Today, we will:

- Continue working on the example problem – the Bénard instability

Recall, we did steps 0 and 1, and started working on step 2. The total flow variables are: \( \vec{u} = u_j, \vec{T} = T(z) + \vec{T}'(x,y,z,t) \), and \( \vec{p} = P(z) + p(x,y,z,t) \). Now plug these into Eqs. (1), (2), and (3).

\[
\frac{\partial u_i}{\partial x_i} = 0 \quad \text{(1a)}
\]

\[
0 = \frac{\partial p}{\partial x_i} - \frac{1}{\rho_0} \frac{\partial P}{\partial x_i} - \frac{1}{\rho_0} \frac{\partial \rho}{\partial x_i} \quad \text{(1b)}
\]

\[
0 = \frac{1}{\rho_0} \frac{\partial \rho}{\partial x_i} - \frac{1}{\rho_0} \frac{\partial P}{\partial x_i} - g \delta_{ij} \left[ 1 - a \left( T - T_0 \right) \right] \quad \text{(1c)}
\]

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x_i} - \frac{1}{\rho_0} \frac{\partial \rho}{\partial x_i} - \frac{1}{\rho_0} \frac{\partial \rho}{\partial x_i} \quad \text{(1d)}
\]

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x_i} - g \delta_{ij} \left[ 1 - a \left( T - T_0 \right) \right] \quad \text{(2a)}
\]

\[
- \frac{\partial u_j}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x_i} + g \delta_{ij} \left[ 1 - a \left( T - T_0 \right) \right] \quad \text{(2b)}
\]

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x_i} + g \delta_{ij} \left[ 1 - a \left( T - T_0 \right) \right] \quad \text{(3a)}
\]

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = K \frac{\partial^2 T}{\partial x_j \partial x_j} + K \frac{\partial^2 T}{\partial x_j \partial x_j} \quad \text{(3b)}
\]

\[
- \frac{\partial u_j}{\partial x_j} = K \frac{\partial^2 T}{\partial x_j \partial x_j} \quad \text{(3c)}
\]

\[
\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} + u_j \frac{\partial T'}{\partial x_j} = K \frac{\partial^2 T'}{\partial x_j \partial x_j} \quad \text{(3d)}
\]

**Step 3:** take (1b)-(1c), (2b)-(3c), (3c)-(3d) — see above ↑

Note some nonlinear terms e.g.
**Step 4:** Linearize the disturbance eqs. \([T' \ll \bar{T}, \rho \ll \rho_0]\)

All terms circled like \(\odot\) are nonlinear – cross them off in above eqs.

\[
(3a) \rightarrow \frac{dT'}{dt} + u_j \frac{dT'}{dx_j} = K \frac{dT'}{dx_j dx_j}
\]

We define \(\Gamma = -\frac{dT}{dt} = -\frac{dT'}{dt}\)

\[
\frac{dT'}{dt} - \Gamma \rho = K \frac{2\nu T'}{dx_j dx_j}
\]

(3b) [on handout]

See handout \(\rightarrow (1c), (3b), (2b)\) are the linearized disturbance eqs.

**Step 5:** Solve the linearized diff. eqs.

\(\Gamma\) unknown, \(\Gamma\) eqs. \(
\begin{align*}
&u_i, \rho, T' \\
&\text{linear diff. eqs.}
\end{align*}
\)

Look at \(i=3\) component of (2b):

\[
\frac{2w}{dt} = -\frac{1}{\rho_0} \frac{dx}{dt} + g \rho \frac{dT'}{dx_3} + \nu \nabla^2 w
\]

\[\nabla^2 w = 0\]

\[
\frac{2}{dt} \nabla^2 w = \left[ -\frac{1}{\rho_0} \frac{dx}{dt} + g \rho \frac{dT}{dx_3} + \nu \nabla^2 w \right]_i
\]

Take divergence of this:

(order of diff. doesn’t matter)

\[
\nabla \cdot \left[ \frac{2}{d x_i (2b)} \right] = \frac{2}{d x_i} \frac{d u_i}{dt} = -\frac{1}{\rho_0} \frac{dx}{dx_i dx_i} + g \rho \frac{dT}{dx_i} + \nu \frac{2}{dx_j dx_j} \frac{d u}{dx_i}
\]

\(\text{(cont)}\)
\[ 0 = -\frac{1}{\rho_0} \frac{\partial^2 \rho}{\partial x \partial x} + g \alpha \sigma_{ij} \frac{\partial T'}{\partial x_i} \]

Laplace: only non-zero when \( i = 3 \)

\[ 0 = -\frac{1}{\rho_0} \Delta \rho + g \alpha \frac{\partial T'}{\partial x} \]

The \( \frac{1}{2} \) of \( \frac{\partial \rho}{\partial t} \), get

\[ 0 = \left\{ -\frac{1}{\rho_0} \Delta \left( \frac{\partial H}{\partial t} \right) \right\} + g \alpha \frac{\partial T'}{\partial x} \quad (5) \]

The two terms \( [\bigbox] \) in \((4)\) i. (5) are identical

Substitute (5) into (4) to eliminate pressure!

\[ \frac{d}{dt} (\nabla^2 w) = g \alpha \left[ \nabla^2 T' - \frac{\partial^2 T'}{\partial x^2} \right] + \nu \nabla^4 w \]

Define \( \nabla_H^2 = \text{Horizontal Laplace} \)

\[ \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

\[ \frac{d}{dt} (\nabla^2 w) = g \alpha \nabla_H^2 T' + \nu \nabla^4 w \quad (6) \quad \text{(an handout)} \]

Recall,

\[ \frac{\partial T'}{\partial t} - \nabla \cdot \mathbf{w} = K \nabla^2 T' \quad (3e) \]

No \( u, v, \) or \( p \) in this set of 2 eqs

2 unknowns

AS UNCOUPLED FROM THE REST OF THE PROBLEM
\text{BC's:} \quad \text{w.r.t. } \frac{z}{\alpha} \text{ to } \quad \text{e.g. higher-order terms } \quad \frac{2^{2^{1}}}{3^{3}} \to \text{needs 2 BCs on } T' \text{ at } z \text{ location.}

\text{e.g.,}

\frac{\partial w}{\partial y} = \text{needs } y \text{ BCs on } W \text{ at } z \text{ location.}

\text{e.g.}

\text{Tangential } \frac{\partial w}{\partial y} = \text{const.} \implies \text{no slip at both walls.}

\implies W = T' = 0 \quad \text{at } z = \pm \frac{1}{2}

\text{Continuity } \frac{\partial u}{\partial x} = \frac{2u}{2x} + \frac{2y}{2y} + \frac{\partial u}{\partial z} = 0 \implies \frac{\partial w}{\partial z} = 0 \quad \text{at the wall.}

\text{At the wall, } u = 0 \implies \frac{\partial u}{\partial x} = 0 \text{ since } u = 0 \text{ anywhere along the wall.}

\text{We also need } 2 \text{ BCs on } T' \text{ at some } x \text{ location.}

\text{and } 2 \text{ BCs on } T' \text{ at some } y \text{ location.}

\text{(But this is nothing special about any } x \text{ or } y \text{ location (infinite plate))}

\text{If we use the method of normal modes (apply infinite waves in } x, y \text{ directions),}

\text{this automatically takes care of these BCs.}
"Semi-normalization" of the equation
Kuhn normality the independent variables only

Let \( x^* = \frac{x}{J}, \quad y^* = \frac{y}{J}, \quad z^* = \frac{z}{J} \)

Time scale \( \rightarrow \) we create time scale \( \frac{J^2}{K} \)

\[ \{ \frac{L^1}{J^2/t} \} = \{ t \} \]

So, we let \( t^* = t \frac{J}{(x^*)^2} \)

Play into (3)

\[ \frac{\partial T'}{\partial x^*} - \frac{\partial y^*}{K} \]

\[ = \nabla^2 T' \]

Kuhn drops the * superscript \( \rightarrow \) See Eq (7) on handout

Likewise for Eq (6)

Define \( Pr = \frac{P_{\text{nat}}}{K} \)

\[ \left( \frac{1}{Pr} \frac{2}{\partial t^*} - \nabla^2 \right) \nabla^2 W = \frac{g^* J^2}{\rho^*} \nabla^2 T' \]

drop * superscript \( \rightarrow \) set \((8)\)

in handout

New normalized BCs:

\[ W = \frac{2w}{\rho} = T' = 0 \quad @ \quad t = \pm \frac{J}{2} \]

\[ \downarrow \]

\[ W = \frac{2w}{\rho} = T' = 0 \quad @ \quad t^* = \pm \frac{1}{2} \]

Drop * again:

\[ W = \frac{2w}{\rho} = T' = 0 \quad @ \quad t = \pm \frac{1}{2} \]