**Today, we will**:

- Finish modeling the tke equation (the additional transport eq. for one-equation turbulence models)
- Discuss closure for two-equation turbulence models, specifically the *k*-*ε* model.
- If time, begin to discuss how to treat the near-wall region.

Recall, the high-Reynolds # form of the tke. eq

Exact:

$$\underbrace{\frac{DK}{Dt}}_{\substack{\text{Total rate of change}\\ \downarrow \\ \text{Leave as is}}} = \underbrace{\frac{\partial}{\partial x_j}\left(\sim\sim\sim\right)}_{\substack{\text{Transport or Diffusion}\\ \downarrow \\ \text{Gradient Diffusion Model}}} - \underbrace{\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}}_{\substack{\text{Production}\\ \downarrow \\ \text{Express in terms of } \nu_e}} - \underbrace{\varepsilon}_{\text{Dissipation}}$$

Modeled:

$$\frac{DK}{Dt} = \frac{\partial}{\partial x_j}\left(\frac{\nu_e}{\sigma_k}\frac{\partial K}{\partial x_j}\right) + \nu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \varepsilon$$

?

recall $\varepsilon \sim \frac{u'^3}{\ell}$ → let $\varepsilon = \underline{\underline{\text{const}}} \cdot \frac{u'^3}{\ell}$

Also let $u' \sim \sqrt{K}$ & $\ell = \underline{\underline{\text{const}}} \cdot \ell_m$ — mixing length

Let $\boxed{\varepsilon = C_D \frac{K^{3/2}}{\ell_m}}$   $C_D$ = Dissipation Coefficient

<u>See Eq (7) on Handout</u> → our "one eq." for one-eq. turb. models

<u>Summary</u> Unknowns: $U_i$, $P$, $K$, $\ell_m$ (6)

4 primary transport eqs for $U_i$, $P$
1 extra transport eq for $K$ } (5)

6th eq is not a transport eq., but must come from experiment → typ. algebraic eq. for $l_m$

Note: $\boxed{\nu_e = C_\mu \sqrt{K}\, l_m}$ + constants $C_\mu$, $C_D$, $\sigma_K$ also determined empirically

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Does it work?

- We have added some "history" or "memory" effects for the turbulence since K is now a transport eq.

  "better" than zero-eq. models

- But still not an ideal model since we need $l_m$ from experiments

$\boxed{\text{one-eq models are not problem independent}}$ ☆

(NOT UNIVERSAL)

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5. Closure for Two-Eq. Turbulence Models

a. Intro → Any model lower than 2-eq. is "incomplete"

Why?

- Turbulence needs a minimum of two scales
  - length scale
  - velocity scale

- zero-eq models → obtain both scales empirically & locally
- one-eq. " → obtain K by a transport eq.

  But $l_m$ = length scale is still obtained empirically & locally

The Lowest level "complete" turbulence model is a two-eq model"

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add two transport eqs for the turbulence

K-$\ell$

K-enstrophy ($\overline{\omega_i \omega_i}$)

## $K$-$\varepsilon$ model (most popular & widely used turbulence model)

↓

- Still an eddy viscosity model

  But → we don't need to supply a mixing length

- Not a mixing length model

$$\boxed{\nu_e = \text{const} \cdot l_m u'}$$ → But we will solve & model both $l_m$ & $u'$ in terms of $K$ & $\varepsilon$ to close the problem

Let $u' = \sqrt{K}$

$$\varepsilon \sim \frac{u'^3}{l} \sim \frac{K^{3/2}}{l} \Rightarrow \text{Let } l_m = \text{const} \cdot l$$

$$l = \frac{K^{3/2}}{\varepsilon}$$

$$\boxed{l_m = \text{const} \frac{K^{3/2}}{\varepsilon}}$$

$$\nu_e = \underbrace{\text{const} \cdot \text{const}}_{= C_\mu} \frac{K^{3/2}}{\varepsilon} K^{1/2}$$

$$\boxed{\nu_e = C_\mu \frac{K^2}{\varepsilon}}$$

Notice — no more mixing length!

[Some people call this $C_\nu$]

$K$-$\varepsilon$ model is still an eddy-viscosity model, but not a mixing-length model.

We already have a modeled transport eq for $K$

We need a modeled transport eq. for $\varepsilon$

b. Exact Transport Eq for $\varepsilon$ is possible

(see Wilcox's book)

Rewrite $\varepsilon$ eq. as

$$\frac{D\varepsilon}{Dt} = \underbrace{\text{Diffusion or Transport of } \varepsilon + \text{Production of } \varepsilon - \text{Dissipation of } \varepsilon}_{\text{Model the entire RHS!}}$$

Total rate of change of $\varepsilon$

Keep LHS

C. Modeled Transport Eq. for $\varepsilon$

Model the $\varepsilon$ eq. after the $K$ eq. $\rightarrow$ make it of the same form

High Reynolds # form of $K$ transport eq.

$$\boxed{\frac{DK}{Dt} = \frac{\partial}{\partial x_j}\left(\frac{\nu_e}{\sigma_K}\frac{\partial K}{\partial x_j}\right) + \nu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \varepsilon} \quad (1)$$

High Reynolds # form of $\varepsilon$ transport Eq. [use the shovel heavily here!]

$$\boxed{\underbrace{\frac{D\varepsilon}{Dt}}_{\text{total rate of change of } \varepsilon} = \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\nu_e}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)}_{\text{Diffusion or transport of } \varepsilon} + \underbrace{\frac{C_{\varepsilon_1}\nu_e\varepsilon}{K}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}_{\text{Production of } \varepsilon} - \underbrace{C_{\varepsilon_2}\frac{\varepsilon^2}{K}}_{\text{Dissipation of } \varepsilon}}$$

[gradient diffusion model]

5 constants in total — Get these from experiments (decay of turbulence downstream of a screen) (BL's) (jets & shear layers)

$$\boxed{\sigma_K = 1.0, \quad \sigma_\varepsilon = 1.3, \quad C_\mu = 0.09, \quad C_{\varepsilon_1} = 1.44, \quad C_{\varepsilon_2} = 1.92}$$

STANDARD $k$-$\varepsilon$ model

[Note: Some authors call it $C_\nu$ instead of $C_\mu$]