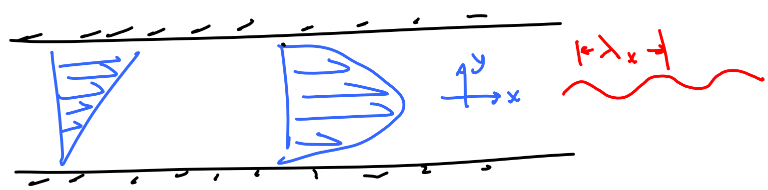
Today, we will:

- Discuss the Method of Normal Modes in more detail for a parallel flow basic state
- Begin an example problem – The Benard problem (thermal instability)
- Do Candy Questions for Candy Friday

C. Method of Normal Modes (continued)

e.g. Basic state is parallel flow $\vec{U} = (U(y), 0, 0)$



In general the disturbance u,

$$\boxed{u(x,y,z,t) = \underline{\underline{\hat{u}(y)}}\, e^{ikx + imz + \sigma t}} \quad [+\cancel{ily}] \; 0 \qquad (1)$$

$\hat{u}(y)$ = complex amplitude

k, l, m = x, y & z components of the wavenumber vector

$\vec{K} = (k, l, m)$ = wavenumber vector [Here $l = 0$]

$k = \dfrac{2\pi}{\lambda_x}$ $\qquad \lambda_x$ = wavelength of the disturbance in x direction

$m = \dfrac{2\pi}{\lambda_z}$ $\qquad \lambda_z$ = " " " " " z

σ = growth rate (in time)

Here, for example k & m are real, but σ may be complex

Complex variables review: $\quad e^{ikx} = \cos kx + i \sin kx$

$e^{-ikx} = \cos kx - i \sin kx$

When k is real → e^{ikx} represents a sinusoidal fluctuation of unity amplitude

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

[$z = x + iy$
= complex #]

Express $\sinh(iz)$ in terms of $\sin z$

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} \frac{i}{i} = \underline{\underline{i \sin z}}$$

TEMPORAL MODE OF INSTABILITY

see Eq. (1)

$$\sigma = \sigma_r + i\sigma_i$$

We have $e^{\sigma t}$ in Eq. (1) $\longrightarrow$ $e^{\sigma t} = e^{\sigma_r t} \, e^{i\sigma_i t}$

This part determines stability

purely periodic in time

Amplitude = 1

[Has nothing to do with stability]

Stability criterion for the temporal mode:

if $\sigma_r > 0$	UNSTABLE	disturbance will grow
if $\sigma_r < 0$	STABLE	" " die off
if $\sigma_r = 0$	Neutrally or Marginally stable	

This is an <u>eigenvalue problem</u>

Eq. (1) (disturbance) is plugged into linearized disturbance eq.

→ get a <u>differential eq.</u> for $\hat{u}(y)$

Nontrivial solns of this diff. eq can be found <u>only</u> for certain values of k, m, & σ

- k, m, & σ are the <u>eigenvalues</u>
- $\hat{u}(y)$ corresponding to these eigenvalues is the <u>eigenfunction</u>

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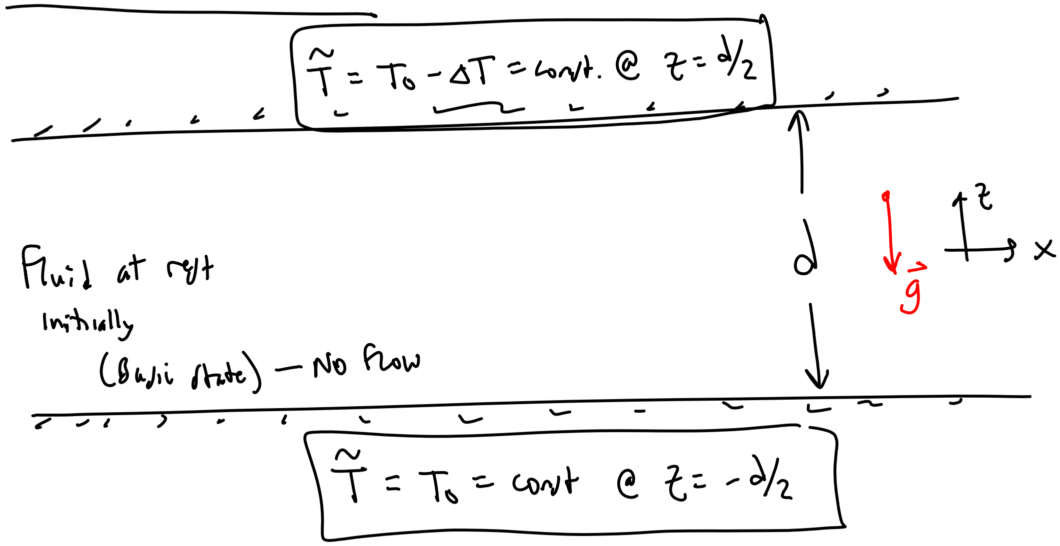
2. <u>The Bénard Problem</u> (Thermal Instability) [Sec. 12.3 Kundu]



<u>Stability analysis</u>

Step 0) → appropriate equations — use <u>Boussinesq approx.</u> (Sec. 4.18)

"quasi-incompressible"

- ρ = const in all terms <u>except</u> the gravity term (buoyancy)
- All other properties (k, μ, etc.) are constant

Continuity

$$\boxed{\frac{\partial \tilde{u}_i}{\partial x_i} = 0} \quad (1)$$

$\left[\rho_0 = \text{constant} = \text{density @ } T = T_0\right]$
$= \text{reference density}$

Momentum

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x_i} + \frac{\tilde{\rho} g_i}{\rho_0} + \nu \nabla^2 \tilde{u}_i$$

Boussinesq → let $\boxed{\tilde{\rho} = \rho_0\left(1 - \alpha\left(\tilde{T} - T_0\right)\right)}$

Note: $g_i = (0, 0, -g) = -g\delta_{i3}$

α = coeft. of thermal expansion $\rightarrow \alpha = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right)_p$

Gravity term becomes $\frac{\tilde{\rho} g_i}{\rho_0} = -g\delta_{i3}\left[1 - \alpha\left(\tilde{T} - T_0\right)\right]$

Mom. eq. is

$$\boxed{\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x_i} - g\delta_{i3}\left[1 - \alpha\left(\tilde{T} - T_0\right)\right] + \nu \nabla^2 \tilde{u}_i} \quad (2)$$

Energy Eq. Boussinesq approx. → use incompressible form

$$\frac{D\tilde{T}}{Dt} = \frac{\partial \tilde{T}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{T}}{\partial x_j} = K\nabla^2 \tilde{T} = K\frac{\partial^2 \tilde{T}}{\partial x_j \partial x_j}$$

where $\boxed{K = \frac{k}{\rho c_p}}$ = thermal diffusivity

(some books use α instead of K)

$$\boxed{\frac{\partial \tilde{T}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{T}}{\partial x_j} = K\frac{\partial^2 \tilde{T}}{\partial x_j \partial x_j}} \quad (3)$$

K is a constant in Boussinesq approx.

5 eqs & 5 unknowns $\rightarrow$ $\tilde{u}_i$ (3 components), $\tilde{p}$, $\tilde{T}$

Step 1 Basic state $\rightarrow$ U_i, $\bar{T}$, P

$\boxed{U_i = 0}$ $(U = V = W = 0)$ No Flow

We need to write eqs (1), (2) & (3) for the basic state

Set $\tilde{u}_i = U_i = 0$

$\tilde{p} = P(x, y, z)$ in general

$\tilde{T} = \bar{T}(x, y, z)$ "

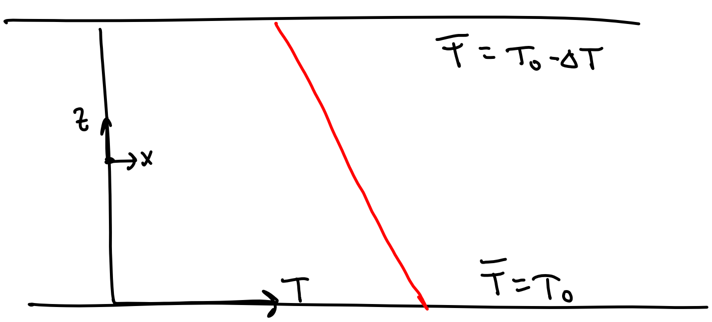
- Continuity: $\dfrac{\partial \cancel{U_i}^{=0}}{\partial x_i} = 0 \rightarrow \boxed{0 = 0}$ (1b)

- Energy: Here $\bar{T} = \bar{T}(z)$ only

$(3) \rightarrow \underbrace{\cancel{\dfrac{\partial \bar{T}}{\partial t}}}_{\text{steady basic state}} + \underbrace{\cancel{U_j}}_{0} \dfrac{\partial \bar{T}}{\partial x_j} = K \dfrac{\partial^2 \bar{T}}{\partial x_j \partial x_j}$ $\boxed{K \dfrac{\partial^2 \bar{T}}{\partial x_j \partial x_j} = 0}$ (3b)

Since $\bar{T} = T(z)$ only, we can integrate

$\boxed{\bar{T}(z) = T_0 - \Gamma\left(z + \frac{d}{2}\right)}$ where $\boxed{\Gamma = -\dfrac{\Delta T}{d}}$



PURE CONDUCTION SINCE NO FLOW

Momentum eq (2) Plug in $\tilde{u}_i = U_i = 0$

$\tilde{p} = P, \quad \tilde{T} = \bar{T}$

$$\boxed{0 = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i} - g\delta_{i3}\left[1 - \alpha\left(\bar{T} - T_0\right)\right]} \qquad (2b)$$

Step 2) Add a disturbance into the original eqs. (1), (2), (3)

$$\tilde{u}_i = \cancel{U_i} + \underbrace{u_i(x,y,z,t)}_{\text{disturbance velocity}}$$

0

$$\tilde{T} = \bar{T}(z) + T'(x,y,z,t)$$

$$\tilde{p} = P(z) + p(x,y,z,t)$$

Plug these into Eq. (1), (2), & (3)

[We will start here on Monday]