Today, we will:

- Discuss Fjortoft's theorem – a further necessary condition for inviscid instability
- Discuss an example problem – instability in a 2-D wake (JMC PhD thesis)

Last time → must be an inflection pt. to have the potential for instability.

This is a necessary but not sufficient condition for instability

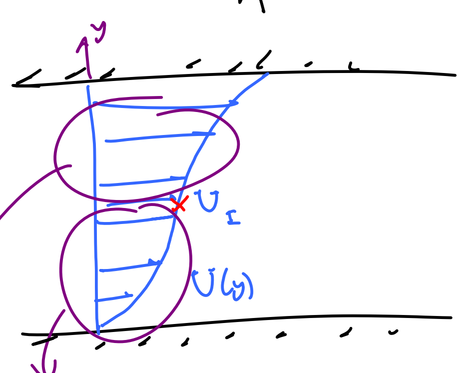
$\boxed{\text{Fjortoft's theorem} \rightarrow \text{A necessary condition for instability is that } U_{yy}(U-U_I) < 0 \text{ somewhere in the basic flow}}$

(U @ inflection pt.)

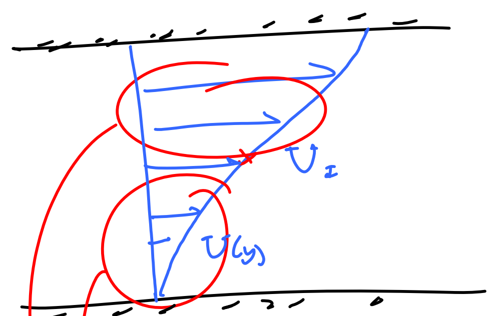
$\boxed{\text{Necessary but not sufficient}}$

e.g. Couette flow profiles – both have inflection pts

A



B



lower half, $U_{yy} < 0$ (slope decreasing with y)

$\therefore\ U-U_I < 0$

$U_{yy}(U-U_I) > 0$

upper half, $U_{yy} > 0$ (slope increasing with y)

$\therefore\ U-U_I > 0$ $\quad U_{yy}(U-U_I) > 0$

$U_{yy}(U-U_I) > 0$ everywhere

$\therefore$ $\boxed{\text{STABLE}}$

lower half $\left.\begin{matrix} U_{yy} > 0 \\ U-U_I < 0 \end{matrix}\right\} U_{yy}(U-U_I) < 0$

upper half $\left.\begin{matrix} U_{yy} < 0 \\ U-U_I > 0 \end{matrix}\right\} U_{yy}(U-U_I) < 0$

$U_{yy}(U-U_I) < 0$ everywhere

$\boxed{\text{Potentially unstable}}$

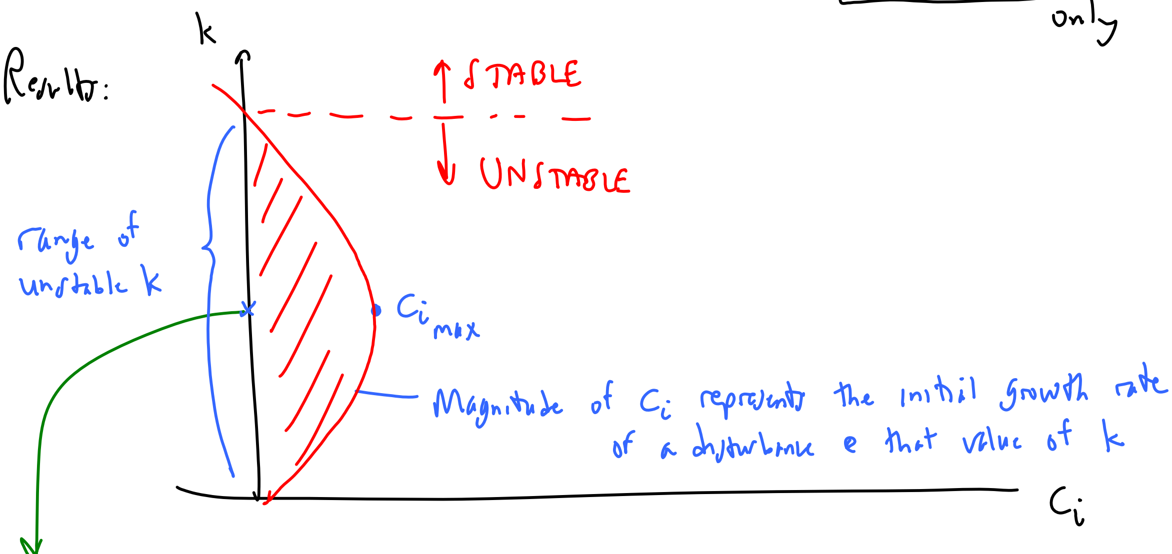
C. Typical Solutions of Rayleigh's eq.

- Consider a basic state that satisfies both Rayleigh & Fjortoft's criteria.

- Solve Rayleigh eq. & find eigenvalues

We are considering the $\boxed{\text{TEMPORAL MODE}}$ only

Results:



This k corresponding to $c_{i_{max}}$ is the one we would expect to see (most likely to be observed)

Typically, very high wavenumbers are stable

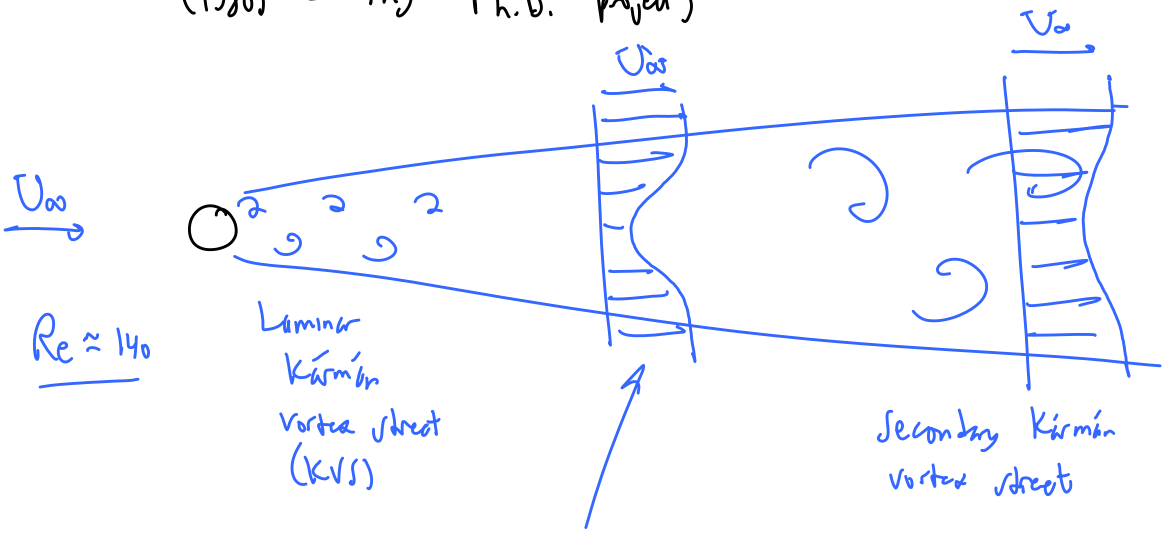
recall $k = \frac{2\pi}{\lambda}$ $\Rightarrow$ high k = wavenumber means short wavelength (or high frequency)

Typically, very short wavenumbers are unstable, but as $k \to 0$, it becomes stable again.

[long wavelength disturbances (low freq. disturbances) are unstable]
very short " " (high " ") are stable ☆

d. Example – far wake of a circular cylinder

(1980's – my Ph.D. project)

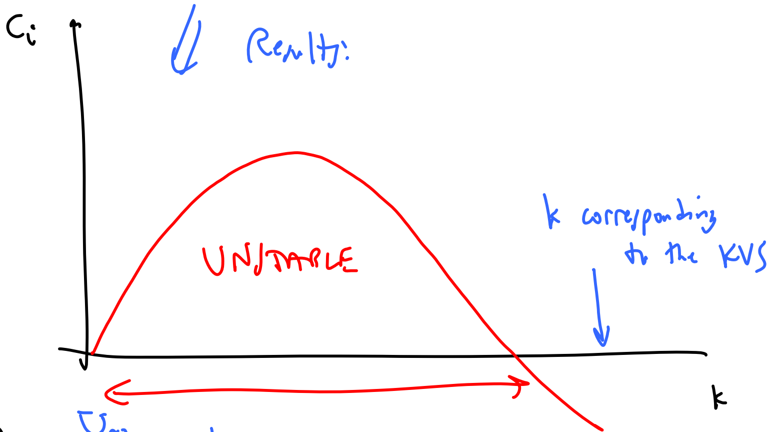


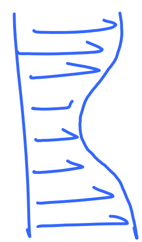
by ≈ 100 diameters downstream, the original KVS disappeared!

Rayleigh Eq → Inviscid stability analysis

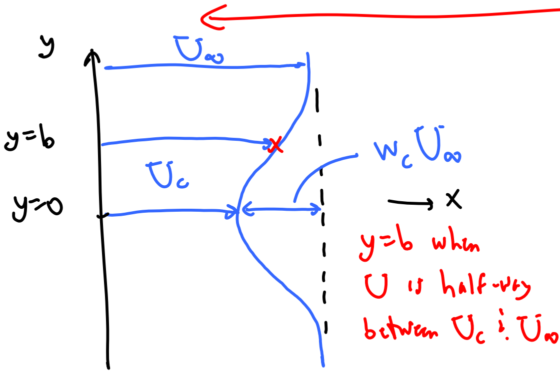
I assumes 2-D, //, inviscid disturbances, linear, temporal mode

⇓ Results:





Basic state is a wake profile



$W_c \equiv$ nondimensional velocity defect @ centerline

U_c = centerline velocity

$$\boxed{W_c = 1 - \frac{U_c}{U_\infty}}$$

Define W = normalized wake defect

$$\boxed{W\left(\frac{y}{b}\right) = \frac{U - U_\infty}{U_c - U_\infty} = e^{-a\left(\frac{y}{b}\right)^2}}$$

where $a = \ln 2 \rightarrow$ so that $W(y/b = 1) = 0.5$

$W(0) = 1$

Gaussian shape is a good curve fit for the wake profile @ any value of x (downstream)

$$U = U_\infty + W(U_c - U_\infty)$$

$$= U_\infty\left[1 + W\left(\frac{U_c}{U_\infty} - 1\right)\right] = U_\infty(1 - w_c W) \rightarrow \boxed{\frac{U}{U_\infty} = 1 - w_c W}$$

Also, like Kundu, let $y = \frac{y}{b}$, $U = \frac{U}{U_\infty}$

$$\therefore \boxed{\begin{aligned} U &= 1 - w_c W \\ U_y &= -w_c W_y \\ U_{yy} &= -w_c W_{yy} \end{aligned}}$$

[y subscript means $\frac{d}{dy}$]

Plug into Rayleigh eq.

Rayleigh eq. $\boxed{(U - c)(\phi_{yy} - k^2\phi) - U_{yy}\phi = 0}$ (1)

$$\left[\underbrace{\frac{1}{w_c}(1 - c)}_{} - W\right](\phi_{yy} - k^2\phi) + W_{yy}\phi = 0$$

let $\boxed{c_u = \frac{1}{w_c}(1 - c)}$ = universal wake wave speed

Universal wake Rayleigh eq. for temporal stability.

$$\star \quad \boxed{(W - c_u)(\phi_{yy} - k^2\phi) - W_{yy}\phi = 0}$$