Today, we will:

- Finish the example problem – secondary flow over a flat plate
- Discuss two other examples of secondary flow – tea cup and cyclone separator

Recall, Blasius laminar boundary layer similarity solution: Let

\[ u = U f'(\eta) \quad \text{where} \quad \eta = y \sqrt{\frac{U}{v x}}. \]

Plug into continuity and x-momentum equations:

\[
v = \frac{1}{2} \sqrt{\frac{U}{x}} (\eta f' - f) \quad \text{and} \quad f''' + \frac{1}{2} f'' = 0,
\]

with BCs \( f(0) = 0, \ f'(0) = 0 \) and \( f'(\infty) = 1 \). We solved using the Runge-Kutta method.
Summary for Method:

\[ f'' + \frac{1}{2} ff'' = 0 \]
\[ h'' + \frac{1}{2} fh' - f'h + 1 - (f')^2 = 0 \]

Bc's:

- \[ f(0) = 0 \]
- \[ h(0) = 0 \]
- \[ f'(0) = 0 \]
- \[ h(\infty) = 0 \]
- \[ f(\infty) = 1 \]

Guess \( f''(0) \)

\( \frac{h}{h'(0)} \)

\( \frac{h}{march} \)

Use RK solution technique

See method file on website [below].
Blasius-secondary flow flat plate boundary layer similarity solution

J. M. Cimbala

The equations to solve are \( f'' + c f' f = 0 \), where prime denotes \( d/d\eta \), and \( h'' + 0.5f h' - fh' + 1 - (f')^2 = 0 \).

Here, let \( c = 1/2 \), following Kundu's book. \( c := 0.5 \)

The boundary conditions are \( f(0)=1, f'(0)=1, f(\infty)=1, h(0)=0 \), and \( h(\infty)=0 \).

Since two of these are at infinity, \( f'(0) \) and \( h'(0) \) need to be guessed until the boundary conditions at infinity are satisfied.

First define a vector \( Y \) which contains five unknowns, \( Y_1 = f' \), \( Y_2 = f \), \( Y_3 = f \), \( Y_4 = h' \), and \( Y_5 = h \).

\[ \text{ORIGIN := 1} \]

Known boundary conditions:

\[ \begin{align*}
Y_2 &= 0 \quad Y_3 = 0 \quad Y_4 = 0
\end{align*} \]

Guessed boundary conditions:

\[ \begin{align*}
Y_1 &= 0.332057 \\
Y_4 &= 1.085973166
\end{align*} \]

Verify the vector:

\[ Y = \frac{0.33206}{0.33206} \]

\[ Y = \frac{0}{0} \]

\[ Y = \frac{1.08597}{1.08597} \]

Now define the derivative vector \( D \) which contains the first derivative with respect to \( \eta \) of each variable in the \( Y \) vector. This derivative vector \( D \) is needed for the Runge-Kutta solution.

Now calculate the solution as \( \eta \) marches from \( \eta_{\text{start}} \) to \( \eta_{\text{end}} \). Here \( Z \) is the solution matrix, where the first column is \( \eta \), the second column is \( \eta' \), the third column is \( \eta'' \), the fourth column is \( \eta'' \), the fifth column is \( \eta' \), and the last column is \( \eta' \).

\[ D(\eta, \eta') = \begin{bmatrix}
-c Y_3 Y_1 \\
Y_1 \\
Y_2 \\
-1 + (Y_2)^2 + Y_2 Y_3 + \frac{1}{2} Y_3 Y_4 \\
Y_4
\end{bmatrix} \]

Top portion of \( Z \):

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<th>( \eta'' )</th>
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Bottom portion of Z (to verify BCs):

\[
f' = \frac{\nu}{h'}
\]

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Now generate a plot of the similarity variables: \( n := 1 \) \( \text{num\_steps} \)

Blasius BL with Secondary Flow

- \( f' \propto u \; \frac{h}{\nu} \propto \text{first part of } w \)
- \( f''(0) \propto C \; \text{for } h(z) \)
- \( f \propto \nu \text{ with } h \propto \text{second part of } w \)
- \( h' \propto \text{second part of } C_2 \)-function
Result: Perspective view:

W goes from $0$ to $0$ to $0$ as we move in $x$-direction.

**Secondary Flow**

Top view:

For any 3-D BL with curved streamlines in the outer flow, fluid inside the BL always moves inward (towards the center of curvature of the outer flow streamlines).

**Secondary Flow**

Fluid particles inside the BL have a more tightly curved path than do the outer flow particles.

**Demonstration - Teacup**

Particles at bottom move towards the center of the cup: [Secondary flow in the BL along the bottom of the cup]

Physical Explanation:
outer flow
\[ \begin{align*}
\vec{V} & \rightarrow R \\
\vec{p}_{\text{high}} & \rightarrow \vec{p}_{\text{low}} \\
\vec{a} & = \vec{a} \text{ caused by pressure gradient} \\
\text{Centripetal force} & \propto \frac{V^2}{R} \\
\text{center of curvature}
\end{align*} \]

in the BL

Analogy to a spacecraft satellite in orbit

The Key is that
\[ \frac{d\vec{V}}{dy} \approx 0 \text{ through the BL} \]

Satellite slows down \( \vec{a} \), therefore trying to veer inward towards the earth

Same pressure difference pushing the fluid particles towards the outer, but smaller velocity, \( \vec{\ddot{g}} = \vec{a} \)

\( \vec{g} \rightarrow \vec{\ddot{g}} \)

tighter radius of curvature makes the BL
d. Practical application of secondary flow

Cyclone separator → to remove particles from the air

- Particles move towards the bottom
  1) Gravity
  2) Secondary flow

Particles drop out the bottom

Flow in
Flow out

Top view
Side view