M E 522 Spring 2008 Professor John M. Cimbala Lecture 27 03/24/2008

Today, we will:

- Discuss the closure problem of turbulence
- Derive and discuss the kinematic Reynolds stress transport equation
- Derive and discuss the mean kinetic energy (MKE) equation and the turbulent kinetic energy (tke) equation

5. The Closure Problem

Consider
- incompressible
- no buoyancy
- energy eq uncoupled from cont & mom.

a. mean flow eqs

cont $\boxed{\frac{\partial U_i}{\partial x_i}=0}$

mom: $\boxed{\rho_0\left[\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}\right] = -\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\mu\frac{\partial U_i}{\partial x_j} - \rho_0\overline{u_i u_j}\right]}$

4 eqs + 4 unk. + 6 new unknowns

Mathematically, the system of eqs is not "closed"

☆ We call this the Closure Problem in turbulence

How do we close the set of eq's?

↓

We can write an exact eq. for the Reynolds stress tensor.

b. Reynolds stress transport eq.

↓

Actually, derive a transport eq. for the kinematic Reynolds stress $\equiv \overline{u_i u_j}$

See Handout

The Turbulence Closure Problem and Derivation of the Reynolds Stress Equation

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1. Introduction – Governing Equations and the Closure Problem

Consider a simple case of turbulent flow – incompressible flow ($\rho = \rho_0$) without gravity. In this situation, the energy equation is *uncoupled* from the continuity and momentum equations. The mean flow equations therefore reduce to:

$$\frac{\partial U_i}{\partial x_i} = 0 \quad (1), \qquad \text{and} \qquad \rho\left(\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial U_i}{\partial x_j} - \rho\overline{u_iu_j}\right) \quad (2).$$

This represents 4 equations and 4 primary unknowns (U_i and P), which are functions of (x_i and t). But, the ***Reynolds stress tensor***, $\tau_{ij} \equiv -\rho\overline{u_iu_j}$ in the momentum equation represents *six additional unknowns*! Note that these equations are exact (within the limits of the approximation of incompressible flow, of course). Thus, *these four equations are insufficient to mathematically close the problem*. This is known as the ***turbulence closure problem***. Somehow we must come up with additional equations or assumptions to mathematically close the problem.

2. Derivation of the [Exact] Kinematic Reynolds Stress Transport Equation

We generate an *exact* transport equation for the Reynolds stresses by clever manipulation of the equations for the turbulent fluctuations (Actually, the equation will be for the ***kinematic Reynolds stress tensor*** $\overline{u_iu_j}$ for convenience.):

- **Step 1.** Start with the momentum equation for the turbulent fluctuations, neglect buoyancy, and use k instead of j as the dummy (repeating) index. This gives $\frac{\partial u_i}{\partial t} + U_k\frac{\partial u_i}{\partial x_k} + \ldots$, which is a *vector* equation since i is a free index.
- **Step 2.** Multiply the equation from Step 1 by u_j. This generates a *tensor* equation since i and j are both unrepeated (free) indices. This gives $u_j\frac{\partial u_i}{\partial t} + U_ku_j\frac{\partial u_i}{\partial x_k} + \ldots$, where k is still a dummy (repeating) index.
- **Step 3.** Interchange subscripts i and j everywhere in the equation from Step 2. (This is permitted since i and j are free indices, as long as it is done in the same manner for every term in the equation.) This gives $u_i\frac{\partial u_j}{\partial t} + U_ku_i\frac{\partial u_j}{\partial x_k} + \ldots$.
- **Step 4.** Add the equations generated by Steps 2 and 3. Utilize the inverse product rule, and make use of the continuity equation for the turbulent fluctuations to simplify some terms. After some algebra, this yields $\frac{\partial(u_iu_j)}{\partial t} + U_k\frac{\partial(u_iu_j)}{\partial x_k} + \ldots$.
- **Step 5.** Take the ensemble average of the equation generated in Step 4. Some terms drop out since $\overline{u_i} = \overline{u_j} = 0$.

  Rearrangement yields the ***Kinematic Reynolds Stress Transport Equation***: *Note*: $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_k\frac{\partial}{\partial x_k}$. Material derivative

$$\underbrace{\frac{D}{Dt}\left(\overline{u_iu_j}\right)}_{\text{I}} = \underbrace{-\left(\overline{u_iu_k}\frac{\partial U_j}{\partial x_k} + \overline{u_ju_k}\frac{\partial U_i}{\partial x_k}\right)}_{\text{II}} + \underbrace{\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{\text{III}} \underbrace{- \frac{\partial}{\partial x_k}\left[\overline{u_iu_ju_k} - \nu\frac{\partial\left(\overline{u_iu_j}\right)}{\partial x_k} + \overline{\frac{p}{\rho}\left(u_i\delta_{jk} + u_j\delta_{ik}\right)}\right]}_{\text{IV}} \underbrace{- 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}}_{\text{V}}.$$

Note: Since i and j are free indices, the above equation represents 9 (6 independent) components. The terms are defined below:

I ***Total rate of change*** of kinematic Reynolds stress following a fluid particle.
II ***Production*** of kinematic Reynolds stress, commonly given the symbol P_{ij}.
III ***Pressure-strain*** term, often called the ***intercomponent transfer*** term, commonly given the symbol π_{ij} or ϕ_{ij}. This term represents a *redistribution* of kinematic Reynolds stress [with no net gain or loss].
IV ***Spatial transport*** (diffusion) of kinematic Reynolds stress by three mechanisms: turbulent fluctuations, molecular (viscous) effects, and pressure fluctuations.

V ***Viscous dissipation*** of kinematic Reynolds stress, commonly given the symbol $-\varepsilon_{ij}$ where $\varepsilon_{ij} \equiv 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$.

This equation is still *exact*, and we now have six extra independent transport equations to mathematically close the problem. However, notice some *new unknowns* in the Reynolds stress transport equation in terms **III**, **IV**, and **V**! These new unknowns make the problem once again mathematically unclosed. Further transport equations for these terms can be developed if desired, e.g., $\frac{D}{Dt}\left(\overline{u_iu_ju_k}\right) = \ldots$, but then *even more new unknowns* would appear! ***The closure problem of turbulence cannot be resolved analytically (exactly)***. This is where ***turbulence modeling*** comes in – to mathematically *close* the equation set.

6. Mean & Turbulent Kinetic Energy Equations

a. MKE Mean Kinetic Energy Eq. (see Kundu Sec. 13.6, 7)

transport eq. for $\frac{1}{2} U_i U_i = \frac{1}{2} U_i^2$ = mean kinetic energy per unit mass

[still implies i summed]

b. TKE turbulent kinetic energy eq.

transport eq. for $\frac{1}{2} \overline{u_i u_i}$ = turbulent k.e. per unit mass

See Handout

Mean Kinetic Energy and Turbulent Kinetic Energy Equations

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1. Mean Kinetic Energy (MKE) Equation

Consider nearly incompressible turbulent flow (Boussinesq approximation). Manipulation of the mean continuity and momentum equations yields an equation for the ***mean kinetic energy per unit mass*** $\tfrac{1}{2}U_i^2$,

$$\frac{D}{Dt}\left(\tfrac{1}{2}U_i^2\right)=\frac{\partial}{\partial x_j}\left[-\frac{PU_j}{\rho_0}+2\nu U_iE_{ij}-\overline{u_iu_j}U_i\right]+\overline{u_iu_j}\frac{\partial U_i}{\partial x_j}-\frac{g}{\rho_0}\bar{\rho}U_3-2\nu E_{ij}E_{ij} \qquad (1)$$

I **II** **III** **IV** **V** **VI** **VII**

where the terms are labeled and defined as follows:

- **I** ***Total rate of change*** of mean kinetic energy per unit mass following a fluid particle.
- **II** Rate of ***spatial transport*** of mean kinetic energy per unit mass by pressure work.
- **III** Rate of ***spatial transport*** of mean kinetic energy per unit mass by mean viscous stresses.
- **IV** Rate of ***spatial transport*** of mean kinetic energy per unit mass by turbulent stresses.
- **V** Rate of ***destruction*** (or production) of mean kinetic energy per unit mass into turbulence.
- **VI** Rate of ***destruction*** (or production) of mean kinetic energy per unit mass into potential energy.
- **VII** Rate of ***viscous dissipation*** of mean kinetic energy per unit mass (turns mean kinetic energy into thermal energy, i.e., heat). *This term is always negative*, indicating a *loss* of mean kinetic energy per unit mass.

Note: In Equation (1), $E_{ij}\equiv\tfrac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)$ = ***mean strain rate tensor***, sometimes called S_{ij} (as in T&L). (2)

2. Turbulent Kinetic Energy (tke) Equation (also called the *tke budget*)

Manipulation of the continuity and momentum equations for the turbulent fluctuations yields an equation for the ***turbulent kinetic energy per unit mass*** $q^2=\tfrac{1}{2}\overline{u_ku_k}$ (*Note*: q^2 is sometimes given the notation K or k or tke),

$$\frac{D}{Dt}\left(q^2\right)=\frac{\partial}{\partial x_j}\left[-\frac{1}{\rho_0}\overline{pu_j}-\tfrac{1}{2}\overline{u_i^2u_j}+2\nu\overline{u_ie_{ij}}\right]-\overline{u_iu_j}\frac{\partial U_i}{\partial x_j}+g\alpha\overline{wT'}-2\nu\overline{e_{ij}e_{ij}} \qquad (3)$$

I **II** **III** **IV** **V** **VI** **VII**

where the terms are labeled and defined as follows:

- **I** ***Total rate of change*** of turbulent kinetic energy per unit mass following a fluid particle.
- **II** Rate of ***spatial transport*** of turbulent kinetic energy per unit mass by pressure work.
- **III** Rate of ***spatial transport*** of turbulent kinetic energy per unit mass by turbulent velocity fluctuations (convective diffusion).
- **IV** Rate of ***spatial transport*** of turbulent kinetic energy per unit mass by turbulent viscous stresses.
- **V** Rate of ***production*** (or destruction) of turbulent kinetic energy per unit mass from mean shear.
- **VI** Rate of ***production*** (or destruction) of turbulent kinetic energy per unit mass from fluctuating potential energy (buoyancy).
- **VII** Rate of ***viscous dissipation*** of turbulent kinetic energy per unit mass (turns tke into thermal energy, i.e., heat). *This term is always negative*, indicating a *loss* of turbulent kinetic energy.

Note: In Equation (3), $e_{ij}\equiv\tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$ = ***fluctuating strain rate tensor***, sometimes called s_{ij} (as in T&L). (4)

3. Rate of Viscous Dissipation of Turbulent Kinetic Energy per Unit Mass

The last term (**VII**) in Equation (3) is extremely important in analysis of turbulent flows. The negative of term **VII** is given a special name and symbol – the ***viscous dissipation rate of turbulent kinetic energy per unit mass***, sometimes called the ***scalar dissipation rate*** or simply the ***dissipation rate***,

$$\varepsilon\equiv2\nu\overline{e_{ij}e_{ij}} \qquad (5)$$

☆

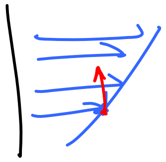
Since $\overline{e_{ij}e_{ij}}$ is positive definite, *the viscous dissipation rate is always a positive quantity* ($\varepsilon>0$).

ε of famous k-ε turbulence model
tke

Comments about the t.k.e. eq.

(1) The turbulence production term $\boxed{-\overline{u_iu_j}\frac{\partial U_i}{\partial x_j}}$ in the tke eq. also appears in the MKE eq., but with opposite sign

2-D: Recall, $\overline{uv}$ typ. <0 when $\frac{\partial U}{\partial y}>0$



& $\overline{uv}$ typ >0 when $\frac{\partial U}{\partial y}<0$

In 2D $\overline{uv}\frac{\partial U}{\partial y}$ typ. <u>negative</u>

$\boxed{\overline{u_iu_j}\frac{\partial U_i}{\partial x_j} \text{ is typically negative}}$

∴ represents
- loss of mean k.e. into turbulence
- gain of t.k.e from the mean flow

(2) <u>Viscous dissipation term</u> $-2\nu\overline{e_{ij}e_{ij}} = -\varepsilon$

$\varepsilon>0$ (the term in tke eq is ∴ ⊖ve)

Always represents a <u>loss</u> of t.k.e. (into heat ultimately)

(3) <u>Notation</u> - be careful!

<u>Author</u>	<u>Viscous transport term</u>	<u>Viscous dissipation term</u>
<u>Kundu</u>	$\frac{\partial}{\partial x_j}\left(2\nu\overline{u_ie_{ij}}\right)$	$-\varepsilon = -2\nu\overline{e_{ij}e_{ij}}$
<u>Tennekes & Lumley</u>	$\frac{\partial}{\partial x_j}\left(2\nu\overline{u_is_{ij}}\right)$	$-\varepsilon = -2\nu\overline{s_{ij}s_{ij}}$

Same as Kundu since $s_{ij}=e_{ij}$

Panton	Same as T & L	Same as T & L
Wilcox	$\frac{\partial}{\partial x_j}\left(\nu \frac{\partial k}{\partial x_j}\right)$ $[k = q^2 = tke]$ Not the same as the others	$-\varepsilon = -\nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}$ Not the same as the others

The sum of these two = the sum of the two terms in the other authors' notation

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(4) **tke notation**

Author	**Reynolds decomposition**	**Defn & symbol of tke**
Kundu	$\tilde{u}_i = U_i + u_i$	$q^2 = \frac{1}{2}\overline{u_i u_i} = tke$
T & L	$\tilde{u}_i = U_i + u_i$	$\frac{1}{2}\overline{q^2} = \frac{1}{2}\overline{u_i u_i} = tke$
☆ Wilcox	$u_i = U_i + u_i'$	$k = \frac{1}{2}\overline{u_i' u_i'} = tke$
Panton	$\tilde{u}_i = U_i + u_i$	$\frac{1}{2} q^2 = \frac{1}{2}\overline{u_i u_i} = tke$
White Viscous Fluid Flow	$u_i = \overline{u_i} + u_i'$	$K = \frac{1}{2}\overline{u_i' u_i'} = tke$

All 5 are different!