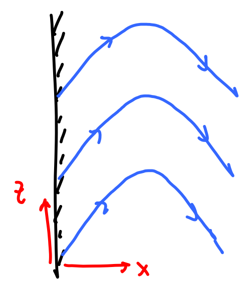
Today, we will:

- Finish the example problem – secondary flow over a flat plate
- Discuss two other examples of secondary flow – tea cup and cyclone separator

Recall, Blasius laminar boundary layer similarity solution: Let $\boxed{u = Uf'(\eta)}$ where $\boxed{\eta = y\sqrt{\frac{U}{\nu x}}}$.

Plug into continuity and *x*-momentum equations: $\boxed{v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left(\eta f' - f\right)}$ and $\boxed{f''' + \frac{1}{2} f f'' = 0}$, with BCs $\boxed{f(0)=0}$, $\boxed{f'(0)=0}$, and $\boxed{f'(\infty)=1}$. We solved using the Runge-Kutta method.

Blasius BL solution (it is identical)



$$\left.\begin{aligned} U &= \text{constant} \\ W &= Ua - Ubx \end{aligned}\right\} \text{outer flow}$$

$$\boxed{\begin{aligned} &@\ y = 0,\ w = 0 \\ &@\ y \to \infty,\ w = W \end{aligned}}$$

z-component of velocity → let's try:

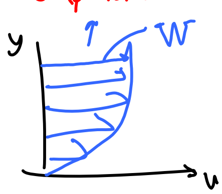
New sim. function

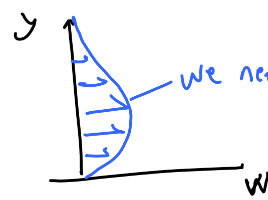
$$\boxed{w = W(x) f'(\eta) - Ubx\, h(\eta)} \qquad (4)$$

a "scaled" Blasius component

component due to the transverse pressure gradient

$$\left[\begin{aligned} &w \to W \\ &@\ \eta \to \infty \end{aligned}\right]$$





we need to calculate $h(\eta)$

Plug into z-mom & do algebra

$$\boxed{h'' + \frac{1}{2} f h' - f' h + 1 - (f')^2 = 0}$$

& BCs → $\boxed{\begin{aligned} h(0) &= 0 \\ h(\infty) &= 0 \end{aligned}}$

Have to guess $h'(0)$ & iterate to find the correct solution.

Summary for Mathcad:

$$\boxed{\begin{aligned} &f''' + \tfrac{1}{2} f f'' = 0 \\ &h'' + \tfrac{1}{2} f h' - f'h + 1 - (f')^2 = 0 \end{aligned}}$$

BC's

$$\boxed{\begin{array}{ll} f(0)=0 & h(0)=0 \\ f'(0)=0 & h(\infty)=0 \\ f(\infty)=1 & \end{array}}$$

Guess $f''(0)$

& $h'(0)$

& march

Use R-K solution technique

See Mathcad file on website & below.

Blasius-secondary flow flat plate boundary layer similarity solution

J. M. Cimbala

The equations to solve are f''' + cff'' = 0, where prime denotes d/dη, and h'' + 0.5fh' - f'h + 1 - (f')^2 = 0.

Here, let c = 1/2, following Kundu's book. $c := 0.5$

The boundary conditions are f'(0)=1, f(0)=1, f'(∞)=1, h(0)=0, and h(∞)=0.

Since two of these are at infinity, f''(0) and h'(0) need to be guessed until the boundary conditions at infinity are satisfied.

First define a vector Y which contains five unknowns, Y_1 = f'', Y_2 = f', Y_3 = f, Y_4 = h', and Y_5 = h.

ORIGIN := 1

Known boundary conditions:

$Y_2 := 0 \quad Y_3 := 0 \quad Y_5 := 0$

Guessed boundary conditions:

$Y_1 := 0.332057 \quad Y_4 := 1.085973166$

see 521 iterate to find

Verify the vector:
$$Y = \begin{pmatrix} 0.33206 \\ 0 \\ 0 \\ 1.08597 \\ 0 \end{pmatrix}$$

Now define the derivative vector D which contains the first derivative with respect to η of each variable in the Y vector. This derivative vector D is needed for the Runge-Kutta solution.

$$D(\eta, Y) := \begin{bmatrix} -c \cdot Y_3 \cdot Y_1 \\ Y_1 \\ Y_2 \\ -1 + (Y_2)^2 + Y_2 \cdot Y_5 - \frac{1}{2} \cdot Y_3 \cdot Y_4 \\ Y_4 \end{bmatrix}$$

Now calculate the solution as η marches from ηstart to ηend. Here Z is the solution matrix, where the first column is η, the second column is Y_1, the third column is Y_2, the fourth column is Y_3, the fifth column is Y_4, and the last column is Y_5.

Top portion of Z:

	η	Y_1 f''	Y_2 f'	Y_3 f	Y_4 h'	Y_5 h
	1	2	3	4	5	6
1	0	0.33206	0	0	1.08597	0
2	$5 \cdot 10^{-3}$	0.33206	$1.66028 \cdot 10^{-3}$	$4.15071 \cdot 10^{-6}$	1.08097	$5.41737 \cdot 10^{-3}$
3	0.01	0.33206	$3.32057 \cdot 10^{-3}$	$1.66028 \cdot 10^{-5}$	1.07597	0.01081
4	0.015	0.33206	$4.98085 \cdot 10^{-3}$	$3.73564 \cdot 10^{-5}$	1.07097	0.01618
5	0.02	0.33206	$6.64114 \cdot 10^{-3}$	$6.64114 \cdot 10^{-5}$	1.06597	0.02152
6	0.025	0.33206	$8.30142 \cdot 10^{-3}$	$1.03768 \cdot 10^{-4}$	1.06098	0.02684
7	0.03	0.33206	$9.96171 \cdot 10^{-3}$	$1.49426 \cdot 10^{-4}$	1.05598	0.03213
8	0.035	0.33206	0.01162	$2.03385 \cdot 10^{-4}$	1.05098	0.0374
9	0.04	0.33206	0.01328	$2.65646 \cdot 10^{-4}$	1.04598	0.04264
10	0.045	0.33206	0.01494	$3.36208 \cdot 10^{-4}$	1.04098	0.04786

Z =

Bottom portion of Z (to verify BCs):

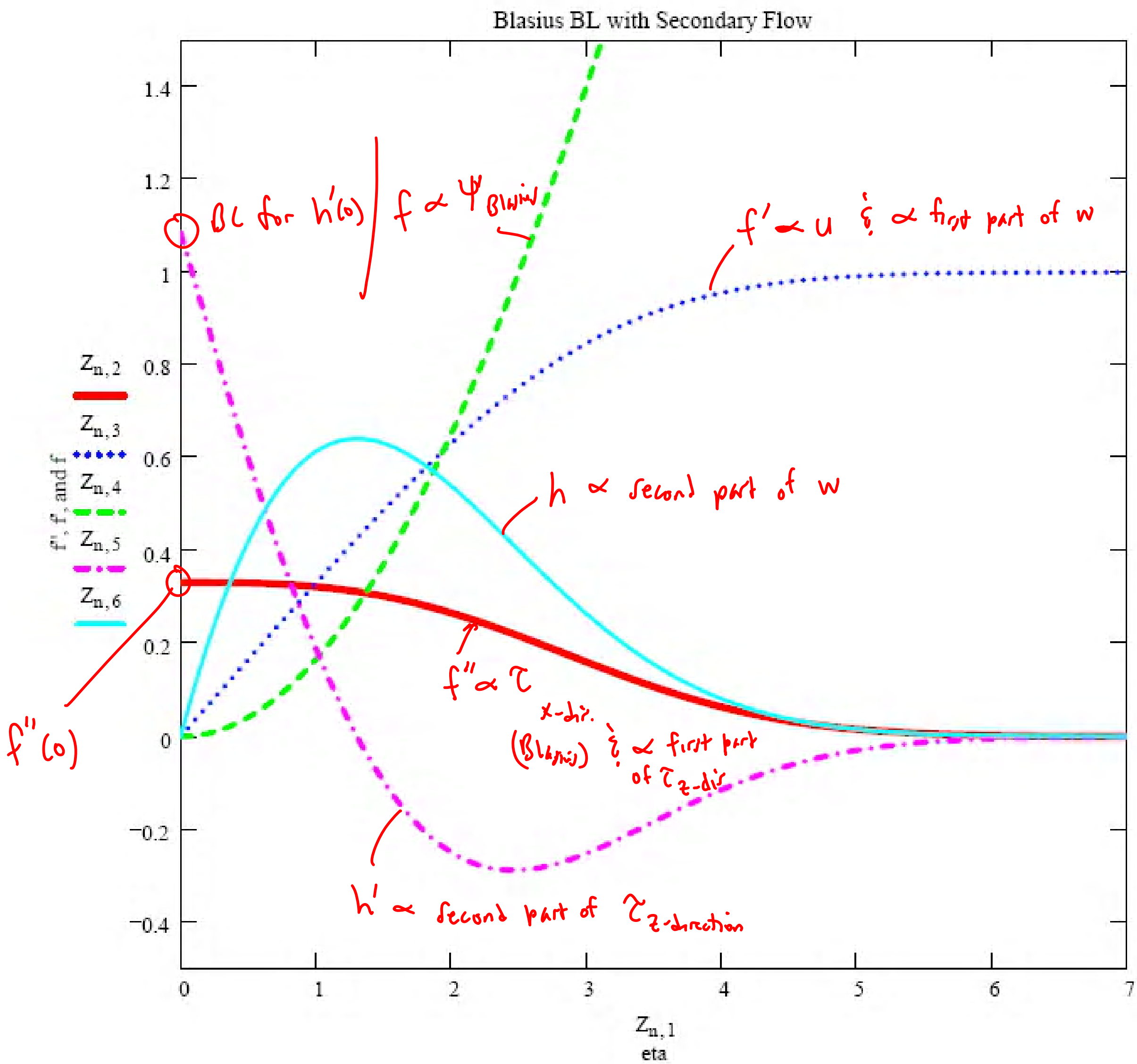
$f' = u/U$ h' h

$Z =$

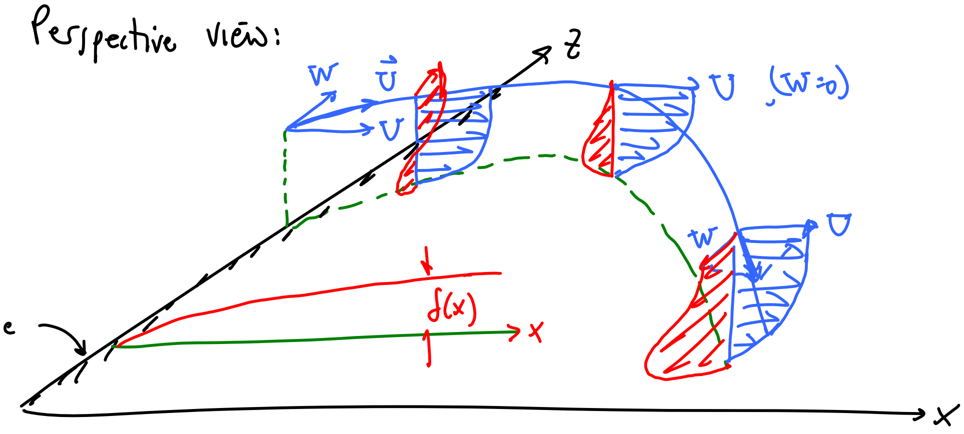
	1	2	3	4	5	6
1991	9.95	$1.03781 \cdot 10^{-8}$	1	8.22921	$-2.12027 \cdot 10^{-9}$	$1.43271 \cdot 10^{-6}$
1992	9.955	$1.01667 \cdot 10^{-8}$	1	8.23421	$-1.69195 \cdot 10^{-9}$	$1.4327 \cdot 10^{-6}$
1993	9.96	$9.95948 \cdot 10^{-9}$	1	8.23921	$-1.27186 \cdot 10^{-9}$	$1.4327 \cdot 10^{-6}$
1994	9.965	$9.75637 \cdot 10^{-9}$	1	8.24421	$3.59864 \cdot 10^{-10}$	$1.43269 \cdot 10^{-6}$
1995	9.97	$9.55729 \cdot 10^{-9}$	1	8.24921	$-4.558 \cdot 10^{-10}$	$1.43269 \cdot 10^{-6}$
1996	9.975	$9.36215 \cdot 10^{-9}$	1	8.25421	$5.95158 \cdot 10^{-11}$	$1.43269 \cdot 10^{-6}$
1997	9.98	$9.17088 \cdot 10^{-9}$	1	8.25921	$3.29137 \cdot 10^{-10}$	$1.43269 \cdot 10^{-6}$
1998	9.985	$8.9834 \cdot 10^{-9}$	1	8.26421	$7.10305 \cdot 10^{-10}$	$1.43269 \cdot 10^{-6}$
1999	9.99	$8.79965 \cdot 10^{-9}$	1	8.26921	$1.08413 \cdot 10^{-9}$	$1.43269 \cdot 10^{-6}$
2000	9.995	$8.61955 \cdot 10^{-9}$	1	8.27421	$1.45076 \cdot 10^{-9}$	$1.4327 \cdot 10^{-6}$
2001	10	$8.44303 \cdot 10^{-9}$	1	8.27921	$1.81033 \cdot 10^{-9}$	$1.43271 \cdot 10^{-6}$

$\eta = 10 \approx$ "∞" $u \to U$ $h(\infty) \to 0$

Now generate a plot of the similarity variables: $n := 1 .. num\_steps$

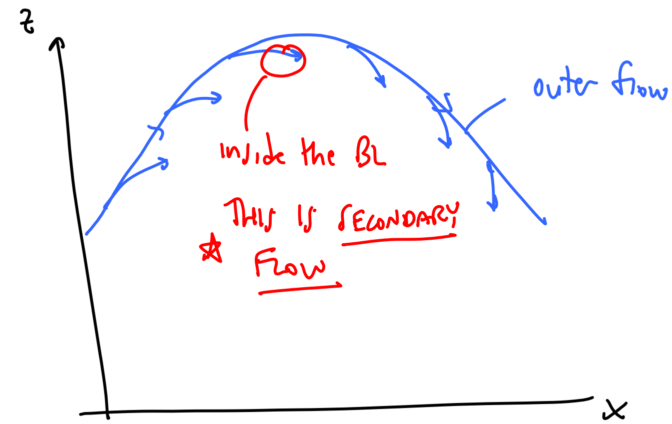


Result: Perspective view:



W goes from (+) to 0 to (−) as we move in x-direction

SECONDARY FLOW Top view



For any 3-D BL with curved streamlines in the outer flow, fluid inside the BL always moves inward (towards the center of curvature of the outer flow streamlines)

SECONDARY FLOW

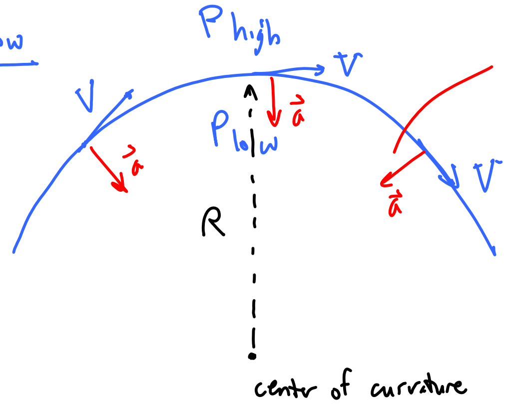
Fluid particles inside the BL have a more tightly curved path than do the outer flow particles

C. Demonstration — teacup → Particles @ bottom move towards the center of the cup!

[SECONDARY FLOW IN THE BL ALONG THE BOTTOM OF THE CUP]

Physical Explanation:

outer flow

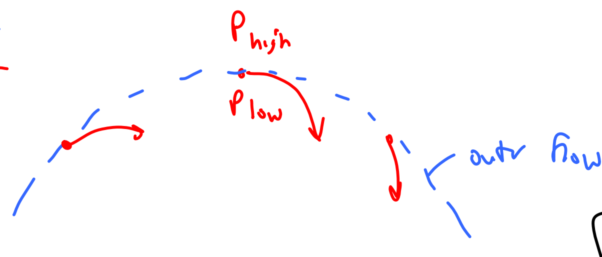


$\vec{a}$ caused by pressure gradient

$\downarrow$

Centripetal force $\propto \dfrac{V^2}{R}$

In the BL

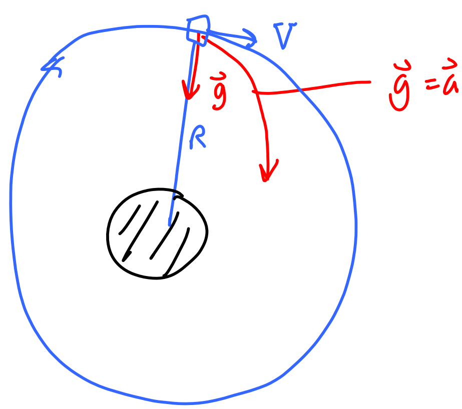


The key is that

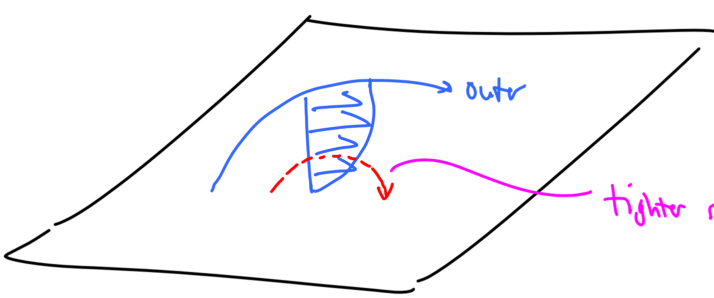
$$\frac{\partial p}{\partial y} \approx 0 \text{ through the BL}$$

Analogy to a spacecraft satellite in orbit

Satellite slows down & therefore begins to veer inward towards the earth →



Same pressure difference pushing the fluid particles towards the center, but smaller velocity, ∴ they veer inward towards the center of the cup

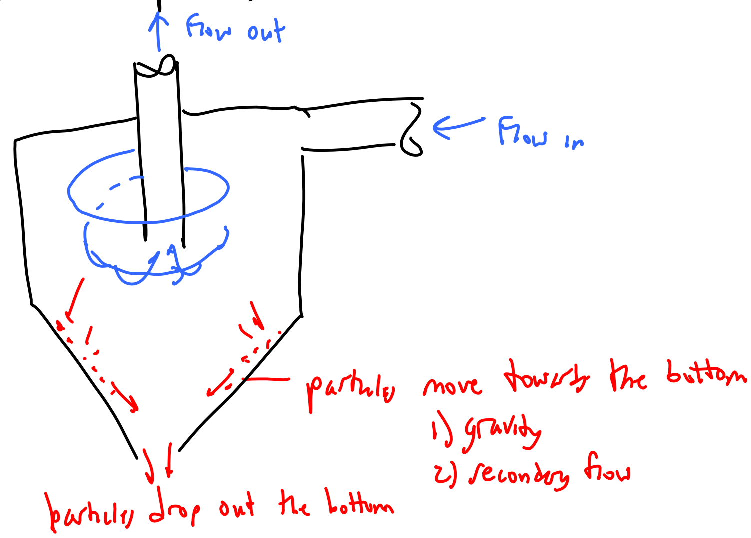


tighter radius of curvature inside the BL

d. Practical application of secondary flow

Cyclone separator → to remove particles from the air

Side view



Top view