Today, we will:

- Discuss the **Eddy Viscosity** – for 2-D boundary layers, and for general 3-D flows
- Discuss closure for algebraic (zero-equation) turbulence models

[E. Turbulence models (continued)
1. Intro – why we need, requirements of, & classification of turbulence models]

2. The Eddy Viscosity [also called the turbulent viscosity]

"nuts & bolts" of turbulence models

Boussinesq (1877)

a. 2-D BL-type flows

2-D, stationary, no gravity
Dominant flow in x direction

x-mom eq. reduces to

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\Bigg[\overbrace{\underbrace{\mu\frac{\partial U}{\partial y}}_{\substack{\text{viscous shear stress} \\ \tau_{\text{laminar}}}} \underbrace{- \rho\overline{uv}}_{\substack{\text{turbulent shear stress} \\ \tau_{\text{turbulent}}}}}^{(\mu+\mu_e)\frac{\partial U}{\partial y}}\Bigg]$$

Eddy viscosity model → assume τ_{turb} is of the same form as τ_{lam}

$$\boxed{\tau_{lam} = \mu\frac{\partial U}{\partial y}} \Rightarrow \boxed{\tau_{turb} = \mu_e\frac{\partial U}{\partial y}} \qquad \mu_e = \text{eddy viscosity}$$

[some authors use μ_t = turbulent viscosity]

μ_e cannot simply be a constant → otherwise, we would simply get laminar flow soln at a lower Re.

i.e., $\boxed{-\rho\overline{uv} = \mu_e \frac{\partial U}{\partial y}}$ or $-\overline{uv} = \nu_e \frac{\partial U}{\partial y}$

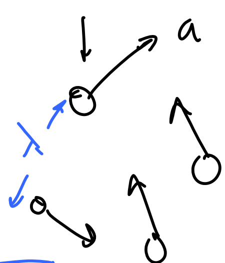
[ν_e = kinematic eddy viscosity $= \frac{\mu_e}{\rho}$]

We have to approx. μ_e.

- Laminar flow (molecular mixing)

a = speed of sound

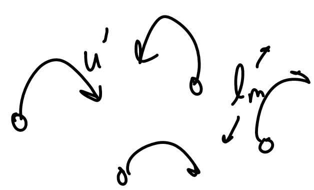
λ = mean free path



Kinetic Theory of gases

Show that $\boxed{\mu = \text{const}\cdot\rho\cdot\lambda\cdot a}$

- Turbulent flow → do a similar analysis
  - use fluid particles instead of molecules



l_m = "mixing length"

u' = char. velocity scale

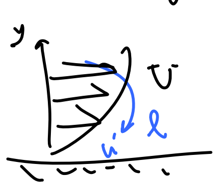
So, let's let

$$\boxed{\mu_e = \text{const}\cdot\rho\, l_m\, u'}$$

$$\therefore \quad \boxed{-\rho\overline{uv} = \text{const}\cdot\rho\cdot l_m\, u' \frac{\partial U}{\partial y}} \quad (2)$$

recall,

The large scale eddies have the same time scale as the mean flow



time scale of the mean flow $\sim \frac{1}{\left|\frac{\partial U}{\partial y}\right|}$

time scale of the largest eddies $\sim \frac{\ell}{u'}$

Equate these two time scales $\rightarrow$ $\boxed{u' \sim \ell \left|\frac{\partial U}{\partial y}\right|}$

Also expect that $\boxed{\ell_m \sim \ell}$

$$\Downarrow$$

$$\therefore \boxed{u' \approx \text{const} \cdot \ell_m \left|\frac{\partial U}{\partial y}\right|} \quad (3)$$

Plug (3) into (2) we get

$$☆ \boxed{-\rho \overline{uv} \approx C_1 \rho \ell_m^2 \left|\frac{\partial U}{\partial y}\right| \frac{\partial U}{\partial y}} ☆$$

☆ PRANDTL's MIXING LENGTH THEORY ☆

but ℓ_m is still not known

OR,

$$\boxed{-\rho\overline{uv} = \mu_e \frac{\partial U}{\partial y} \quad \text{where} \quad \mu_e = \rho \ell_m^2 \left|\frac{\partial U}{\partial y}\right|} ☆$$

b. General 3-D flows

3-D mean mom. eq.:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial}{\partial x_j}\Big[\underbrace{-P\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)}_{\substack{\text{laminar stress tensor} \\ \tau_{ij\,\text{lam}}}} \underbrace{- \rho\overline{u_i u_j}}_{\substack{\text{call this the turbulent} \\ \text{stress tensor } \tau_{ij\,\text{turb}}}}\Big]$$

We need to model this turbulent stress tensor

i.e., Let $\boxed{-\rho\overline{u_i u_j} = \underbrace{\boxed{}}_{\text{normal part}}\delta_{ij} + \mu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)}$

Model the normal part $\square$ assuming isotropic turbulence

Recall, for isotropic turbulence $\left[K = q^2 = tke\right]$

$$-\rho\overline{u_i u_j} = -\rho \begin{bmatrix} \frac{2}{3}K & 0 & 0 \\ 0 & \frac{2}{3}K & 0 \\ 0 & 0 & \frac{2}{3}K \end{bmatrix}$$

$\therefore$ Let

$$\boxed{-\rho\overline{u_i u_j} = \underbrace{-\frac{2}{3}\rho K \delta_{ij}}_{\text{additional normal stress due to the mixing}} + \underbrace{\mu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)}_{\text{additional shear stress due to the mixing}}}$$

Mean strain rate tensor E_{ij} or $S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$

$$\boxed{\begin{aligned} \tau_{ij_{lam}} &= -P\delta_{ij} + 2\mu E_{ij} \\ \tau_{ij_{turb}} &= -\frac{2}{3}\rho K \delta_{ij} + 2\mu_e E_{ij} \end{aligned}}$$ ✩

[we can't use $\mu_e = \rho l_m^2 \left|\frac{\partial U}{\partial y}\right|$]

Go back to $\boxed{\mu_e = \text{const}\,\rho\, l_m u'}$

we must approx. l_m & u'

Let $u' \sim \sqrt{K}$

l_m is treated differently in different turbulence models

Eg. let l_m = an algebraic eq. & solve a transport eq. for K

[one-eq turbulence model]

OR, can write a transport eq. for l_m itself & for K

[2-eq turb. model → K-l turb. model]

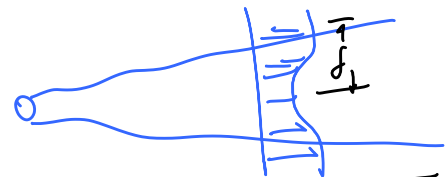
---

3. Closure for Algebraic (zero eq.) Turbulence Models

⇓

Good only for 2-D shear flows

eg.



Recall,

$$\boxed{\mu_e = \rho l_m^2 \left|\frac{\partial U}{\partial y}\right|} \qquad \boxed{\tau_{12} = -\rho\overline{u_1 u_2} = \mu_e \frac{\partial U}{\partial y}}$$

To close the problem, we need an expression for l_m

b. Examples:

(1) For free shear flows (jets, wakes, mixing layers) (no wall)

$l_m \approx$ constant at any x location → $\boxed{l_m = \underbrace{\text{const}}_{\text{empirically determined}} \cdot \delta}$

(2) Wall-bounded shear flows [eg. BL on a flat plate]

↓

typically use a 3-piece or 3-layer model

$$\boxed{\begin{array}{ll} l_m = \text{const} \cdot y^2 & \text{in the inner layer} \\ l_m = \underbrace{\kappa}_{\text{kappa}} y & \text{in the overlap layer} \\ l_m = \text{const} \cdot \delta & \text{in the outer layer} \end{array}}$$

(3) Von-Karman's mixing length (1930)

First attempt at a universal self-closing turbulence model

$$\boxed{l_m = \text{const}\,\frac{|\partial U/\partial y|}{|\partial^2 U/\partial y^2|}}$$

Problem – not universal

– Blows up at inflection pts

[not physical]

from experiment

$\simeq$ 0.4 for a flat plate turb. BL

[Von Karman's constant]