M E 522 Spring 2008 Professor John M. Cimbala Lecture 04 01/23/2008

Today, we will:

- Continue our discussion about the general axisymmetric boundary layer equations
- Do an example problem – the thin axisymmetric incompressible jet
- ~~If time, begin to discuss three-dimensional boundary layers~~

4. General Axisymmetric BL Equations

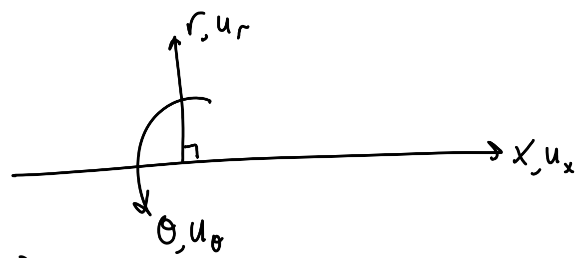
a. Intro

b. Derivation of Axisymm. BL Eq's

Start w/ NS → App. B-1

use (r, θ, x) & (u_r, u_θ, u_x) ← streamwise u

instead of (r, θ, z)



Simplify: 1) Axisymmetric → $\frac{\partial}{\partial \theta} = 0$ & $u_\theta = 0$ (no swirl)

2) BL Approx → similar to 2-D generation of BL eq's

Let $\frac{\partial}{\partial x} << \frac{\partial}{\partial r}$ & Large Re → thin BL

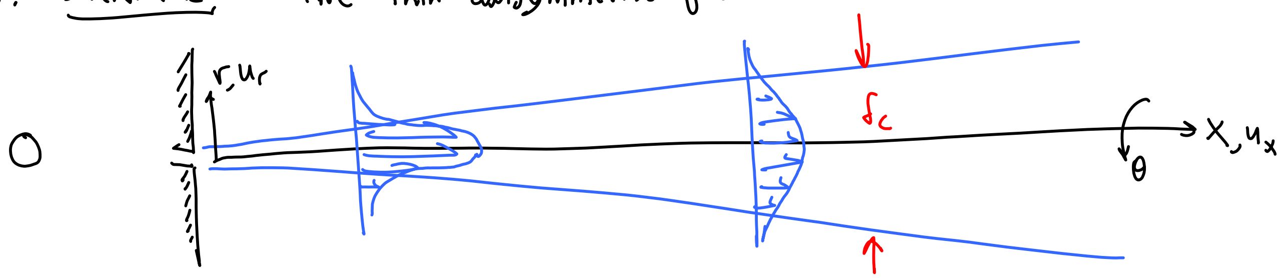
Do an order of magnitude (o.o.m.) analysis

↓ similar to 2-D case

General Axisymmetric BL Eqs (incompressible, steady)

$$\boxed{\begin{aligned} &\frac{\partial u_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0 \qquad (1) \\ &u_x \frac{\partial u_x}{\partial x} + u_r \frac{\partial u_x}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_x}{\partial r}\right) \qquad (2) \end{aligned}}$$

5. EXAMPLE – The thin axisymmetric jet



Assumptions / Approximations

- incompressible
- steady
- laminar
- p = constant everywhere
- axisymmetric
- high Re so that BL approx. holds (BL is thin)

Eq.s (1) & (2) become

$$\boxed{\begin{aligned} &\frac{\partial u_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0 && (1) \\ &u_x \frac{\partial u_x}{\partial x} + u_r \frac{\partial u_x}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) && (2) \end{aligned}}$$

Boundary Conditions: 1) BC on u_r at some r for all x

Set $\boxed{u_r = 0 \text{ @ } r = 0 \text{ for all } x}$

2) BC on u_x at some r for all x

Set u_x = a maximum @ $r=0$ (symmetry)

$$\boxed{\frac{\partial u_x}{\partial r} = 0 \text{ @ } r = 0 \text{ for all } x}$$

3) Another BC on u_x at some r for all x

Set $\boxed{u_x = 0 \text{ as } r \to \infty \text{ for all } x}$

4) A starting profile for u_x at some x for all r

[we could specify a starting profile @ $x=0$ & march downstream]

But we expect a similarity solution

Set our BC based on the C.V. analysis of momentum

let $\boxed{M = \text{momentum flux} = 2\pi\rho\int_0^\infty u_x^2\, r\,dr = \text{constant}}$ (independent of x)

[Recall, this comes from a C.V. momentum analysis of the jet]

Similarity Solution [Schlichting] — (1933)

solve using a stream function

For axisymm. flow, $\boxed{u_x = \frac{1}{r}\frac{\partial\psi}{\partial r} \qquad u_r = -\frac{1}{r}\frac{\partial\psi}{\partial x}}$

Plug into continuity eq (1), it is exactly satisfied.

Assume that the jet grows linearly $\delta_c \sim x$

so let $\boxed{\eta = \frac{r}{x}}$ & let $\boxed{\psi = \nu x f(\eta)}$ → try this to see if similarity soln is available

$$\frac{\partial\eta}{\partial x} = -\frac{r}{x^2} = \frac{-\eta}{x} \qquad u_x = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{\nu}{r}x f'\frac{1}{x} \rightarrow \boxed{u_x = \frac{\nu f'}{r}}$$

$$\frac{\partial\eta}{\partial r} = \frac{1}{x} \qquad u_r = -\frac{1}{r}\frac{\partial\psi}{\partial x} = \frac{-\nu}{r}\left[x f'\left(\frac{-\eta}{x}\right) + f\right]$$

$$\boxed{u_r = \frac{\nu}{r}(\eta f' - f)}$$

Plug these into Eq (2)

$\Big\downarrow$ (algebra)

result: $$\frac{ff'}{\eta^2} - \frac{(f')^2}{\eta} - \frac{ff''}{\eta} = f''' - \frac{f''}{\eta} + \frac{f'}{\eta^2}$$

clever: $$-\frac{d}{d\eta}\left(\frac{ff'}{\eta}\right) = \frac{d}{d\eta}\left[f'' - \frac{f'}{\eta}\right]$$

Integrate once: $$\boxed{-\frac{ff'}{\eta} = f'' - \frac{f'}{\eta} + \underline{\underline{const}}} \quad (3)$$

2nd-order ODE of $f(\eta)$ → similarity has been achieved provided that the BCs work out

BC's → $\boxed{\begin{array}{l} f(0) = 0 \\ f'(0) = 0 \\ f'(\infty) = 0 \end{array}}$ → if $f'(\infty) = 0$, $f''(\infty) = 0$

Eq (3) @ $\eta \to \infty$ $$\cancel{f''} - \cancel{\frac{f'}{\eta}} + \cancel{\frac{ff'}{\eta}} = const = 0$$

Finally, $$\boxed{f'' - \frac{f'}{\eta} + \frac{ff'}{\eta} = 0} \quad (4)$$

with BC's $\boxed{\begin{array}{l} f(0) = 0 \\ f'(0) = 0 \end{array}}$

SIMILARITY HAS BEEN ACHIEVED ✪

Schlichting's solution $\rightarrow$ $\boxed{f(\eta) = \dfrac{\gamma^2 \eta^2}{1 + \frac{1}{4}\gamma^2\eta^2}}$ where γ is some new constant

A family of solutions, depending on the value of γ

Any γ will work $\rightarrow$ γ must be tied to the mom. flux M to distinguish one jet from another

Now use $\boxed{M = 2\pi\rho \int_0^\infty u_x^2 \, r \, dr = \text{constant at any } x}$

Plug our sol'n. into here

result is $\boxed{M = \frac{16}{3}\pi\rho\gamma^2\nu^2}$ $\quad [\gamma \propto \sqrt{M}]$

or $\boxed{\gamma = \sqrt{\dfrac{3M}{16\pi\rho\nu^2}}}$

Let's calculate the mass flux, $\dot{m}$

$$\boxed{\dot{m} = 2\pi\rho \int_0^\infty u_x \, r \, dr}$$

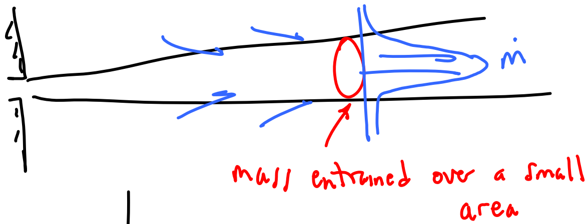
algebra

$$\boxed{\dot{m} = 8\pi\mu x}$$

$\dot{m}$ is independent of M !

?? what does this mean ??

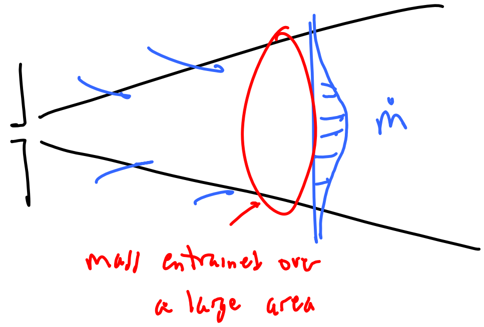
Large M



Thin BL → BL approx is good

<u>But</u> → jet goes turbulent quickly since Re is large

Small M



Thin BL approx is not good

<u>But</u> → Re is small, so it stays laminar

high velocity, small cross-sectional area ⟶ $\dot m$

low velocity, high cross-sectional area ⟶ $\dot m$

☆ It turns out that $\dot m$ is the same in either case!

Bottom line — this soln does not work well for small M or large M