Today, we will:

- Finally finish the example problem – the Bénard instability
- Discuss some other stability example problems
- Do Candy Questions for Candy Friday

Start with $\boxed{W(z) = A\cos(q_0 z) + B\cosh(qz) + C\cosh(q^* z)}$ (14)

[even mode of disturbance] Sol'n for marginal stability

- Use BCs: $\boxed{@\, z = +/- \tfrac{1}{2}}$ $\boxed{W = 0}$ (15)

$$\boxed{DW = \frac{dW}{dz} = 0} \quad (16)$$

& $\hat{T} = 0$

$$\left[(D^2 - K^2)^2 W = Ra K^2 \hat{T} \quad (12\ m) \right]$$

BC becomes $\boxed{(D^2 - K^2)^2 W = 0}$ (17)

Plug into (14):

@ $z = \pm \tfrac{1}{2}$, (15) → $\boxed{A\cos\frac{q_0}{2} + B\cosh\frac{q}{2} + C\cosh\frac{q^*}{2} = 0}$ (15 bc)

@ $z = \pm \tfrac{1}{2}$ (16) → $\boxed{A\left(-q_0 \sin\frac{q_0}{2}\right) + B\left(q\sinh\frac{q}{2}\right) + C\left(q^*\sinh\frac{q^*}{2}\right) = 0}$ (16 bc)

@ $z = \pm \tfrac{1}{2}$ (17) → (17 bc)

Let's combine (15 bc) – (17 bc) into matrix form

$$\begin{bmatrix} \cos\frac{q_0}{2} & \cosh\frac{q}{2} & \cosh\frac{q^*}{2} \\ -q_0 \sin\frac{q_0}{2} & q\sinh\frac{q}{2} & q^*\sinh\frac{q^*}{2} \\ (q_0^2+K^2)^2\cos\frac{q_0}{2} & (q^2-K^2)^2\cosh\frac{q}{2} & (q^{*2}-K^2)^2\cosh\frac{q^*}{2} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This is where the <u>eigenvalue</u> nature of the problem pops up.

Non-trivial sol'ns for A, B, C (non-zero) <u>only</u> if det [matrix] = 0

This will happen only @ certain values of eigenvalue K

& for various values R_a

See mathcad sol'n on website & next pg.

[I solved the above matrix eq. using Mathcad]

Benard Convection Problem - Eigenvalue Solution for Marginal Stability

J. M. Cimbala

First define the roots q0, q, and qs as functions of wavenumber K and Rayleigh number Ra:

$$q0(K,Ra) := K\cdot\sqrt{\left[\left(\frac{Ra}{K^4}\right)^{\frac{1}{3}} - 1\right]} \quad q(K,Ra) := \sqrt{K^2\cdot\left[1 + \frac{1}{2}\cdot\left(\frac{Ra}{K^4}\right)^{\frac{1}{3}}\cdot\left(1+\sqrt{3}\cdot i\right)\right]} \quad qs(K,Ra) := \sqrt{K^2\cdot\left[1 + \frac{1}{2}\cdot\left(\frac{Ra}{K^4}\right)^{\frac{1}{3}}\cdot\left(1-\sqrt{3}\cdot i\right)\right]}$$

Now define the matrix as given in the class notes (Matrix M is also a function of K and Ra):

$$M(K,Ra) := \begin{bmatrix} \cos\left(\frac{q0(K,Ra)}{2}\right) & \cosh\left(\frac{q(K,Ra)}{2}\right) & \cosh\left(\frac{qs(K,Ra)}{2}\right) \\ -q0(K,Ra)\cdot\sin\left(\frac{q0(K,Ra)}{2}\right) & q(K,Ra)\cdot\sinh\left(\frac{q(K,Ra)}{2}\right) & qs(K,Ra)\cdot\sinh\left(\frac{qs(K,Ra)}{2}\right) \\ \left(q0(K,Ra)^2 + K^2\right)^2\cdot\cos\left(\frac{q0(K,Ra)}{2}\right) & \left(q(K,Ra)^2 - K^2\right)^2\cdot\cosh\left(\frac{q(K,Ra)}{2}\right) & \left(qs(K,Ra)^2 - K^2\right)^2\cdot\cosh\left(\frac{qs(K,Ra)}{2}\right) \end{bmatrix}$$

Define the determinant of this matrix, and its root as functions of K and Ra. *Note*: In order to use the root function, an initial guess for Ra must be specified. I also had to lower the tolerance for the root function from the default:

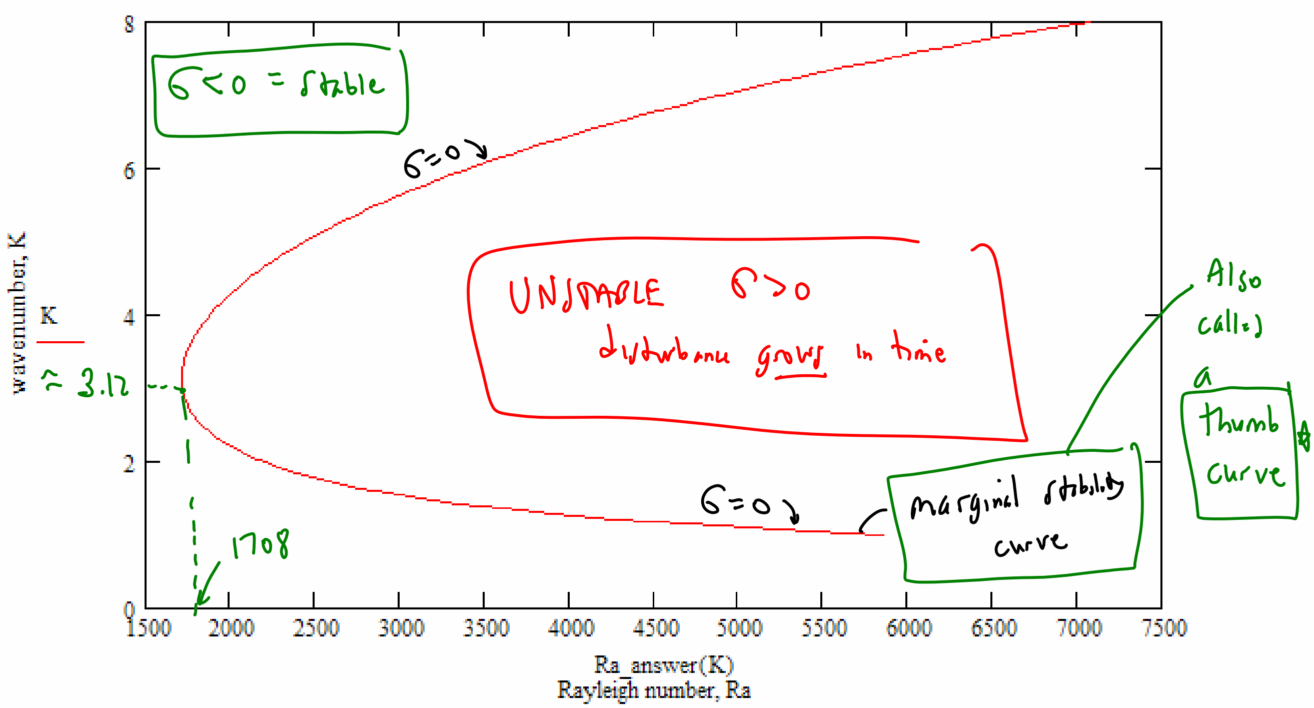
$$Ra\_guess := 4000 \quad Ra := Ra\_guess \quad determ(K,Ra) := |M(K,Ra)| \quad Ra\_answer(K) := root(determ(K,Ra),Ra)$$

Verify for some initial K: $TOL = 1\times 10^{-3}$ $\quad TOL := 1.0\cdot 10^{-6}$ $\quad K := 1$ $\quad Ra\_answer(K) = 5.854\times 10^3$

Specify a range of wavenumber K, and solve for the corresponding Ra which yields non-zero roots.

$$ntotal := 100 \quad Kmin := 1 \quad Kmax := 8 \quad K := Kmin, Kmin + \frac{(Kmax - Kmin)}{ntotal} .. Kmax$$

Plot the marginal (neutral) stability curve:



Write the minimum Ra for stability, along with its wavenumber, K (I'm not sure why this works in Mathcad):

$\underline{K = 3.111}$ $\quad \underline{Ra\_answer(K) = 1.708\times 10^3}$ → Minimum Ra is 1708 @ K = 3.111

(Kundu gives 3.12 as the K value)

$$\boxed{Ra_{critical} = 1708}$$ → This is the minimum Ra at which any kind of instability can occur

So – we predict that this is the instability that we would observe in an experiment

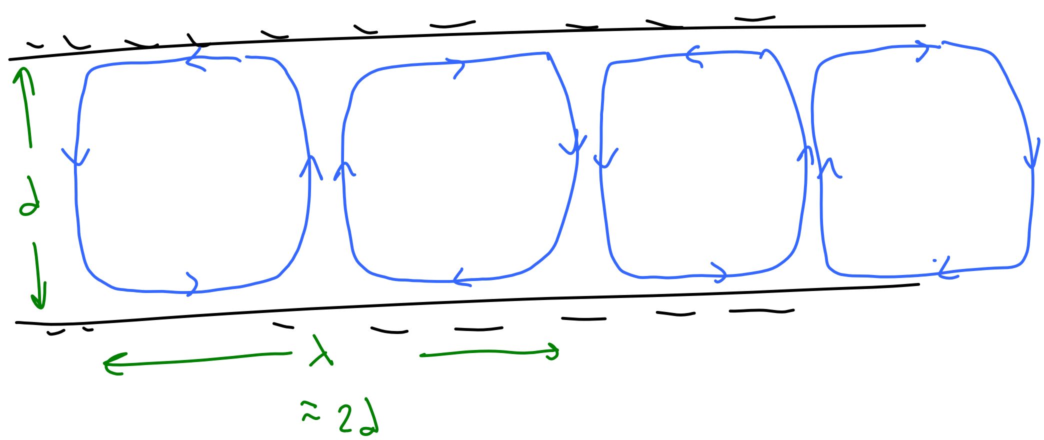
also, $K = \frac{2\pi}{\lambda}$ → $\therefore \lambda$ = wavelength $= \frac{2\pi}{K} = \frac{2\pi}{3.12} = 2.014$

So $\boxed{\lambda \simeq 2.0}$

this λ is actually $\lambda^* = \frac{\lambda}{d}$

Prediction: At $Ra \geq 1708$ we should see cells with $\lambda/d \simeq 2.0$

We expect nearly square cells



Experimentally → This prediction is a great success!!

[See photos on website]

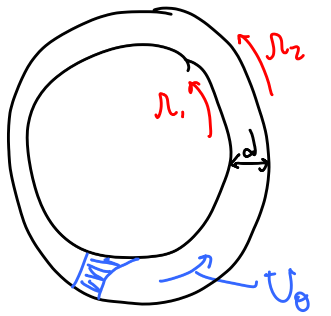
nearly square cells do indeed form when $Ra \gtrsim 1708$ and λ turns out to be nearly $2d$, just as predicted!

3. Other examples of instabilities

a. Centrifugal instability

two coaxial rotating cylinders with a small gap in between

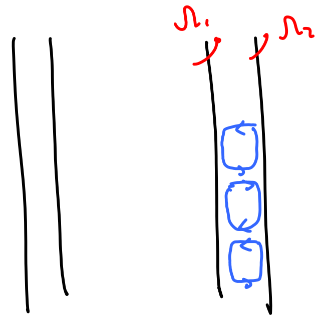
Top view:



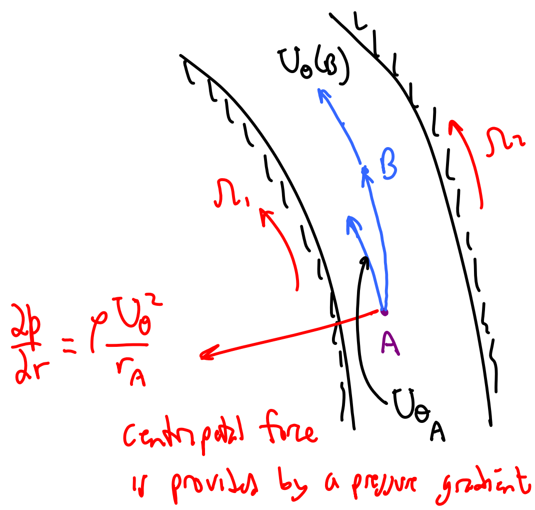
$U_\theta(r)$ = basic state

can be obtained analytically

Side view



Why & under what conditions is this flow unstable?



Basic state is $\boxed{U_\theta(r)}$

purely tangential (circumferential) flow

(no U_r component)

Add a disturbance so that the fluid particle moves outward to point B

where $U_\theta = U_{\theta_B}$

Next time, we will show that the basic state is stable for solid body rotation & for the case with $\Omega_2 > 0$ but $\Omega_1 = 0$,

but $\boxed{\text{it is unstable when } \Omega_1 > 0 \text{ \& } \Omega_2 = 0 \text{ (inner cyl. rotating)}}$ ☆