M E 522 Spring 2008 Professor John M. Cimbala Lecture 13 02/13/2008

Today, we will:

- Continue with the example problem – the Bénard problem – temperature instability between two infinite plates: Apply the method of normal modes and solve.

We begin with Eqs. (7) and (8) on the handout (semi-normalized eqs. for the disturbance variables w and T'): $\boxed{\frac{\partial T'}{\partial t} - \frac{\Gamma d^2}{\kappa} w = \nabla^2 T'}$ (7), and $\boxed{\left(\frac{1}{\Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = \frac{g\alpha d^2}{\nu}\nabla_H{}^2 T'}$ (8) .

along with their BCs: $w = \frac{\partial w}{\partial z} = T' = 0$ at $z = \pm\frac{1}{2}$.

Apply the Method of Normal Modes → disturbances are periodic in x & y

may be periodic and/or growing in t

Let $w = \hat{w}(z)\, e^{ikx + ily + \sigma t}$

$T' = \hat{T}(z)$ "

Where • $\hat{w}(z)$, $\hat{T}(z)$ are complex amplitudes

• k, l are x, y components of wavenumber vector $\vec{K} = (k, l, \cancel{m})$ 0

Here • k & l must be real → no growth in x & y directions

• σ = growth rate = $\sigma_r + i\sigma_i$ in general (may be growing or decaying in time)

[temporal stability analysis]

Plug these into Eqs (7) & (8)

(7) → $\frac{\partial T'}{\partial t} - \frac{\Gamma d^2}{\kappa} w = \nabla^2 T'$

$$\hat{T}(z)\, e^{ikx + ily + \sigma t}\,\sigma - \frac{\Gamma d^2}{\kappa}\hat{w}(z)\, e^{(\quad)} = \hat{T}(z)\, e^{(\quad)} (ik)^2 + \dots$$

$i^2 k^2 = -k^2$

every term has $e^{(\quad)}$ so we can cancel it out

also, define $K = \sqrt{k^2 + l^2}$ = magnitude of the wavenumber vector

We get $\boxed{\left[\sigma - \left(\frac{d^2}{dz^2} - K^2\right)\right]\hat{T} = \frac{\Gamma d^2}{\kappa}\hat{w}}$ (9)

Let $\boxed{D = \frac{d}{dz}}$ $\boxed{W = \frac{\Gamma d^2}{\kappa}\hat{w}}$

$$\boxed{Ra = \text{Rayleigh \#} = \frac{g\alpha|\Delta T|d^3}{\kappa\nu} = \frac{g\alpha\Gamma d^4}{\kappa\nu}}$$

since $\Gamma = \frac{\Delta T}{d}$

(9) becomes Eq (11) on handout

& (8) " " (12) " "

BC's become

$$\boxed{W = DW = \hat{T} = 0 \text{ @ } z = \pm 1/2}$$

Comments: 1) No longer any distinction between x & y directions

k = wavenumber in x-direction
l = " " y- "
} combined into $K = \sqrt{k^2 + l^2}$

(because nothing special about x or y directions)

2) x, y, & t have dropped out! z is the only remaining indep. variable

Now we have ODEs!

originally,

5 pdes 5 unknowns u_i, T', p as fncs of x, y, z, t	$\Rightarrow$	2 pdes 2 unknowns w, T' as fncs of x, y, z, t	$\Rightarrow$	2 odes 2 unknowns $\hat{W}(z)$ & $\hat{T}(z)$

3) It can be proven [see Kundu, pg. 460-461] that σ must be real

$$\sigma = \sigma_r + i\underbrace{\cancel{\sigma_i}}_{0} = \sigma_r = \text{real}$$

recall $\quad w = \hat{w}(z)\, e^{ikx + ily + \sigma t}$

sinusoidal waves in x & y directions

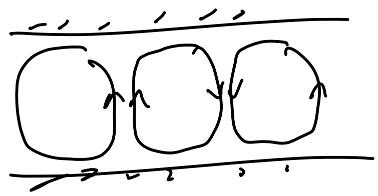
growth & decay in time + periodicity in time

But $\quad \boxed{e^{\sigma t} = e^{\sigma_r t}}\, {}^{+ i \underbrace{\cancel{\sigma_i}}_{0} t}$

$\therefore$ NO PERIODICITY IN TIME

$\therefore$ we conclude that the instability will cause the basic state to jump to another steady state rather than an oscillating state

agrees with experiment $\rightarrow$ get cells



We call this: The principle of exchange of stabilities ☆

Also, since $\sigma = \sigma_r = \text{real}$, σ_r determines the stability

$$e^{\sigma t} \rightarrow \begin{array}{lll} \text{if } \sigma < 0 & \text{stable} \\ \text{if } \sigma > 0 & \text{unstable} \\ \text{if } \sigma = 0 & \text{neutrally or marginally stable} \end{array}$$

☆ Let's examine the marginal stability case

☆ Let's set or force $\sigma = 0$ to find the marginal stability condition

Eq's (11) & (12) $\Rightarrow$

$$(D^2 - K^2)\hat{T} = -\hat{W} \qquad (11\ m)$$

$$(D^2 - K^2)^2 \hat{W} = Ra\, K^2 \hat{T} \qquad (12\ m)$$

marginal

To solve (11 m) & (12 m): Take (D^2-K^2) of (12 m):

$$(D^2-K^2)^3 W = Ra K^2 \underbrace{(D^2-K^2)\hat{T}}_{= -W \text{ from Eq. (11 m)}}$$

$$\therefore \quad \boxed{(D^2-K^2)^3 W = -Ra K^2 W}$$ → ! we've eliminated T'

ONE ODE FOR W(z)

$$\boxed{\left[(D^2-K^2)^3 + Ra K^2\right] W(z) = 0} \qquad (13 \text{ m})$$

☆ a linear, homogeneous o.de. with constant coefficients

Solve using the "auxiliary eq." for q

$$\boxed{(q^2-K^2)^3 + Ra K^2 = 0}$$ → Find the roots of this eq.

6^{th}-order → ∴ get 6 roots $q_1, q_2, q_3 \cdots q_6$

Solution is $W(z) = c_1 e^{q_1 z} + c_2 e^{q_2 z} + \cdots c_6 e^{q_6 z}$

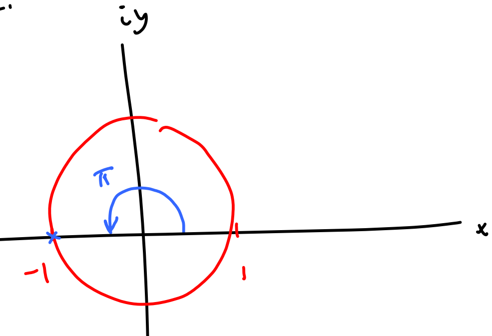
or → if any roots are repeated, e.g., if q_1 is repeated

$$W(z) = c_1 e^{q_1 z} + c_2 z e^{q_1 z} + \cdots$$

Here, we get 6 distinct roots ⇒ $(q^2-K^2)^3 = -Ra K^2$

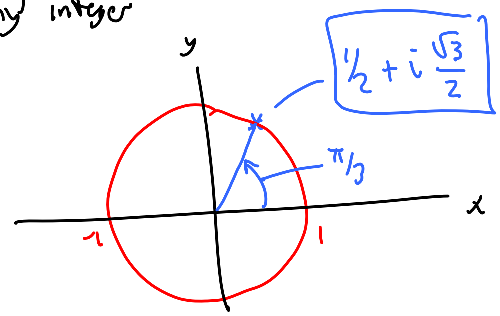
$$\boxed{q = \pm K \sqrt{1 + \left(\frac{Ra}{K^4}\right)^{1/3} (-1)^{1/3}}} \qquad (13 \text{ r})$$

recall, $-1 = e^{i\pi} = e^{3i\pi}$, etc.



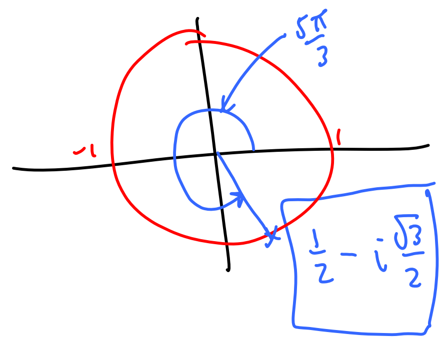
i.e., $\boxed{-1 = e^{i\pi(1+2m)}}$ where $m =$ any integer

when $m=0$ $\therefore$ $(-1)^{1/3} = \left(e^{i\pi}\right)^{1/3} = e^{i\pi/3}$



when $m=1$ $(-1)^{1/3} = \left(e^{3i\pi}\right)^{1/3} = e^{i\pi} = \boxed{-1}$

when $m=2$ $(-1)^{1/3} = \left(e^{5i\pi}\right)^{1/3} = e^{\frac{5}{3}i\pi}$



$\therefore$ 6 distinct independent roots of Eq. 13 r

Kundu calls them $\pm i q_0$, $\pm q$, & $\pm q^*$

where $\boxed{q_0 = K\sqrt{\left(\frac{Ra}{K^4}\right)^{1/3} - 1}}$

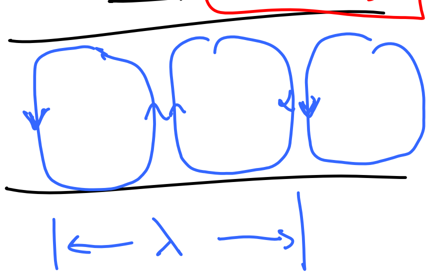
$\boxed{q = K\sqrt{1 + \frac{1}{2}\left(\frac{Ra}{K^4}\right)^{1/3}(1 + i\sqrt{3})}}$

$\boxed{q^* = \ldots\ldots\ldots - \ldots}$

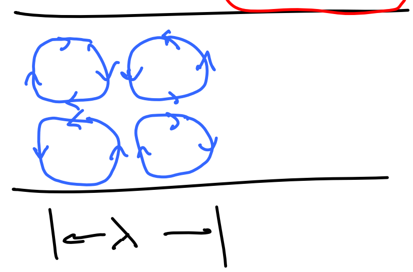
So finally
$$\boxed{W(z) = \underbrace{C_1 e^{iq_0 z} + C_2 e^{-iq_0 z}}_{C_1 = C_2 \text{ to be even}} + \underbrace{C_3 e^{qz} + C_4 e^{-qz}}_{C_3 \text{ must } = C_4 \text{ to be even}} + \underbrace{C_5 e^{q^* z} + C_6 e^{-q^* z}}_{C_5 \text{ must } = C_6 \text{ to be even}}}$$

Also, see Kundu → odd mode is less likely than the even mode

Even — most likely



ODD — Less likely



So, combine into $\cos(q_0 z)$ & $\cosh(qz)$

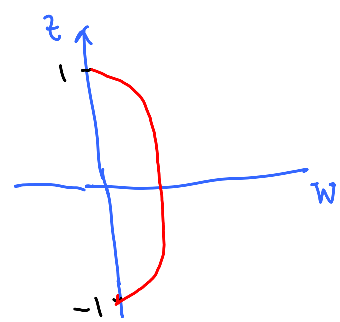
$$\therefore \quad \boxed{W(z) = A\cos(q_0 z) + B\cosh(qz) + C\cosh(q^* z)}$$

where $A, B, \& \, C$ = new constants

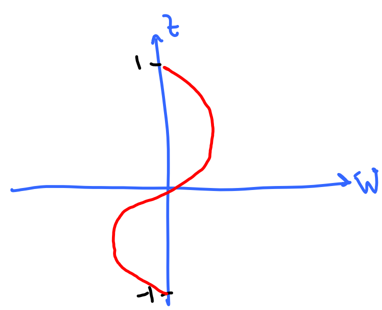
A — even mode only

NOTE: AFTER CLASS THERE WAS A QUESTION ABOUT HOW WE ARE DEFINING ODD & EVEN HERE.

In terms of $W(z)$, the even mode has



The odd mode has



[as can be verified on the cell patterns above]