M E 522 Spring 2008 Professor John M. Cimbala Lecture 26 03/21/2008

Today, we will:

- Discuss Reynolds decomposition and the appearance of the Reynolds stress
- Discuss the Reynolds stress tensor
- If time, begin to discuss the turbulence closure problem
- Do Candy Questions for Candy Friday

Example: Given: Measurements in a wind tunnel

$\tilde{u}$ & $\tilde{v}$ measured vs time

we calculate $U = 5.530\ \mathrm{m/s}$

$V = 3.470\ \mathrm{m/s}$

$\tilde{u}_{rms} = 5.5470\ \mathrm{m/s}$

$\tilde{v}_{rms} = 3.3413\ \mathrm{m/s}$

$\overline{\tilde{u}\tilde{v}} = 19.163\ \mathrm{m^2/s^2}$

To do: Calc. $\overline{uv}$

$$\overline{\tilde{u}\tilde{v}} = UV + \overline{uv}$$

$$\overline{uv} = \overline{\tilde{u}\tilde{v}} - UV = \boxed{-0.0261\ \mathrm{m^2/s^2}}$$

notice it is negative

(we will discuss why later)

2. Reynolds Decomposition – See handout (also on next page →)

Reynolds Decomposition

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1. Introduction

- Start with the Boussinesq equations (Navier-Stokes equations for buoyant flows) for *total* flow variables ($\tilde{q}$):

$$\boxed{\frac{\partial \tilde{u}_i}{\partial x_i} = 0}\ (1),\quad \boxed{\left(\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}\right) = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x_i} - g\delta_{i3}\left[1-\alpha\left(\tilde{T}-T_0\right)\right] + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}}\ (2), \text{ and }\ \boxed{\frac{\partial \tilde{T}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{T}}{\partial x_j} = \kappa \frac{\partial^2 \tilde{T}}{\partial x_j \partial x_j}}\ (3).$$

- This represents 5 equations and 5 unknowns ($\tilde{u}_i$, $\tilde{p}$, $\tilde{T}$), functions of (x_i, t). Note that these equations are *exact* (within the limits of the Boussinesq approximation of nearly incompressible flow, of course).

2. Procedure

- **Step 1.** Substitute Reynolds decomposition $\boxed{\tilde{q} = Q + q}$ into (1), (2), and (3): This generates ***instantaneous equations***, (1i), (2i), and (3i).
- **Step 2.** Take the ensemble average of the instantaneous equations: This generates the ***Reynolds averaged equations***, or the ***equations for the mean flow***, (1m), (2m), and (3m).

Summary of rules for ensemble averaging (considering two generic flow variables, $\tilde{q} = Q + q$ and $\tilde{p} = P + p$):

$\overline{\tilde{q}} = Q$	Ensemble average of a total flow quantity equals the mean flow quantity
$\overline{C_1\tilde{q}} = C_1\overline{\tilde{q}} = C_1 Q$	A constant (C_1) is not affected by an ensemble average
$\overline{q} = 0$	Ensemble average of a fluctuating quantity is identically zero (by definition)
$\overline{Q} = Q$	Ensemble average of a mean quantity does not change that mean quantity
$\overline{\tilde{q}P} = QP$	Ensemble average of a total flow quantity times a mean quantity is the product of the two mean quantities (P is already ensemble averaged, so it doesn't change further)
$\overline{C_1 q} = C_1\overline{q} = C_1 \cdot 0 = 0$	Ensemble average of a constant (C_1) times a fluctuating quantity is identically zero
$\overline{qP} = \overline{q}P = 0 \cdot P = 0$	Ensemble average of a fluctuating quantity times a mean quantity is identically zero
$\overline{\tilde{q}\tilde{p}} = QP + \overline{qp}$	Ensemble average of the product of two total flow quantities yields two terms
$\overline{\tilde{q} + \tilde{p}} = \overline{\tilde{q}} + \overline{\tilde{p}} = Q + P$	The order (sequence) of addition or ensemble average does not matter
$\overline{\frac{\partial \tilde{q}}{\partial x_i}} = \frac{\partial \overline{\tilde{q}}}{\partial x_i} = \frac{\partial Q}{\partial x_i}$	The order (sequence) of spatial derivation or ensemble average does not matter. *Note*: Same thing holds with *time* derivatives.
$\overline{\int \tilde{q}dx} = \int \overline{\tilde{q}}dx$	The order (sequence) of spatial integration or ensemble average does not matter. *Note*: Same thing holds with *time* integration.

- **Step 3.** Subtract the mean flow equations from the instantaneous equations to generate the ***equations for the turbulent fluctuations***, (1f), (2f), and (3f).

Example

Consider the continuity equation, Equation (1). Let's work through the three steps above:

- **Step 1.** $\frac{\partial\left(U_i + u_i\right)}{\partial x_i} = \frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0$, or $\boxed{\frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0}$ (1i)

  Equation (1i) is the ***instantaneous continuity equation***.

- **Step 2.** $\overline{\frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}} = \overline{\frac{\partial U_i}{\partial x_i}} + \overline{\frac{\partial u_i}{\partial x_i}} = \frac{\partial U_i}{\partial x_i} + \cancel{\frac{\partial \overline{u_i}}{\partial x_i}} = \frac{\partial U_i}{\partial x_i} = 0$, or $\boxed{\frac{\partial U_i}{\partial x_i} = 0}$ (1m)

  Equation (1m) is the ***mean continuity equation***, or the ***Reynolds averaged continuity equation***.

- **Step 3.** Subtracting (1m) from (1i) yields $\boxed{\frac{\partial u_i}{\partial x_i} = 0}$ (1f)

  Equation (1f) is the ***continuity equation for the turbulent fluctuations***.

Note: All three equations above are still *exact*, since we haven't made any further simplifications, like linearization, etc. The same procedure must be done on the momentum and energy equations, (2) and (3) respectively.

[See HW # 8]

3. Comparison of Laminar & Turbulent Mean Flow Eqs.

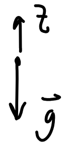
Compare laminar flow & turbulent flow with identical BCs

Let U_{ℓ_i}, P_ℓ, $\bar{T}_\ell$ = laminar flow solution (no fluctuations)

Let U_i, P, T = mean flow solution for the turbulent problem

Cont: Lam $\boxed{\frac{\partial U_{\ell_i}}{\partial x_i} = 0}$ Turb $\boxed{\frac{\partial U_i}{\partial x_i} = 0}$

Identical



Mom: Note mult. by ρ_0, $\mu = \nu \rho_0$

Laminar:

$$\boxed{\rho_0\left[\frac{\partial U_{\ell_i}}{\partial t} + U_{\ell_j}\frac{\partial U_{\ell_i}}{\partial x_j}\right] = -\frac{\partial P_\ell}{\partial x_i} - \rho_0 g\left[1-\alpha\left(\bar{T}_\ell - T_0\right)\right]\delta_{i3} + \frac{\partial}{\partial x_j}\left[\mu\frac{\partial U_{\ell_i}}{\partial x_j}\right]}$$

Turbulent (2m)

$$\boxed{\rho_0\left[\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j}\right] = -\frac{\partial P}{\partial x_i} - \rho_0 g\left[1-\alpha\left(\bar{T} - T_0\right)\right]\delta_{i3} + \frac{\partial}{\partial x_j}\left[\mu\frac{\partial U_i}{\partial x_j} - \rho_0\overline{u_i u_j}\right]}$$

Notice the "extra" term

Reynolds stress tensor $\equiv -\rho_0 \overline{u_i u_j}$ ☆

[We call it a "stress" & we put it on the RHS of the eq.
But – it actually comes from the advective terms on LHS]

☆ Turbulent flow is not merely laminar flow with added fluctuations

Mean flow eq for momentum contains this "extra" term

$$U_i \neq U_{e_i} \qquad P \neq P_e \quad \text{etc.}$$

☆ It is through the Reynolds stress tensor that turbulence affects the mean flow

Specifically, $-\rho_0 \overline{u_i u_j}$ exchanges momentum between the turbulence and the mean flow

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Acts like an additional "stress" that diffuses momentum

→ Viscosity is a microscopic stress (mixing at the molecular level)

Reynolds stress is a macroscopic stress (mixing due to turbulent eddies)

Energy: → Also an "extra" term → $\rho_0 c_p \overline{u_j T'}$

= turbulent heat flux vector

4. Some comments about the Reynolds Stress Tensor $\left[-\rho_0 \overline{u_i u_j}\right]$

a. It is a 9-component tensor

$$-\rho_0 \overline{u_i u_j} = -\rho_0 \begin{bmatrix} \overline{u^2} & \overline{uv} & \overline{uw} \\ \overline{vu} & \overline{v^2} & \overline{vw} \\ \overline{wu} & \overline{wv} & \overline{w^2} \end{bmatrix}$$

Diagonal components are normal stresses

off-diagonals are shear stresses

The trace = sum of diagonals = $-\rho_0 \overline{u_i u_i}$

recall $q^2 = k = \text{tke} = \frac{1}{2} \overline{u_i u_i}$

$$\boxed{\text{trace} = -2\rho_0 q^2}$$

b. It is symmetric $\overline{uv} = \overline{vu}$, etc.

↳ $\boxed{\text{6 independent components in the Reynolds stress}}$

c. Sign of the normal components

$\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ are always ≥ 0

The normal Reynolds stress components are always ≤ 0

$$\left(-\rho_0 \overline{u^2},\ -\rho_0 \overline{v^2},\ -\rho_0 \overline{w^2}\right)$$

d. Special case — isotropic turbulence (no preferred direction)

e.g. perfectly random turbulence → u & v are not correlated

$$\therefore\ \boxed{\overline{uv} = \overline{uw} = \overline{vw} = 0 \text{ for isotropic turbulence}}\ \star$$

Also, for isotropic turbulence, $\boxed{\overline{u^2} = \overline{v^2} = \overline{w^2}}$

We can write the Reynolds stress tensor as

$$-\rho_0 \overline{u_i u_j} = \begin{bmatrix} \boxed{-\frac{2}{3}\rho_0 k} & 0 & 0 \\ 0 & \boxed{-\frac{2}{3}\rho_0 k} & 0 \\ 0 & 0 & \boxed{-\frac{2}{3}\rho_0 k} \end{bmatrix}$$

Isotropic turbulence models

- need k (one parameter)

$$k = \frac{1}{2}\left(\overline{u^2} + \overline{v^2} + \overline{w^2}\right) = \frac{3}{2}\overline{u^2} \rightarrow \underline{\overline{u^2} = \frac{2}{3}k}$$

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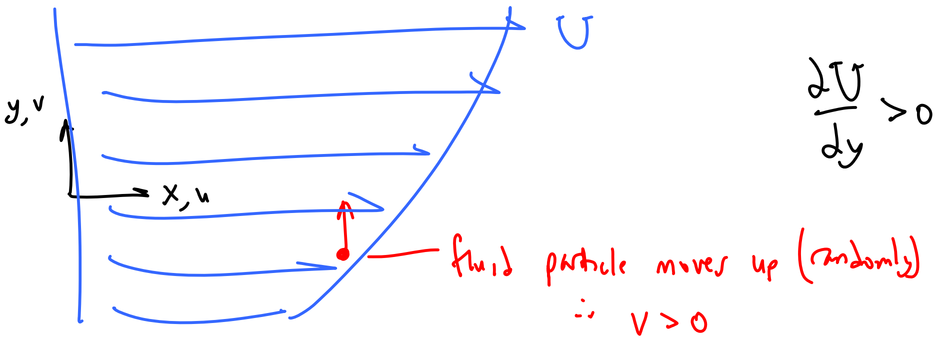
e. Sign of the shear stresses

E.g. Flows with a mean velocity gradient (BLs, wakes, jets, homogeneous shear flow)



$$\frac{\partial U}{dy} > 0$$

particle is now moving slower than its surroundings ∴ $\underline{u < 0}$

∴ $\underline{\overline{uv} < 0}$ typically

The Reynolds shear stress $-\rho_0 \overline{uv}$ is ⊕ve for a ⊕ve mean shear

f. Notation

Some authors define $\underline{\underline{\tau_{ij} = -\rho_0 \overline{u_i u_j}}}$ [T & L]

〃 〃 〃 call $-\rho_0 \overline{uv}$ the "Reynolds stress" for 2-D flow

〃 〃 drop the ρ → $-\overline{u_i u_j}$ = "Reynolds stress"

$$\boxed{\text{Let's call } -\overline{u_i u_j} \text{ the } \underline{\underline{\text{kinematic Reynolds stress}}}}$$

〃 〃 drop the ⊖ ve sign !

I added this after class was over:

Here is the notation we will use in this course — [agrees with the majority of turbulence books & articles]

$$\boxed{\begin{array}{l} \text{Reynolds Stress Tensor} \equiv -\rho_0 \overline{u_i u_j} \\ \\ \text{Kinematic Reynolds Stress Tensor} \equiv \overline{u_i u_j} \end{array}}$$

★

(notice - we dropped the ⊖ sign. I don't like this, but it is the most common notation)