Today, we will:

- Discuss the notation associated with stability theory
- Discuss linear stability theory
- Do an example problem – linear stability theory

INSTABILITY (continued)

A. Introduction

1. examples of instabilities

2. Notation (I follow Kundu's notation here)

- Basic state → use upper case letters → U, V, W, P
  use overbar for Greek & temperature $\bar{\rho}, \bar{T}$

- Disturbance → use lower case letters → u, v, w, p, ρ, T'
  For temp. use T'

- Total flow variable (Basic state + disturbance)
  → use tilda

$$\tilde{u}_i = U_i + u_i$$
$$\tilde{p} = P + p$$
$$\tilde{T} = \bar{T} + T'$$
$$\tilde{\rho} = \bar{\rho} + \rho$$

- Amplitudes → use a "hat", e.g., $\hat{u}$

B. Linear Stability Theory:

1. Definition

For a given basic state, the state is stable if the system returns to that basic state when an infinitesimally small disturbance is applied

Limitations :

- If it is unstable, this technique can <u>not</u> predict the new state.

  (e.g. → may lead to a different basic state
  → may lead to turbulence)

- We can't predict the new state, but we <u>can</u> predict which frequencies of the disturbance are stable or unstable.

We <u>can</u> predict the <u>onset</u> of turbulence or a jump to another state.

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2. <u>Procedure</u>

<u>Step 0)</u> Identify the differential equation(s) to examine

In general, for some quantity $\tilde{q}$

$$\boxed{\mathcal{D}(\tilde{q}) = 0} \qquad (1)$$

means a differential eq.

<u>Step 1)</u> Write the diff. eq. for a <u>basic state Q</u> (some known soln of (1))

[Q satisfies (1) exactly]

$$\boxed{\mathcal{D}(Q) = 0} \qquad (1\,b)$$

"basic"

<u>Step 2)</u> Add a disturbance q → Substitute $\underline{\tilde{q} = Q + q}$ into (1)

Linear approx. ☆ $\boxed{\text{Here, } q \ll Q}$

$$\mathcal{D}(\tilde{q}) = \mathcal{D}(Q+q) = 0 \qquad (1\,t)$$

"total"

Step 3) $\underbrace{\text{Subtract (1b) from (1t)}}$

The result is the $\boxed{\text{disturbance equation}}$ (1d)

d — disturbance

[NOT, necessarily, $\mathcal{D}(q)=0$]

Typically nonlinear

Step 4) Linearize the disturbance eq. $\rightarrow$ neglect all higher-order terms like q^2, q^3, pq, p^2q, etc.

[p is some other disturbance variable, $p \ll P$]

Result is the $\boxed{\text{linearized disturbance eq.}}$ (1ℓ)

ℓ — linearized

Step 5) Solve the linearized disturbance eq. for q

[any way you can!]

Step 6 Examine the stability

- if q grows (in space and/or time), basic state Q is unstable
- if q decays (in space and time), basic state Q is stable

Often, all we can get is the neutrally stable case

Mathematically $\rightarrow$ This is often an eigenvalue problem.

↳ e.g. high frequency disturbances may be stable
low frequency " " " unstable

} frequency is the eigenvalue

Amplitude of q is the corresponding eigenfunction

3. Simple example $\left(\tilde{q} = \tilde{q}(t)\right)$

Step 0) → Consider $\boxed{\mathcal{D}(\tilde{q}) = \frac{d\tilde{q}}{dt} + k\tilde{q}^2 - c = 0}$ (1)

Let's examine the stability of this eq. for some basic state Q

Step 1) Find a basic state → call it $\boxed{Q(t) = \text{basic state}}$

[$Q(t)$ satisfies Eq. (1) exactly.]

$$\therefore \quad \boxed{\frac{dQ}{dt} + kQ^2 - c = 0} \qquad (1b)$$

Step 2) Add a disturbance q, $q = q(t)$ → $\tilde{q} = Q + q$

$$(1) \rightarrow \quad \boxed{\frac{dQ}{dt} + \frac{dq}{dt} + k\left(Q^2 + 2Qq + q^2\right) - c = 0} \qquad (1t)$$

Step 3) Subtract (1b) from (1t)

get
$$\boxed{\frac{dq}{dt} + 2kQq + kq^2 = 0} \qquad (1d)$$

[nonlinear disturbance eq.]

Step 4) Linearize $\quad q^2$ is negligible compared to Qq

$$\boxed{\frac{dq}{dt} + 2kQq = 0} \quad \text{Linearized dist. eq.} \qquad (1\ell)$$

Notice – (1ℓ) is not the same as (1)

Step 5) Solve (1ℓ) for q

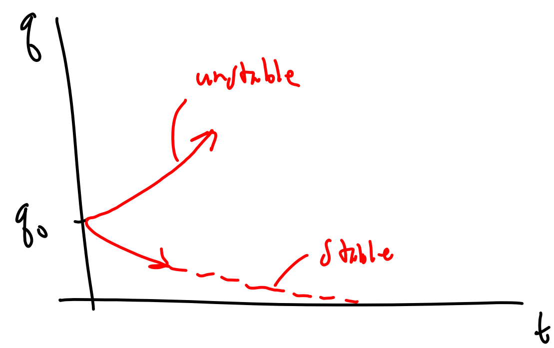
Separate variables → $\frac{dq}{q} = -2kQ\,dt$

Integrate → $\boxed{q = q_0\, e^{-2k\int Q\, dt}}$ (k is a constant)

q_0 = initial condition on q
[$q_0 = q$ @ $t=0$]

<u>Step 6)</u> <u>Examine the stability</u>

if $-2k\int Q\, dt$ grows with time, Q is <u>unstable</u>



k is a parameter → may influence the stability

C. Method of Normal Modes

1. <u>Introduction</u> → The <u>linearized disturbance eq.</u> is <u>linear</u>! & <u>homogeneous</u>

∴ Can use <u>superposition</u>

↓ So, we can choose the type of disturbance we apply

Recall → We can decompose any arbitrary disturbance into a <u>Fourier Series</u>

The <u>modes</u> in the Fourier series do not interact with each other — they behave independently

∴ $\boxed{\text{We can examine one mode at a time!}}$ ☆

☆ Examine the stability of a sinusoidal disturbance of some wavelength and/or frequency → Leads to eigenvalue problem