Today, we will:

- Continue to discuss the stability of nearly parallel flows – the Orr-Sommerfeld Eq.
- Look at some qualitative example problems – Solutions of the Orr-Sommerfeld Eq.

We left off here last time …

- **Step 4.** *Linearize* the disturbance equations to generate the *linearized disturbance equations* $\boxed{\frac{\partial u_i}{\partial x_i} = 0}$ (1l) and

$$\boxed{\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{dU}{dy} = -\frac{\partial p}{\partial x} + \frac{1}{\text{Re}}\nabla^2 u}, \boxed{\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{\text{Re}}\nabla^2 v}, \boxed{\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \frac{1}{\text{Re}}\nabla^2 w} \text{ (2l).}$$

This still represents 4 equations and 4 unknowns, but the equations are now *linear*. (*Note*: the *disturbance* variables are now the unknowns since the basic state is known.) These are still p.d.e.s since u, v, w, and p are functions of (x, y, z, t).

- **Step 5.** Solve the linearized disturbance equations (1l) and (2l): We use the **method of normal modes**.

Method of Normal Modes: Assume disturbances that are *periodic* in x and z, but *not growing or decaying* in x or z, and *may be periodic* and *may be growing or decaying* in t. (*temporal* instability.) Specifically, let the disturbances be of the form $\boxed{u(x,y,z,t) = \hat{u}(y)e^{i(kx+mz-kct)}}$, $\boxed{v(x,y,z,t) = \hat{v}(y)e^{i(kx+mz-kct)}}$, $\boxed{w(x,y,z,t) = \hat{w}(y)e^{i(kx+mz-kct)}}$, and $\boxed{p(x,y,z,t) = \hat{p}(y)e^{i(kx+mz-kct)}}$, where variables with hats are **complex amplitudes**. k and m are the x and z components, respectively, of **wavenumber vector** $\vec{K}$. For temporal stability analysis, both k and m must be *real*, while **complex wave speed** c can be *complex*. (Otherwise spatial instability would also be possible.) Plug these disturbances into Eqs. (1l) and (2l) to get the *normal mode equations*: If $c_i < 0$ stable If $c_i > 0$ unstable – temporal growth

As previously, when taking derivatives, $e^{(\;)}$ in every term → so, it cancels out

$$\boxed{ik\hat{u} + \hat{v}_y + im\hat{w} = 0} \text{ (1n)}, \boxed{ik(U-c)\hat{u} + \hat{v}U_y = -ik\hat{p} + \frac{1}{\text{Re}}\left[\hat{u}_{yy} - \left(k^2+m^2\right)\hat{u}\right]},$$

(n = normal mode eq's)

$$\boxed{ik(U-c)\hat{v} = -\hat{p}_y + \frac{1}{\text{Re}}\left[\hat{v}_{yy} - \left(k^2+m^2\right)\hat{v}\right]}, \text{ and } \boxed{ik(U-c)\hat{w} = -im\hat{p} + \frac{1}{\text{Re}}\left[\hat{w}_{yy} - \left(k^2+m^2\right)\hat{w}\right]} \text{ (2n).}$$

$= -\frac{d\hat{p}}{dy}$

Note: For convenience in Eqs. (1n) and (2n), subscript y denotes differentiation with respect to y. We are now down to 4 o.d.e.s and 4 unknowns since $U(y)$ is known, along with its derivatives. [Now ODEs, not PDEs

Squire's Theorem: In 2-D parallel flow, for each unstable 3-D disturbance, there corresponds a *more* unstable 2-D disturbance. In other words, the *most unstable* case is the 2-D one: $\boxed{m=0}$ & $\boxed{\hat{w}=0}$. The normal mode equations simplify:

$$\boxed{ik\hat{u} + \hat{v}_y = 0} \text{ (4)}, \boxed{ik(U-c)\hat{u} + \hat{v}U_y = -ik\hat{p} + \frac{1}{\text{Re}}\left[\hat{u}_{yy} - k^2\hat{u}\right]} \text{ (5), and } \boxed{ik(U-c)\hat{v} = -\hat{p}_y + \frac{1}{\text{Re}}\left[\hat{v}_{yy} - k^2\hat{v}\right]} \text{ (6).}$$

We are now down to 3 o.d.e.s and 3 unknowns, $\hat{u}(y)$, $\hat{v}(y)$, and $\hat{p}(y)$.

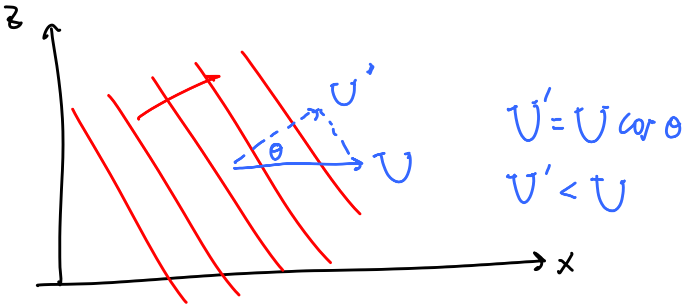
Orr-Sommerfeld Equation: Define a disturbance stream function, $\boxed{\psi(x,y,t) = \phi(y)e^{ik(x-ct)}}$. *Note*: $\phi(y)$ is *not* a velocity potential function, but simply the magnitude of the disturbance stream function. Plugging this into Eqs. (4) to (6) yields *one* o.d.e. and *one* unknown:

$$\boxed{(U-c)\left(\phi_{yy} - k^2\phi\right) - U_{yy}\phi = \frac{1}{ik\,\text{Re}}\left[\phi_{yyyy} - 2k^2\phi_{yy} + k^4\phi\right]} \text{ (7), the **Orr-Sommerfeld equation**.}$$

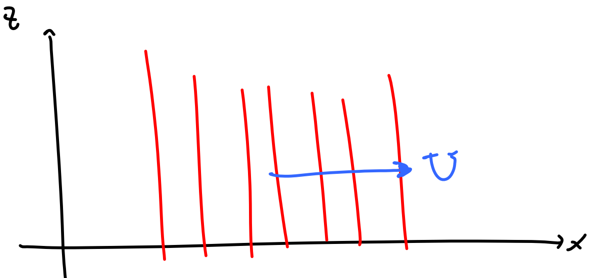
- **Step 6.** Examine stability: Finally, we examine solutions of the Orr-Sommerfeld equation (to be done in class).

3. SQUIRE'S THEOREM (1933)

For each unstable 3-D disturbance (oblique wave), there corresponds a more unstable 2-D disturbance oblique disturbances



2-D disturbances



The effective Reynolds # of the oblique case is <u>smaller</u> than that of the 2-D case

$\therefore$ We can let $m=0$ in Eqs. 1l & 2l
& let $\hat{w}=0$ for 2-D disturbances (waves)

MOST LIKELY TO BE OBSERVED

This is experimentally verified

[See pictures on website]

Plug in Squire's thm $\rightarrow$ set $\hat{w}=0$, $m=0$ into eqs (1n) & (2n)

& algebra

Get $\rightarrow$ Eqs. (4), (5), (6) on handout

Now we have

$$\boxed{\begin{aligned} u &= \hat{u}(y)\, e^{ik(x-ct)} \\ v &= \hat{v}(y)\, e^{ik(x-ct)} \\ p &= \hat{p}(y)\, e^{ik(x-ct)} \end{aligned}}$$

4. Orr-Sommerfeld Eq.

a. Derivation → combine the three eqs (4), (5), (6) by eliminating pressure

→ defined a disturbance stream function ψ

$$\boxed{u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}}$$

∴ Normal modes → let $\boxed{\psi = \hat{\psi}(y)\, e^{ik(x-ct)}}$

Kundu's notation → He lets $\boxed{\psi = \phi(y)\, e^{ik(x-ct)}}$

$\phi(y)$ is the complex amplitude of disturbance ψ

Differentiate to get $\hat{u}$ & $\hat{v}$:

$\boxed{\hat{u} = \phi_y}$ $\boxed{\hat{v} = -ik\phi}$ (the $e^{(\ )}$ cancels out)

After some algebra combine (4), (5), (6) into one eq.

$$\boxed{(U-c)(\phi_{yy} - k^2\phi) - U_{yy}\,\phi = \frac{1}{ikRe}\left[\phi_{yyyy} - 2k^2\phi_{yy} + k^4\phi\right]} \qquad (7)$$

THE ORR-SOMMERFELD EQ.

OR, in standard form:

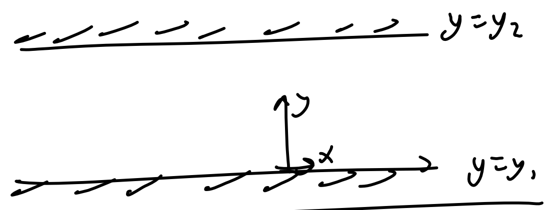
$$\boxed{\phi_{yyyy} - \left[ikRe(U-c) + 2k^2\right]\phi_{yy} + \left[ikRe\,U_{yy} + k^4 + ik^3 Re(U-c)\right]\phi = 0}$$

O-S eq. is linear, ode, 4th-order, homogeneous, with non-constant coefficients $[U = U(y)]$

★ 1 Eq & 1 Unknown! $\phi(y)$

b. BCs: Need 4

<u>e.g.</u> flow between // walls



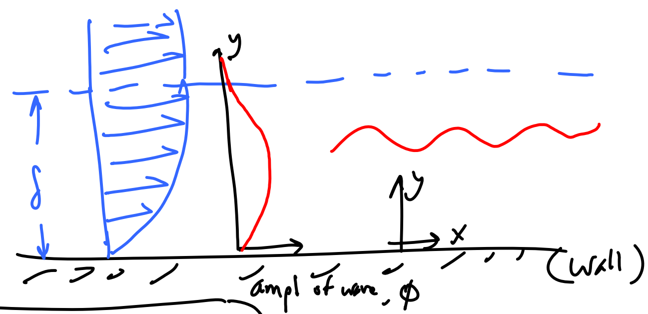
$u = v = 0$ @ $y = y_1$ & $y = y_2$

$u = 0$ when $\phi_y = 0$
$v = 0$ when $\phi = 0$

$\therefore$ $\boxed{\phi = \phi_y = 0 \text{ @ } y = y_1 \text{ \& } y = y_2}$ 4 BC

<u>E.g. BL flow</u>

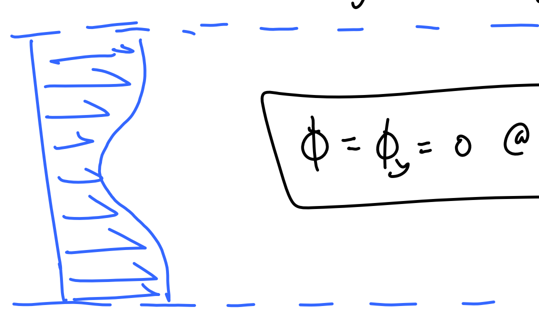
<u>nearly parallel flow</u>



BCs → $\phi = \phi_y = 0$ @ $y = 0$ (no slip)
& $\phi = \phi_y = 0$ @ $y \to \infty$ [disturbances must die off beyond the "edge" of the BL]

<u>E.g. free shear layers</u>

nearly parallel flow <u>locally</u>



$\boxed{\phi = \phi_y = 0 \text{ @ } y \to \pm\infty}$

This is an <u>eigenvalue problem</u>:

$U(y)$ is known (basic state)

Re = Reynolds # = $\dfrac{U_0 L}{\nu}$ = parameter in the problem

At some value of Re, there are only certain combinations of <u>wavenumber k</u> & <u>complex wave speed c</u> that satisfy the O-S eq. and its BCs & yield nontrivial solutions

$\therefore$ k & c are the eigenvalues, $\phi(y)$ is the associated eigenfunction ✱

C. Solutions & Examples

Typically - we need to use a "searching method" or a "shooting method" to obtain a solution

Aersp 514 → how to do this

Qualitative Results: Stability Curves

recall, $\psi = \phi e^{ik(x-ct)}$ = disturbance

For temporal mode → k = real = wavenumber
(oscillatory in x, but no growth in x)

c = complex wave speed

$$c = c_r + i c_i$$

c_r: wave speed

c_i: growth rate

if $c_i < 0$ stable
if $c_i = 0$ marginally stable (neutrally stable)
if $c_i > 0$ unstable

To solve:

- Pick a Re
- Pick a k

} solve for c that gives a non-trivial sol'n.

- march to a new k & repeat