial mass flux of the jet (per unit width into the page).

Solution:
Similarly, we integrate at $x = 0$, the origin of the jet, $\dot{m}_0 = \int_{-\infty}^{\infty} \rho u dy = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} U_0 dy = \rho U_0 h$. Again, since the jet is initially uniform, the mass flux per unit width into the page is simply $\dot{m}_0 = \rho U_0 h$. This mass flux is not constant with $x$ location, as discussed in class – in fact, the mass flux increases downstream due to entrainment of the surrounding ambient fluid.

c) If the jet remains steady, laminar, and 2-D, the boundary layer solution discussed in class is applicable. Namely, as $x$ increases, the jet entrains ambient fluid like $x^{1/3}$, and the polluted water becomes diluted. Assuming that at any $x$ location the fluid inside the jet is fully mixed (well-mixed), generate an expression for how far downstream one must go in order that the jet contains only 1% of the initial effluent, and 99% ambient water.

Solution:
At any $x$-location, the mass flux is given by $\dot{m} = 3.302(\rho \mu M)^{1/3} x^{1/3}$. So we need to find $x$ where $\dot{m}_0/\dot{m} = 0.01$, or $3.302(\rho \mu M)^{1/3} x^{1/3} = 100\dot{m}_0$, or $3.302(\rho^2 \mu U_0^2 h)^{1/3} = 100\rho U_0 h$. Solving for $x$ gives $x = \left( \frac{100}{3.302} \right) \frac{h^2 U_0}{\nu}$, where $\nu \equiv \mu / \rho$ is the kinematic viscosity. (Notice that the answer is independent of density.)

d) For $h = 5.00$ cm, $U_0 = 10.0$ cm/s, and $\nu = 0.0100$ cm$^2$/s, what downstream distance is required for the conditions of Part (c)?

Solution:
Plugging in the numbers, we get $x = 6.9 \times 10^8$ cm, or approximately 4300 miles!

Discussion: Clearly this is not a realistic answer! Why not? What have we assumed incorrectly? What would happen in real life?
Uncertain because the jet will go turbulent (will not remain laminar!)

Define local Reynolds at a given x location:

\[ Re = \frac{U_{\text{max}} \, \delta_c}{\nu} \sim \frac{x^{-\frac{1}{3}}}{\frac{2}{3} x^{\frac{2}{3}}} \sim x^{\frac{1}{3}} \]

\[ Re \text{ increases with } x \]

Jet goes turbulent at \( Re \approx 30 \text{ or } 40 \)

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G. Axisymmetric BLs

1. Intro rotationally symmetric \( \Rightarrow \frac{\partial}{\partial \theta} = 0 \) but \( U_\theta \) can be non-zero (there can be swirl)

\[ \star \text{ Axisymmetric} \rightarrow \frac{1}{r} \frac{\partial}{\partial \theta} = 0 \text{ and } U_\theta = 0 \text{ (no swirl)} \]

We are considering axisymmetric BL on a body

X = distance from stagnation point along the wall

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\[ u_0 \rightarrow \frac{f}{\theta} \]
We are tempted to use the 2-D BL egs. for this case

\[ \frac{f}{\ell} \ll 1 \]
\[ \frac{f}{r_0} \ll 1 \]

*THIN BL APPROX. FOR AXISYM. BL

This is not correct

Conv. of mass reqg. that \( r_0(x) \) appear in the continuity eg.

even when \( f \ll r_0 \)

Consider a 2-D i.e., an axisym. body of the same cross-sectional shape.

Look at a region where the streamlines outside the BL are parallel
to the body surface.
2-D

Why is increasing BL increasing? Is that streamlining are // outside the BL

Combines effect is that streamlining are // outside the BL.

$\text{Area} = h \cdot b$

$b = \text{depth into the page}$

Area $A = \text{Area } B$ for 2-D case

Axym:

Area $A \neq \text{Area } B$

Area $B > \text{Area } A$ even in this case where $h$ is the same.

The BL thickness in the axym. case must be smaller than that in the 2-D case for the same cross-sectional area (streamline must converge towards the wall in axi. case.)
Back of body

Bottom line:
- The axi-symmetric BL eqs. must contain \( r_0 \) even when \( S/r_0 \ll 1 \)
- You cannot simply apply the 2D BL eqs. to an axi-symmetric body even if \( S/r_0 \ll 1 \)

Another case - Stagnation point flow

2D

BL thickness must decrease with \( x \)