Today, we will:

- Continue to work through the example problem – fully developed turbulent channel flow
- Discuss the 3-layer model for wall-bounded flows – the law of the wall

Recall, $x$-momentum reduced to

$$0 = -\frac{\partial P}{\partial x} + \frac{d}{dy} \left[ \mu \frac{dU}{dy} - \rho \bar{uv} \right]$$

(1)

and $y$-momentum reduced to

$$P = P_w(x) - \rho \bar{v}^2(y)$$

(2)

\[ \frac{\partial P_w}{\partial x} = \text{constant} \quad \frac{\partial P_w}{\partial x} \text{ vary linearly with } x \]

\[ \frac{d}{dy} \left[ \mu \frac{dU}{dy} - \rho \bar{uv} \right] = \text{constant} = \frac{dP_w}{dx} \cdot \left( \frac{2P}{dP} \right) \text{ since } P = P_w - \rho \bar{v}^2 \]

integrate:

$$\mu \frac{dU}{dy} - \rho \bar{uv} = y \frac{dP}{dx} + C_1$$

Define:

$$\bar{C} = \text{mean total stress} \equiv \mu \frac{dU}{dy} - \rho \bar{uv} = \text{func. of } y \text{ only}$$

(fully developed)

$$\bar{C} = y \frac{dP}{dx} + C_1$$

(5)

To find $C_1$, apply BC1: at wall, $U = 0$, $\bar{uv} = 0$, $P = P_w$
\[ \bar{y} = -b \quad \text{M.} \quad \frac{dU}{dy} \quad \text{at} \quad y = -b \]

Call this \( \bar{\omega} \) \quad \text{[Known call it \( \bar{\omega}_0 \)]}

\[ \bar{\omega} \quad \text{lower wall} = \text{positive} \]

(5) boundary

\[ \bar{\omega} = -b \frac{d\rho}{dx} + c_1 \quad \text{(6)} \]

(6) boundary

\[ \bar{\omega} = b \frac{d\rho}{dx} + c_i \quad \text{(7)} \]

Solve (6) & (7) simultaneously for unknown \( \bar{\omega} \) i. e.

\[ c_i = 0 \quad \therefore \quad \bar{\omega} = -b \frac{d\rho}{dx} \]

(5) boundary

\[ \bar{C} = M \frac{dU}{dy} - \rho \bar{w} = y \frac{d\rho}{dx} \quad \text{(8a)} \]

or, in terms of \( \bar{\omega} \)

\[ \bar{C} = M \frac{dU}{dy} - \rho \bar{w} = -\bar{\omega} \frac{y}{b} \quad \text{(8b)} \]

\( \bar{\omega} \) vary linearly across the channel!

\[ \bar{C} \quad \text{Vary linearly across the channel!} \]

This is identical to the Sommer channel flow result.
\[ \overline{c_2} = \mu \frac{d \overline{u}}{dy} - \rho \overline{uv} \]

Both laminar and turbulent flows are nonlinear in \( y \), but their sum ends up being linear in \( y \).

We can integrate again:

\[ \int \frac{d \overline{u}}{\overline{v}} = \frac{1}{\overline{v}} \int \overline{uv} \, dy + \frac{1}{\mu} \int \frac{2p}{2x} \, dy \]

Int. from \( y = -b \) to \( y = y \)

\[ \overline{U(y)} - \overline{U(-b)} = \frac{1}{\overline{v}} \int_{y=-b}^{y} \overline{uv} \, dy + \frac{1}{2\mu} \int_{y=-b}^{y} \frac{2p}{2x} \, dy \]

\[ \overline{U(y)} = \frac{1}{\overline{v}} \int_{y=-b}^{y} \overline{uv} \, dy + \frac{1}{2\mu} \int_{y=-b}^{y} \frac{2p}{2x} \, dy \]

This is as far as we can go analytically → need to model \( \overline{uv} \)

Can solve for \( \overline{U(y)} \) if you model \( \overline{uv} \) with a turbulence model

b. Approximate Analysis using the Layered Approach

Split the lower half of the channel into 3 layers.
At \( z = 0 \), \( z_{turb} \gg z_{lam} \)

- Outer layer
- Core region
- Turbulent shear \( \gg \) laminar shear

- Overlap layer
- Initial sublayer
- Near (laminar) shear
  \(~ \approx \) turbulent shear

At the wall, all of \( \bar{c} \) is laminar \((\bar{u}v = 0)\)

Inner layer: expect \( U = U(u, \rho, Z_w, (y+b)) \)

[Note: the flow in the inner layer, very close to the wall, does not "know" or "care" about \( U_e \) \((U \in y=0)\)]

Dimensional Analysis:

\[ \Pi_1 = \frac{U}{\sqrt{Z_w/\rho}} \]
\[ \Pi_2 = \frac{(y+b)\sqrt{Z_w/\rho}}{u^*} \]

Define Friction Velocity = \( u^* = \sqrt{Z_w/\rho} \)

\[ \Pi_2 = \frac{(y+b)u^*}{Z_w} \]

(Note: When \( v^* \)

\[ u \approx u_2 \]
Define "Inner variables"

Let \( \Pi_1 = \frac{U}{U^*} = U^* \) = inner velocity variable

Let \( \Pi_2 = \frac{(y+1)u^*}{U^*} = y^+ \) = inner coordinate or wall coordinate

O.m. anal. \( \Pi_1 = \text{fnc}(\Pi_2) \) or vise-versa.

\( U^+ = f(y^+) \) in the inner layer

"Law of the wall" → "universal" for any turbulent flow near a smooth solid wall

Eq. (8b), recall, \( \mu \frac{dU}{dy} - p\overline{uv} = -2 \frac{\gamma}{b} \) (8b)

In terms of inner variables:

we get

\[
\frac{du^+}{dy^+} - \frac{\overline{uv}}{u^*} = 1 - \frac{y^+}{R^*} \tag{11}
\]

where \( R^* = \frac{u^* b}{\nu} = y^+ @ \text{centerline of the channel} \)

In the inner layer, \( y^+ \ll R^* \)

In inner layer (11) becomes

\[
\frac{du^+}{dy^+} - \frac{\overline{uv}}{u^*} = 1 \tag{11 i}
\]

Let’s split the inner layer itself into two layers:
In the **viscous sublayer**, (11 i) becomes:

\[
\frac{du^+}{dy^+} - \frac{\overline{uu}}{u^+} \approx 1 \Rightarrow du^+=dy^+ \quad \text{for } y^+ \geq 0
\]

Integrate:

\[
U^+ = y^+ + \text{const}
\]

\(\text{At } y^+ = 0 \text{ (at wall)}, \quad y^+ = 0, \quad U^+ = 0, \quad \text{const} = 0\)

\(U^+ = y^+ \text{ in the viscous sublayer of the inner layer}\)

\(\text{Also called the inner sublayer or laminar sublayer}\)

\(U \text{ varies linearly with } y\)

Experimentally, the viscous sublayer extends to approx. 5 wall units from the wall:

\[U^+ \approx y^+ \text{ for } 0 \leq y^+ \leq 5\]