Today, we will:

- Continue working on the example problem – the Bénard instability

Recall, we did steps 0 and 1, and started working on step 2. The total flow variables are: $\tilde{u}_i = u_i$, $\tilde{T} = \bar{T}(z) + T'(x,y,z,t)$, and $\tilde{p} = P(z) + p(x,y,z,t)$. Now plug these into Eqs. (1), (2), and (3).

(1) $\boxed{\dfrac{\partial u_i}{\partial x_i} = 0}$ (1t)

$-$ $0 = 0$ (1b)

$\boxed{\dfrac{\partial u_i}{\partial x_i} = 0}$ (1d)

(2) $\dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j} = -\dfrac{1}{\rho_0}\dfrac{\partial P}{\partial x_i} - \dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x_i} - g\delta_{i3}\left[1 - \alpha\left(\bar{T} + T' - T_0\right)\right] + \nu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j}$ (2t)

$-$ $0 = -\dfrac{1}{\rho_0}\dfrac{\partial P}{\partial x_i} - g\delta_{i3}\left[1 - \alpha\left(\bar{T} - T_0\right)\right]$ (2b)

$\boxed{\dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j} = -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x_i} + g\delta_{i3}\alpha T' + \nu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j}}$ (2d)

(3) $\cancel{\dfrac{\partial \bar{T}}{\partial t}} + \dfrac{\partial T'}{\partial t} + u_j \dfrac{\partial \bar{T}}{\partial x_j} + u_j \dfrac{\partial T'}{\partial x_j} = K \dfrac{\partial^2 \bar{T}}{\partial x_j \partial x_j} + K \dfrac{\partial^2 T'}{\partial x_j \partial x_j}$ (3t)

$\bar{T}$ = steady

$-$ $0 = K \dfrac{\partial^2 \bar{T}}{\partial x_j \partial x_j}$ (3b)

$\boxed{\dfrac{\partial T'}{\partial t} + u_j \dfrac{\partial \bar{T}}{\partial x_j} + u_j \dfrac{\partial T'}{\partial x_j} = K \dfrac{\partial^2 T'}{\partial x_j \partial x_j}}$ (3d)

Step 3: take (1t)-(1b), (2t)-(2b), (3t)-(3b) → see above ↑

Notice some nonlinear terms, e.g.

Step 4: Linearize the disturbance eq's $\quad [T' \ll \bar{T},\ p \ll P]$

All terms circled like ◌ are nonlinear — cross them off in above eqs

$$(3l) \rightarrow \frac{\partial T'}{\partial t} + \underbrace{u_j \frac{\partial \bar{T}}{\partial x_j}}_{u \cancel{\frac{\partial \bar{T}}{\partial x}} + v \cancel{\frac{\partial \bar{T}}{\partial y}} + w \frac{d\bar{T}}{dz}} = K \frac{\partial^2 T'}{\partial x_j \partial x_j}$$

$\bar{T} = \bar{T}(z)$ only

we defined $\Gamma = -\dfrac{d\bar{T}}{dz} = \dfrac{\Delta T}{d}$

$$\boxed{\frac{\partial T'}{\partial t} - \Gamma w = K \frac{\partial^2 T'}{\partial x_j \partial x_j}} \quad (3l) \quad [\text{on handout}]$$

See Handout → (1l), (2l), (3l) are the linearized disturbance eqs.

Step 5) Solve the linearized dist. eq's

$$\left.\begin{array}{l} 5 \text{ unknowns, } 5 \text{ eq's} \\ u_i,\ p,\ T' \end{array}\right\} \text{linear diff. eq's}$$

Look @ $i=3$ component of (2l):

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\alpha T' + \nu \nabla^2 w$$

- Take Laplacian of this: (order of diff. doesn't matter)

$$\boxed{\frac{\partial}{\partial t}\nabla^2 w = -\frac{1}{\rho_0}\nabla^2\left(\frac{\partial p}{\partial z}\right) + g\alpha \nabla^2 T' + \nu \nabla^4 w} \quad (4)$$

- Take divergence of (2l) $\quad \left[\dfrac{\partial}{\partial x_i}\left[(2l)\right]\right]$

$$\frac{\partial}{\partial t}\underbrace{\cancel{\frac{\partial u_i}{\partial x_i}}}_{0\ (\text{cont})} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x_i \partial x_i} + g\alpha \delta_{i3}\frac{\partial T'}{\partial x_i} + \nu \frac{\partial^2}{\partial x_j \partial x_j}\overbrace{\cancel{\frac{\partial u_i}{\partial x_i}}}^{0\ (\text{cont})}$$

$$0 = -\frac{1}{\rho_0}\underbrace{\frac{\partial^2 p}{\partial x_i \partial x_i}}_{\downarrow\ \text{Laplacian}} + \underbrace{g\alpha\delta_{i3}\frac{\partial T'}{\partial x_i}}_{\text{only non-zero when } i=3}$$

$$0 = -\frac{1}{\rho_0}\nabla^2 p + g\alpha\frac{\partial T'}{\partial z}$$

Take $\frac{\partial}{\partial z}$ of this eq, Get $\quad 0 = \boxed{-\frac{1}{\rho}\nabla^2\left(\frac{\partial p}{\partial z}\right)} + g\alpha\frac{\partial^2 T'}{\partial z^2} \qquad (5)$

The two terms $\boxed{}$ in (4) & (5) are identical

Substitute (5) into (4) to eliminate pressure!

$$\frac{\partial}{\partial t}(\nabla^2 w) = g\alpha\underbrace{\left[\nabla^2 T' - \frac{\partial^2 T'}{\partial z^2}\right]}_{\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2}} + \nu\nabla^4 w$$

Define ∇_H^2 = Horizontal Laplacian $\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$$\boxed{\frac{\partial}{\partial t}(\nabla^2 w) = g\alpha\nabla_H^2 T' + \nu\nabla^4 w} \qquad (6) \text{ (on handout)}$$

recall,

$$\boxed{\frac{\partial T'}{\partial t} - \Gamma w = K\nabla^2 T'} \qquad (3e)$$

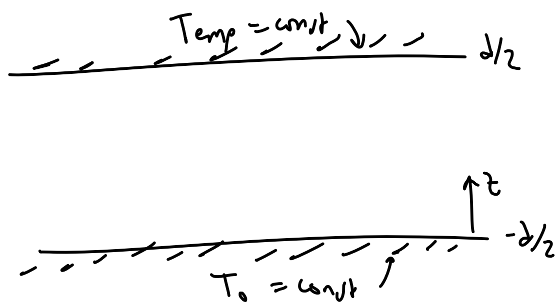
no u, v, or p in this set of 2 eqs
2 unknowns
$[T'$ & $w]$

☆ UNCOUPLED FROM THE REST OF THE PROBLEM

BC's: w.r.t. z → e.g. highest-order terms $\frac{\partial^2 T'}{\partial z^2}$ → need 2 BCs on T' at z locations

e.g. $\frac{\partial^4 w}{\partial z^4}$ → need 4 BCs on w at z locations

⋮ etc.



@ both walls, must satisfy no slip

$$\therefore \boxed{w = T' = 0 \quad @ \quad z = \pm d/2}$$

Continuity eq → $\frac{\partial u_i}{\partial x_i} = \cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial v}{\partial y}} + \frac{\partial w}{\partial z} = 0 \Rightarrow \boxed{\frac{\partial w}{\partial z} = 0 \quad @ \quad z = \pm d/2}$

at the wall

At the walls, $u = 0$

∴ $\frac{\partial u}{\partial x} = 0$ since $u = 0$ anywhere along the wall

We also need 2 BCs on T' @ some x locations
4 " " w " " "

2 BCs on T' @ some y locations
4 " " w " " "

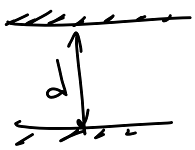
But – there is nothing special about any x or y location (infinite plates)

→ If we use the method of normal modes (apply infinite waves in x & y directions),

this automatically takes care of these BCs

"Semi-normalization" of the equations

Kundu normalizes the independent variables only



Let $\boxed{x^* = \frac{x}{d}, \quad y^* = \frac{y}{d}, \quad z^* = \frac{z}{d}}$

time scale → we create time scale $= \frac{d^2}{K}$ $\qquad \left\{ \frac{L^2}{L^2/t} \right\} = \{t\}$

So, we let $\boxed{t^* = t\,\frac{K}{d^2}}$

$$\nabla^{*2} = \frac{\partial^2}{\partial x_i^* \partial x_i^*}$$

Plug into (3e) → $$\boxed{\frac{\partial T'}{\partial t^*} - \frac{\Gamma d^2}{K} w = \nabla^{*2} T'} \qquad (7)$$

Kundu drops the * superscripts → $\boxed{\text{See Eq (7) on handout}}$

Likewise for Eq (6)

Define $\boxed{Pr = \text{Prandtl \#} \equiv \frac{\nu}{K}}$

$$\boxed{\left(\frac{1}{Pr} \frac{\partial}{\partial t^*} - \nabla^{*2} \right) \nabla^{*2} w = \frac{g\alpha d^2}{\nu} \nabla_H^{*2} T'} \quad (8)$$

drop * superscript to get (8) on handout

New normalized BCs :

$$w = \frac{\partial w}{\partial z} = T' = 0 \quad @ \quad z = \pm \frac{d}{2}$$

$$\Downarrow$$

$$w = \frac{\partial w}{\partial z^*} = T' = 0 \quad @ \quad z^* = \pm \frac{1}{2}$$

drop * again → $$\boxed{w = \frac{\partial w}{\partial z} = T' = 0 \quad @ \quad z = \pm 1/2}$$