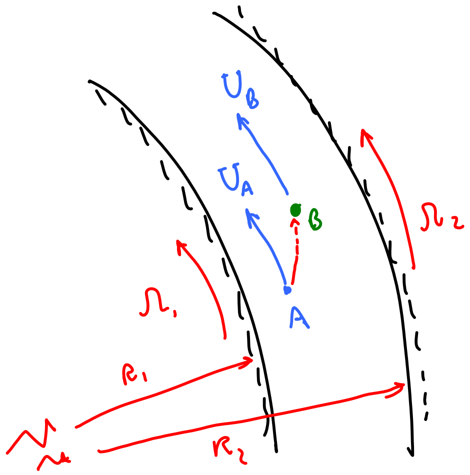
Today, we will:

- Continue to discuss the centrifugal instability (Taylor instability)
- Begin a discussion about stability of locally parallel flows – the Orr-Sommerfeld Eq.



Basic state $U_\theta(r)$ for a given Ω_1 & Ω_2

Angular momentum: $r\tilde{u}_\theta$ = constant

$r_A U_{\theta_A} = r_B \tilde{u}_{\theta_B}$ (ignoring viscous effects)

$\therefore \boxed{\tilde{u}_{\theta_B} = \frac{r_A U_{\theta_A}}{r_B}}$

But, at radius B, $U_\theta = U_{\theta_B}$

Stability → if the actual $\boxed{\tilde{u}_{\theta_B} > U_{\theta_B}}$ then the flow is potentially unstable

Stability criterion is $\boxed{\text{if } r_A U_{\theta_A} > r_B U_{\theta_B}, \text{ it is potentially unstable}}$ ☆

Viscous effects will try to keep it stable

Cases:

1) Solid Body rotation $U_\theta = r\Omega$ $(\Omega_1 = \Omega_2 = \Omega)$

$\boxed{r_A^2 \Omega < r_B^2 \Omega}$ since $r_B > r_A$

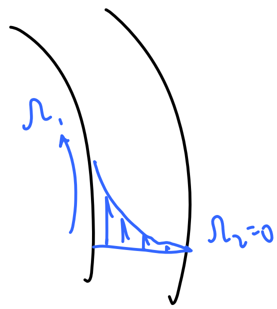
This basic state is stable

2) Outer cyl rotating, inner cyl. stationary

$U_{\theta_B} > U_{\theta_A}$ & $r_B > r_A$ → $r_A U_{\theta_A} < r_B U_{\theta_B}$

Stable

3) Inner cyl. rotating, outer cyl. stationary



$r_A U_{\theta_A} \ ? \ r_B U_{\theta_B}$

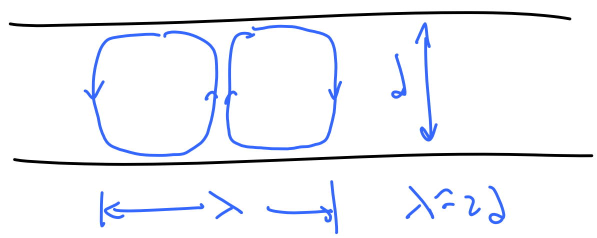
$r_A < r_B$

But $U_{\theta_A} > U_{\theta_B}$ $\Big\}$ Potentially unstable

Define a Taylor #
$$\boxed{Ta = 2\left(\frac{V_1 d}{\nu}\right)^2 \frac{d}{R_1}}$$

$V_1 = \Omega_1 R_1 \qquad d = R_2 - R_1$ = gap thickness

There is a critical Ta for which flow is unstable



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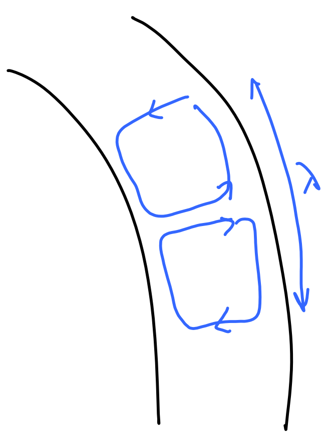


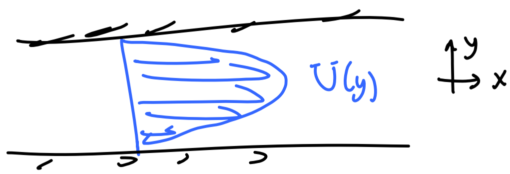
$\lambda \simeq 2d$

Taylor

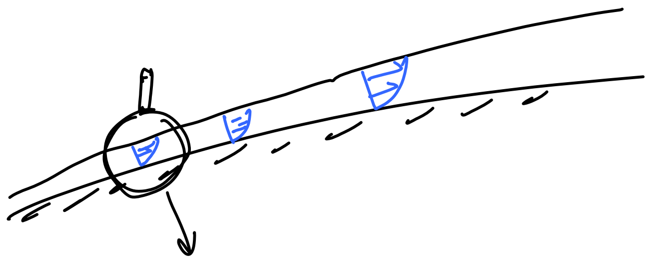
See Kundu for details

D. Stability of locally parallel flows (Kundu - Sec. 12.8)

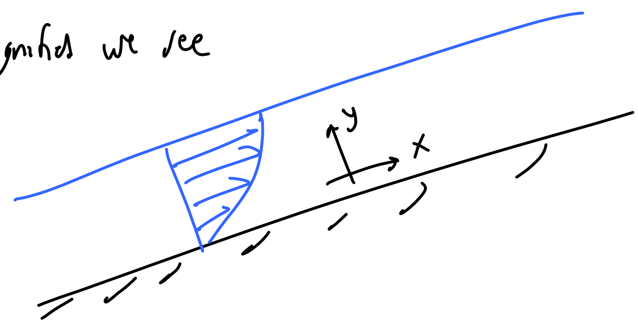
1. Intro: Some flows are parallel, e.g. fully developed channel flow, Couette flow, etc.



Other flows are nearly parallel, e.g. thin BLs, thin wakes, jets, etc.



Magnified we see

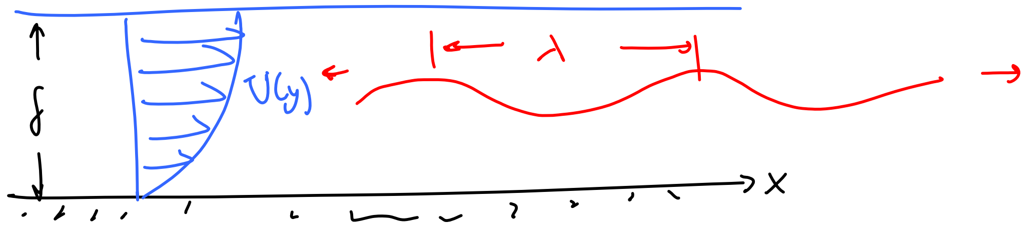


Approx. the basic state as $\vec{U} = (U(y), 0, 0)$ — V & W are very small compared to U

Locally Parallel Flow Approximation $\rightarrow$ $U \neq$ fnc. of x

(local analysis only)

Advantage
- Simpler math
- We can examine temporal stability with the method of normal modes



2. Linear Stability Analysis [temporal analysis]

- Step 0 → set of eq's → Use N-S & ignore gravity

notation → $\tilde{u}_{d_i}$ = • total velocity (tilde)
• dimensional (subscript d)
• i component (tensor notation)

Cont $$\boxed{\frac{\partial \tilde{u}_{d_i}}{\partial x_{d_i}} = 0}$$ (1 d)

4 eqs, 4 unknowns

Mom $$\boxed{\frac{\partial \tilde{u}_{d_i}}{\partial t_d} + \tilde{u}_{d_j}\frac{\partial \tilde{u}_{d_i}}{\partial x_{d_j}} = -\frac{1}{\rho}\frac{\partial \tilde{p}_d}{\partial x_{d_i}} + \nu \frac{\partial^2 \tilde{u}_{d_i}}{\partial x_{d_j}\partial x_{d_j}}}$$ (2 d)

Nondimensionalize everything

Let L = characteristic length scale → $\boxed{x_i = \frac{x_{d_i}}{L}}$

U_0 = " velocity " $\boxed{\tilde{u}_i = \frac{\tilde{u}_{d_i}}{U_0}}$

$\frac{L}{U_0}$ = " time " $\boxed{t = t_d \frac{U_0}{L}}$

ρU_0^2 = pressure $\boxed{\tilde{p} = \frac{\tilde{p}_d}{\rho U_0^2}}$

Plug these into (1d) & (2d) $\Big\}\downarrow$ See handout, get Eqs (1) & (2)

Step 1 Basic State $\rightarrow \vec{U} = (U(y), 0, 0)$

pressure = P

Plug into eqs (1) & (2) $\rightarrow$ get (1b) & (2b) on handout

Cont $\rightarrow \dfrac{\partial U_i}{\partial x_i} = 0 \rightarrow \cancel{\dfrac{\partial U}{\partial x}} + \cancel{\dfrac{\partial V}{\partial y}} + \cancel{\dfrac{\partial W}{\partial z}} = 0 \rightarrow \underline{0=0}$ (1b)

$U = U(y)$

x-mom $\rightarrow \cancel{\dfrac{\partial U}{\partial t}} + \cancel{U\dfrac{\partial U}{\partial x}} + \cancel{V\dfrac{\partial U}{\partial y}} + \cancel{W\dfrac{\partial U}{\partial z}} = \boxed{-\dfrac{\partial P}{\partial x} + \dfrac{1}{Re}\nabla^2 U = 0}$ (2b)

y mom $\rightarrow 0 = -\dfrac{\partial P}{\partial y} \rightarrow P \neq$ fnc of y

z-mom $\rightarrow P \neq$ fnc of z

$\Big\}$ $P = P(x)$ only

So $\boxed{-\dfrac{dP}{dx} + \dfrac{1}{Re}\nabla^2 U = 0}$ (2b)

Step 2 Add disturbance

$$\boxed{\begin{aligned}\tilde{u} &= U + u\\ \tilde{v} &= 0 + v\\ \tilde{w} &= 0 + w\\ \tilde{p} &= P + p\end{aligned}}$$

lower case without tilde = disturbance

in general fncs of (x, y, z, t)

Plug these into (1) & (2) to get (1t) & (2t)

Step 3 $\rightarrow$ subtract (1t) − (1b), (2t) − (2b) $\rightarrow$ **Step 4** Linearize $\rightarrow$ get (1l) & (2l)

Temporal Stability of Locally Parallel Flow – The Orr-Sommerfeld Equation

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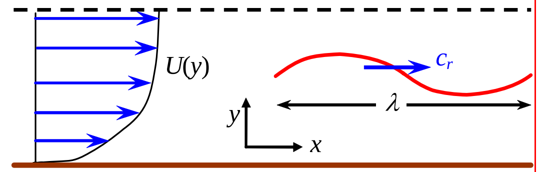
1. Problem Setup:



Consider an incompressible, locally parallel flow with some known steady basic state $U(y)$. We examine this problem using linear stability analysis. *Note*: A boundary layer type of flow is sketched, but the procedure applies to *any* kind of parallel or nearly parallelflow.

2. Summary of Linear Stability Analysis:

The in-class analysis follows Kundu, Section 12.8 closely, filling in some of the details. Start with the normalized incompressible Navier-Stokes equations for *total* flow variables ($\tilde{q}$):

$\boxed{\frac{\partial \tilde{u}_i}{\partial x_i} = 0}$ (1) and $\boxed{\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{\mathrm{Re}} \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}}$ (2) where $\boxed{\mathrm{Re} \equiv \frac{U_0 L}{\nu}}$ and U_0 and L are a characteristic velocity and a characteristic length, respectively. This represents 4 equations and 4 unknowns, nonlinear p.d.e.s.

- **Step 1.** Start with the basic state (Q): $U_i = (U(y), 0, 0)$. Continuity yields $\boxed{0 = 0}$ (1b). The y and z momentum equations show that $P = P(x)$ only, and the x-momentum reduces to $\boxed{0 = -\frac{dP}{dx} + \frac{1}{\mathrm{Re}} \nabla^2 U}$ (2b).
- **Step 2.** Add disturbances ($\tilde{u} = U + u$, $\tilde{v} = 0 + v$, $\tilde{w} = 0 + w$, $\tilde{p} = P + p$), and plug them into (1) & (2): This generates the *total equations* (1t) and (2t).
- **Step 3.** Subtract the basic state equations from the total equations: This generates the *disturbance equations* (1d) and (2d).
- **Step 4.** *Linearize* the disturbance equations to generate the *linearized disturbance equations* $\boxed{\frac{\partial u_i}{\partial x_i} = 0}$ (1l) and

  $\boxed{\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + v \frac{dU}{dy} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}} \nabla^2 u}$, $\boxed{\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}} \nabla^2 v}$, $\boxed{\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}} \nabla^2 w}$ (2l).

  This still represents 4 equations and 4 unknowns, but the equations are now *linear*. (*Note*: the *disturbance* variables are now the unknowns since the basic state is known.) These are still p.d.e.s since u, v, w, and p are functions of (x, y, z, t).
- **Step 5.** Solve the linearized disturbance equations (1l) and (2l): We use the **method of normal modes**.

  Method of Normal Modes: Assume disturbances that are *periodic* in x and z, but *not growing or decaying* in x or z, and *may be periodic* and *may be growing or decaying* in t. (*temporal* instability.) Specifically, let the disturbances be of the form $\boxed{u(x,y,z,t) = \hat{u}(y) e^{i(kx+mz-kct)}}$, $\boxed{v(x,y,z,t) = \hat{v}(y) e^{i(kx+mz-kct)}}$, $\boxed{w(x,y,z,t) = \hat{w}(y) e^{i(kx+mz-kct)}}$, and here $\boxed{\sigma = -ikc}$ $\boxed{p(x,y,z,t) = \hat{p}(y) e^{i(kx+mz-kct)}}$, where variables with hats are **complex amplitudes**. k and m are the x and z components, respectively, of **wavenumber vector** $\vec{K}$. For temporal stability analysis, both k and m must be *real*, while **complex wave speed** c can be *complex*. (Otherwise spatial instability would also be possible.) Plug these disturbances into Eqs. (1l) and (2l) to get the *normal mode equations*: $\boxed{ik\hat{u} + \hat{v}_y + im\hat{w} = 0}$ (1n), $\boxed{ik(U-c)\hat{u} + \hat{v}U_y = -ik\hat{p} + \frac{1}{\mathrm{Re}}\left[\hat{u}_{yy} - \left(k^2 + m^2\right)\hat{u}\right]}$, $\boxed{ik(U-c)\hat{v} = -\hat{p}_y + \frac{1}{\mathrm{Re}}\left[\hat{v}_{yy} - \left(k^2 + m^2\right)\hat{v}\right]}$, and $\boxed{ik(U-c)\hat{w} = -im\hat{p} + \frac{1}{\mathrm{Re}}\left[\hat{w}_{yy} - \left(k^2 + m^2\right)\hat{w}\right]}$ (2n).

  Note: For convenience in Eqs. (1n) and (2n), subscript y denotes differentiation with respect to y. We are now down to 4 o.d.e.s and 4 unknowns since $U(y)$ is known, along with its derivatives.

  Squire's Theorem: In 2-D parallel flow, for each unstable 3-D disturbance, there corresponds a *more* unstable 2-D disturbance. In other words, the *most unstable* case is the 2-D one: $\boxed{m = 0}$ & $\boxed{\hat{w} = 0}$. The normal mode equations simplify: $\boxed{ik\hat{u} + \hat{v}_y = 0}$ (4), $\boxed{ik(U-c)\hat{u} + \hat{v}U_y = -ik\hat{p} + \frac{1}{\mathrm{Re}}\left[\hat{u}_{yy} - k^2\hat{u}\right]}$ (5), and $\boxed{ik(U-c)\hat{v} = -\hat{p}_y + \frac{1}{\mathrm{Re}}\left[\hat{v}_{yy} - k^2\hat{v}\right]}$ (6).

  We are now down to 3 o.d.e.s and 3 unknowns, $\hat{u}(y)$, $\hat{v}(y)$, and $\hat{p}(y)$.

  Orr-Sommerfeld Equation: Define a disturbance stream function, $\boxed{\psi(x,y,t) = \phi(y) e^{ik(x-ct)}}$. *Note*: $\phi(y)$ is *not* a velocity potential function, but simply the magnitude of the disturbance stream function. Plugging this into Eqs. (4) to (6) yields *one* o.d.e. and *one* unknown: $\boxed{(U-c)\left(\phi_{yy} - k^2\phi\right) - U_{yy}\phi = \frac{1}{ik\,\mathrm{Re}}\left[\phi_{yyyy} - 2k^2\phi_{yy} + k^4\phi\right]}$ (7), the **Orr-Sommerfeld equation**.
- **Step 6.** Examine stability: Finally, we examine solutions of the Orr-Sommerfeld equation (to be done in class).

c = complex wave speed $\quad \boxed{c = c_r + i c_i}$

For temporal instability analysis, k & m are <u>real</u>, but c may be complex

examine $e^{-ikct} \rightarrow$ $e^{-ikct} = e^{-ikc_r t}\, e^{kc_i t}$

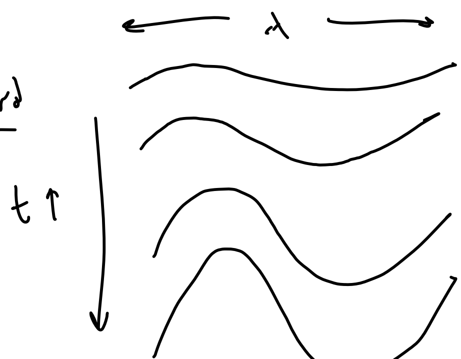
$e^{-ikc_r t}$: oscillating in time (periodic in time)

$e^{kc_i t}$: represents the growth or decay — the <u>stability</u>

$$\boxed{\begin{array}{l} \text{if } c_i < 0 \text{ stable} \\ \text{if } c_i > 0 \text{ unstable} \end{array}}$$

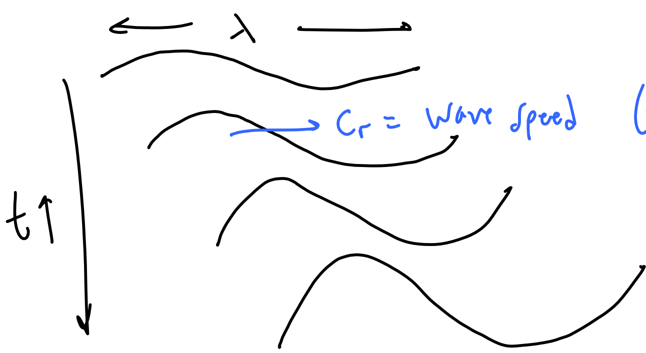
c_r represents the <u>wave propagation speed</u>.

<u>Benard</u>



→ x

<u>Here</u>



→ c_r = wave speed (wave is propagating in x-direction)

$$\boxed{\begin{array}{l} c_r \text{ determines the wave speed} \\ c_i \text{ determines the stability} \end{array}}$$