Today, we will:

- Discuss more examples of solutions to the O-S equation
- Discuss the Rayleigh eq. (inviscid O-S equation) in more detail, and look at some solutions

Last time: Examples of "thumb curves" (O-S solutions) (TEMPORAL ANALYSIS)

a) Smooth mixing layer

b) BL with an adverse pressure gradient (or similar behavior for wake flow)

Rayleigh eq. solution (inviscid)
c) BL with favorable pressure gradient (e.g. front part of a body)

At \( Re \to \infty \) we prove that the basic state is stable for any disturbance.

Why does 0-5 solution predict that higher velocity fluid (lower \( Re \) case) is more unstable? [See mathematical argument in Kandus book]

- We approximate flow as parallel - it is not in reality
- Temporal instability may - there may be spatial instability
- Linear stability - nonlinear effects may be important
- 2-D - there may be 2-D effects
- To get to high \( Re \), typically must pass through lower \( Re \) (where it is unstable)
5. The Rayleigh Eq. (Inviscid form of OJ eq.)

a. Derivation

Consider 2-D parallel flow \( U = U(y) \)

\[ \text{O-J eq.} \quad (U-c)(\phi_{yy} - k^2 \phi) - U_{yy} \phi = \frac{1}{ik\text{Re}} \left[ \phi_{yyyy} - 2k^2 \phi_{yy} + k^4 \phi \right] \]

As \( \text{Re} \to \infty, \quad \text{RHS} \ll \text{LHS} \to \text{let's ignore the RHS} \)

rewrite as

\[ \phi_{yy} - \left( \frac{U_{yy}}{U-c} + k^2 \right) \phi = 0 \quad \text{(1)} \]

Rayleigh Eq.

Note: We are assuming that the disturbances behave inviscidly, but the basic state itself can be a viscous flow

\[ \text{e.g.} \quad \begin{array}{c}
\includegraphics[width=2cm]{example1.png} \\
\includegraphics[width=2cm]{example2.png}
\end{array} \]

Need 2 BCs since 2nd order

We can no longer enforce the no-slip condition on the disturbances

\[ \text{BCs} \quad \hat{V}(y_1) = \hat{V}(y_2) = 0 \quad \text{for flow between ducts} \]

\[ \phi(y_1) = \phi(y_2) = 0 \]

b. Conditions for Instability (of the Rayleigh Eq.)

(1) Rayleigh's inflection pt. criterion

"manipulate" on Eq. (1)

\[ C_i \int_{y_1}^{y_2} \frac{U_{yy} |\phi|^2}{|U-c|^2} \, dy = 0 \quad \text{(2)} \]
(2) can be satisfied two ways:

(a) if \( C_i = 0 \) \( \rightarrow \) neutrally stable, then instability is not possible

(b) if \( \int_0^\infty \frac{U_{yy} |\phi|^2}{|U-c|^2} \, dy = 0 \) \( \Leftrightarrow C_i = 0 \)

\# Necessary condition for instability

[The only way that \( C_i \) can be \( > 0 \) (unstable) is if this integral \( = 0 \)]

\( |\phi|^2 \) is positive definite (always positive)

\( |U-c|^2 \)

\( U_{yy} \) can be \( \Theta \) or \( \Theta \)

\( \int \) can be \( = 0 \) only if:

- \( U_{yy} = 0 \) everywhere (trivial case – shear layer with constant slope \( U_y = \text{const} \))

or

- \( U_{yy} \) must change sign somewhere within \( y_1 \leq y \leq y_2 \)

\( U_{yy} = \frac{d^2U}{dy^2} \) \( \rightarrow \) when \( \frac{d^2U}{dy^2} < 0 \), this is an inflection \( \uparrow^i \)

in the basic state velocity profile.

\[
\frac{U_{yy} |\phi|^2}{|U-c|^2}
\]

There must be at least one inflection point in the basic state velocity profile for the flow to be unstable.
e.g. Stable Basic States (no instability pt.)

- 2-D channel flow
- BL with favorable p.g.

Inviscid stability analysis (Rayleigh Eq.) predicts that these basic states are stable for all wavenumber k.

Potentially Unstable Basic States

- July
- Sharlaye
- West
- BL with adverse p.g.

Note: Rayleigh inflection pt. criterion is necessary but not sufficient for instability.