Today, we will:

- Continue our discussion of three-dimensional boundary layer coordinate systems
- Discuss secondary flow in three-D boundary layers
- Do an example problem – secondary flow on a flat plate

E.g.: a cone → can solve as a developable surface

b. Intrinsic coordinate – a “trick”

- Solve for \( U \) in any coord system
- Pick the B.L. coordinate system such that
  - \( x, \bar{v} \parallel \text{outer streamlining} \)
  - \( \bar{z} \parallel \text{normal} \)
- Transform \( U \) into \( x, \bar{v}, \bar{z} \) B.L. coordinates

Advantage – B.C.s are easy

\[ U = \frac{\bar{U}}{U} \]

\[ W = 0 \]

Disadvantage – \( h_x, h_z \) will not necessarily = 1

B.L. eqs more difficult to solve

4. Summary – Why 3-D B.L.s are harder than 2-D B.L.s

a. Outer flow (harder to calculate)

→ two outer flow components \( U(x, \bar{z}) \) instead of \( U(x) \)

b. Coordinate system

- not readily apparent – there are options

C. B.L. eqs themselves are more complicated
2 mom. eqs. \((x, \tau)\) w.r.t. of one \((x)\)  
more terms \(\rightarrow\) more complicated terms if \(h_x, h_\tau \neq 1\)  
\(h_x, h_\tau\) are embedded in the eqs.  
\(\Rightarrow\) they change with location on the body! 

4. Secondary flow can occur  
The velocity within the BL does not have to be  
in the same direction as the outer flow 

5. Secondary Flow:  
   a. Defn.:  
   When the flow inside the BL is in a different direction 
   than the outer flow  

Sketch:  
\(\vec{U}, \vec{U}_w, \vec{U}_n\)  
\(\vec{U}\) in fact \(\vec{C}\) anywhere in the BL  
can all be in different directions  

Secondary flow  
\(\vec{C}_{w_x} = M \frac{\partial U}{\partial y}\)  
\(\vec{C}_{w_z} = M \frac{\partial U}{\partial z}\)  
\(\vec{C}_{w_\tau} = \frac{\partial C_w}{\partial \tau}\)  
\(\tau\)-component  
favorable p.g.  
\(x\)-component  
adv. p.g.
6. Example Problem

Given:
- Semi-infinite flat plate with a constant transverse pressure gradient
  \( \frac{dP}{dz} = \text{constant} \)
- Zero pressure gradient in x-direction

Top view:

- Nothing special about any \( z \)-location

Outer Flow Velocity:

\[
U = \text{constant} \\
W = U(1 - bx)
\]
(a and b are constants)

Generates parabolic streamlines as shown.

This is a 2-D geometry, but it is a 2-D problem; will have secondary flow.

- Pick a BL coordinate system → choose \( x, y, z \) on the plate

\[
h_x = h_y = h_z = 1 \rightarrow \text{BL eqs are easy}
\]

BL eqs:

\[
\begin{align*}
\text{Cont:} & \quad \frac{2u}{dx} + \frac{2v}{dy} + \frac{2w}{dz} = 0 \\
\text{x-mom:} & \quad u \frac{2u}{dx} + v \frac{du}{dy} + W \frac{dx}{dz} = -\frac{1}{\rho} \frac{dP}{dy} + \nu \frac{2u}{dy^2} \\
\text{z-mom:} & \quad u \frac{2w}{dx} + v \frac{2w}{dy} + W \frac{dz}{dz} = -\frac{1}{\rho} \frac{dP}{dz} + \nu \frac{2w}{dy^2} \\
\text{y-mom:} & \quad \frac{2v}{dy} = 0 \quad \text{through the BL}
\end{align*}
\]

We expect that \( \frac{d}{dz} (\text{any velocity}) = 0 \) but \( \frac{dP}{dz} \neq 0 \)}
Euler's Eq. for the outer flow:

\[
\begin{aligned}
\text{x-mom: } & \quad U \frac{dU}{dx} + W \frac{dW}{dt} = -\frac{1}{\rho} \frac{dB}{dx} \\
& \quad U = \text{const} \\
\text{t-mom: } & \quad U \frac{dW}{dx} + W \frac{dW}{dt} = -\frac{1}{\rho} \frac{dB}{dt} \\
\text{at } W &= U(a-bx) \\
\frac{dW}{dx} &= -Ub \\
\end{aligned}
\]

BL eq. i. BCs:

\[
\begin{aligned}
\text{cont: } & \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
\text{x-mom: } & \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \\
\text{t-mom: } & \quad u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} = -U_b + \nu \frac{\partial^2 w}{\partial y^2}
\end{aligned}
\]

BCs:

no slip \rightarrow U = V = W = 0 \quad \text{at } y = 0

\text{edge } \rightarrow U \rightarrow U \quad \text{at } y \rightarrow \infty

W \rightarrow W = U(a-bx) \quad \text{at } y \rightarrow \infty

W is uncoupled from Eq. (1) \Rightarrow We \text{ can solve (1) i. (2) first. Then get } U, V, \text{ then solve for } W \text{ later.}

(1 i. (2) + the BCs for } U, V \text{ are identical to Blasius BL problem!}

\text{u \& v components are already known!}

(similarity soln)