Today, we will:

- Continue to discuss the centrifugal instability (Taylor instability)
- Begin a discussion about stability of locally parallel flows – the Orr-Sommerfeld Eq.
3) Inner cyl. rotating, outer cyl. stationary

\[ r_A \omega_{A} \neq r_0 \omega_0 \]
\[ r_A < r_0 \]
\[ \text{But } \omega_{A} > \omega_0 \]

Potentially unstable

Define a Taylor number

\[ \overline{Ta} = 2 \left( \frac{V_1 d}{\nu} \right)^2 \frac{1}{R_i} \]

\[ V_1 = \pi \cdot R \cdot \omega \]
\[ d = R_2 - R_1 \text{ = gap thickness} \]

That is a critical \( Ta \) for which flow is unstable

\[ \text{Bénard} \]
\[ \lambda = 2d \]

\[ \text{Taylor} \]
\[ \lambda = 2d \]

See Kundu for details
D. Stability of locally parallel flows (Kundu - sec. 12.8)

1. Intro: Some flows are parallel e.g. fully developed channel flow, Couette flow, etc.

Other flows are nearly parallel e.g. thin BLs, thin vortices, etc.

Approx. the basic state as \( \mathbf{U} = (U(y), 0, 0) \) \( V \) and \( W \) are very small compared to \( U \)

Locally Parallel Flow Approximation \( \to U \approx \) inc. of \( x \)

(local analysis only)

Advantage: Simpler mathematical model.

We can examine temporal stability with the method of normal modes.
2. **Linear Stability Analysis**

Step 0: set of eqs. - Use NSE: no gravity

**Notation**
- \( \tilde{u}_i \): Total velocity
- \( \tilde{u}_i \): Dimensional \( i \) component

\[
\text{Cont} \quad \frac{\partial \tilde{u}_i}{\partial x_i} = 0 \quad \text{4 eqs, 4 unknowns} \tag{1.2}
\]

\[
\text{Mom} \quad \frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_i^2} \tag{2.1}
\]

Non-dimensionalize everything

Let \( L = \text{characteristic length scale} \rightarrow X_i = \frac{x_i}{L} \)

\( U_0 = \) velocity

\( \tilde{u}_i = \) \( \frac{\tilde{u}_i}{U_0} \)

\( \frac{L}{U_0} = \) time

\( \frac{p U_0^2}{\rho} = \) pressure

\( \tilde{p} = \) \( \frac{\tilde{p}}{\rho U_0^2} \)
Step 1  Basic State  \[ \vec{U} = (U(x), 0, 0) \]

\[ p_{beam} = P \]

Plug into eqs (1a) & (1b)  \[ \rightarrow \text{get (1b) i.e. (2b) on handout} \]

\[ \begin{align*}
\text{Conservation} & \quad \frac{\partial U_i}{\partial x_i} = 0 \quad \rightarrow \quad \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0 \quad \rightarrow \quad 0 = 0 \\
U &= U(x) \\
\end{align*} \]

\[ \begin{align*}
\text{Mom} & \quad \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = -\frac{\partial P}{\partial x} + \frac{1}{Re} \nabla^2 U = 0 \\
\end{align*} \]

\[ \begin{align*}
\text{Mom} & \quad 0 = -\frac{\partial P}{\partial y} \quad \rightarrow \quad P \neq \text{const of } y \\
\text{Mom} & \quad 0 = -\frac{\partial P}{\partial z} \quad \rightarrow \quad P \neq \text{const of } z \\
\end{align*} \]

\[ \begin{align*}
\text{So} & \quad -\frac{\partial P}{\partial x} + \frac{1}{Re} \nabla^2 U = 0 \\
\end{align*} \]

Step 2  All disturbance  \[ \tilde{U} = U + U' \]

\[ \begin{align*}
\tilde{U} = U + U' \\
\tilde{V} = 0 + V' \\
\tilde{W} = 0 + W' \\
\tilde{P} = P + p' \\
\end{align*} \]

Plug these into (1a) & (1b) to get (2a) & (2b)
1. Problem Setup:
Consider an incompressible, locally parallel flow with some known steady basic state \( U(y) \). We examine this problem using linear stability analysis. Note: A boundary layer type of flow is sketched, but the procedure applies to any kind of parallel or nearly parallel flow.

2. Summary of Linear Stability Analysis:
The in-class analysis follows Kundu, Section 12.8 closely, filling in some of the details. Start with the normalized incompressible Navier-Stokes equations for total flow variables (\( \tilde{q} \)) :

\[
\frac{\partial \tilde{u}_i}{\partial x_i} = 0 \quad (1) \text{ and } \frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} \quad (2) \text{ where } Re \equiv \frac{U_0 L}{\nu} \text{ and } U_0 \text{ and } L \text{ are a characteristic velocity and a characteristic length, respectively. This represents 4 equations and 4 unknowns, nonlinear p.d.e.s.}
\]

- **Step 1.** Start with the basic state (\( Q \)) : \( U = (U(y), 0, 0) \). Continuity yields \( \frac{\partial}{\partial x} = 0 \) (1b). The \( y \) and \( z \) momentum equations show that \( \frac{\partial p}{\partial x} + \frac{1}{Re} \nabla^2 U = 0 \) (2b).

- **Step 2.** Add disturbances (\( \tilde{u} = U + u, \tilde{v} = 0 + v, \tilde{w} = 0 + w, \tilde{p} = P + p \)), and plug them into (1) & (2): This generates the total equations (1t) and (2t).

- **Step 3.** Subtract the basic state equations from the total equations: This generates the disturbance equations (1d) and (2d).

- **Step 4.** Linearize the disturbance equations to generate the linearized disturbance equations \( \frac{\partial \tilde{u}_i}{\partial x_i} = 0 \) (1l) and \( \frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} \quad (2l). \)

This still represents 4 equations and 4 unknowns, but the equations are now linear. (Note: the disturbance variables are now the unknowns since the basic state is known.) These are still p.d.e.s since \( u, v, w \), and \( p \) are functions of \( x, y, z, t \).

- **Step 5.** Solve the linearized disturbance equations (1l) and (2l): We use the method of normal modes.

**Method of Normal Modes:** Assume disturbances that are periodic in \( x \) and \( z \), but not growing or decaying in \( x \) or \( z \), and may be periodic and may be growing or decaying in \( t \) (temporal instability.) Specifically, let the disturbances be of the form

\[
u(x,y,z,t) = \tilde{u}(y)e^{i(kx+\zeta z-\omega t)} , \quad v(x,y,z,t) = \tilde{v}(y)e^{i(kx+\zeta z-\omega t)} , \quad w(x,y,z,t) = \tilde{w}(y)e^{i(kx+\zeta z-\omega t)}, \quad p(x,y,z,t) = \tilde{p}(y)e^{i(kx+\zeta z-\omega t)}, \quad \text{and} \quad c = \sqrt{\sigma^2 - \zeta k c}
\]

where variables with hats are complex amplitudes. \( k \) and \( \zeta \) are the \( x \) and \( z \) components, respectively, of wavenumber vector \( \vec{K} \). For temporal stability analysis, both \( k \) and \( \zeta \) must be real, while complex wave speed \( c \) can be complex. (Otherwise spatial instability would also be possible.) Plugging these disturbances into Eqs. (1l) and (2l) to get the normal mode equations:

\[
(ik-c)\tilde{v}_y = -i\tilde{p}_y + \frac{1}{Re} \left[ \tilde{\nu}_y - (k^2 + m^2) \tilde{u}\right], \quad (1n)
\]

\[
(ik-c)\tilde{u}_y = i\tilde{\nu}_y - \frac{1}{Re} \left[ \tilde{u}_y - (k^2 + m^2) \tilde{v}\right], \quad (2n)
\]

**Note:** For convenience in Eqs. (1n) and (2n), subscript \( y \) denotes differentiation with respect to \( y \). We are now down to 4 o.d.e.s and 4 unknowns since \( U(y) \) is known, along with its derivatives.

**Squire’s Theorem:** In 2-D parallel flow, for each unstable 3-D disturbance, there corresponds a more unstable 2-D disturbance. In other words, the most unstable case is the 2-D one: \( m = 0 \) \& \( \tilde{w} = 0 \). The normal mode equations simplify:

\[
(ik-c)\tilde{u}_y = -i\tilde{\nu}_y + \frac{1}{Re} \left[ \tilde{u}_y - k^2 \tilde{u}\right], \quad (5)
\]

\[
(ik-c)\tilde{v}_y = -i\tilde{p}_y + \frac{1}{Re} \left[ \tilde{v}_y - k^2 \tilde{v}\right], \quad (6)
\]

We are now down to 3 o.d.e.s and 3 unknowns, \( \tilde{u}(y), \tilde{v}(y), \) and \( \tilde{p}(y) \).

**Orr-Sommerfeld Equation:** Define a disturbance stream function, \( \psi(x,y,t) = \phi(y)e^{i(kx-\omega t)} \) Note: \( \phi(y) \) is not a velocity potential function, but simply the magnitude of the disturbance stream function. Plugging this into Eqs. (4) to (6) yields one o.d.e. and one unknown:

\[
(U-c)(\phi_{yy} - k^2 \phi) - U_{yy} \phi = \frac{1}{ik Re} \left[ \phi_{yyyy} - 2k^2 \phi_{yy} + k^4 \phi \right], \quad (7)
\]

**Note:** The Orr-Sommerfeld equation (to be done in class).
C = complex wave speed \[ C = C_r + iC_i \]

For temporal instability analysis, \( k \) and \( m \) are real, but \( C \) may be complex.

Examine \( e^{-ikct} \rightarrow e^{-ikct} \).

\[ e = e^{-ikct} e^{kct} \]

\( Cr \) represents the wave propagation speed.

\( Cr \) represents the wave propagation speed (wave is propagating in x-direction).

\( C_r \) determines the wave speed.

\( C_i \) determines the stability.

If \( C_i < 0 \) stable.

If \( C_i > 0 \) unstable.

Oscillating in time (periodic in time).

Represents the growth or decay - the stability of the system.