Today, we will:

- Continue to work through the example problem – fully developed turbulent channel flow
- Discuss the 3-layer model for wall-bounded flows – the law of the wall

Recall, *x*-momentum reduced to $\boxed{0=-\frac{\partial P}{dx}+\frac{d}{dy}\left[\mu\frac{dU}{dy}-\rho\overline{uv}\right]}$ (1)

and *y*-momentum reduced to $\boxed{P=P_w(x)-\rho\overline{v^2}(y)}$ (3)

Plug (3) into (1): $\frac{\partial}{\partial x}(\text{fnc. of } y)=0$

$$0=-\frac{dP_w}{dx}+\frac{d}{dy}\left[\mu\frac{dU}{dy}-\rho\overline{uv}\right]$$

$$\frac{dP_w}{dx}=\frac{d}{dy}\underbrace{\left[\mu\frac{dU}{dy}-\rho\overline{uv}\right]} \qquad (4)$$

fnc. of x only = fnc. of y only   MUST = CONSTANT

$\frac{dP_w}{dx}$ = constant P_w varies linearly with x

[same as in laminar flow]

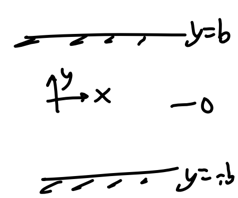
$$\frac{d}{dy}\left[\mu\frac{dU}{dy}-\rho\overline{uv}\right]=\text{constant}=\frac{dP_w}{dx}\text{ or }\frac{\partial P}{dx}\quad\text{since } P=P_w-\rho\overline{v^2}$$

NOT A FUNC. OF X

integrate: $\boxed{\mu\frac{dU}{dy}-\rho\overline{uv}=y\frac{\partial P}{dx}+C_1}$

Define $\boxed{\bar{\tau}=\text{mean total stress}\equiv\mu\frac{dU}{dy}-\rho\overline{uv}}$ = fnc. of y only (fully developed)

$$\boxed{\bar{\tau}=y\frac{\partial P}{dx}+C_1} \qquad (5)$$

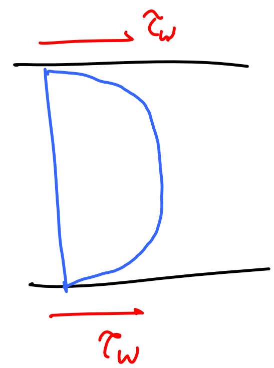


To find C_1, apply BCs → @ walls, $U=0$, $\overline{uv}=0$, $P=P_w$

@ $y = -b$ $\quad \underbrace{\mu \left.\frac{dU}{dy}\right)_{y=-b}} = \text{constant}$

Call this τ_w [Kundu calls it τ_0]

$\bar{\tau}$ @ lower wall = positive



(5) becomes
$$\boxed{\tau_w = -b \frac{\partial P}{dx} + c_1} \qquad (6)$$

@ $y = b$, $\overline{uv} = 0$ $\quad \mu \left.\frac{dU}{dy}\right)_{y=b} = \text{const}$, call this $-\tau_w$

(5) becomes
$$\boxed{-\tau_w = b \frac{\partial P}{dx} + c_1} \qquad (7)$$

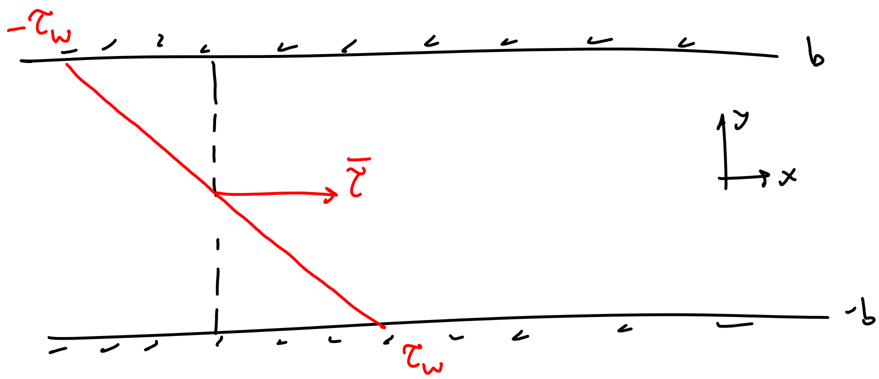
Solve (6) & (7) simultaneously for unknowns τ_w & c_1

$$\boxed{c_1 = 0} \quad \& \quad \boxed{\tau_w = -b \frac{\partial P}{dx}} \quad \star$$

so, (5) becomes
$$\boxed{\bar{\tau} = \mu \frac{dU}{dy} - \rho \overline{uv} = y \frac{\partial P}{dx}} \qquad (8a)$$

or, in terms of τ_w
$$\boxed{\bar{\tau} = \underbrace{\mu \frac{dU}{dy}}_{\text{lam}} \underbrace{- \rho \overline{uv}}_{\text{turb}} = -\tau_w \frac{y}{b}} \qquad (8b)$$

$\bar{\tau}$ varies linearly across the channel!



This is identical to the laminar channel flow result

$$\bar{\tau} = \underbrace{\mu \frac{dU}{dy}}_{\text{laminar shear stress}} \underbrace{- \rho \overline{uv}}_{\text{turbulent shear stress}}$$

Both lam & turb. shear stresses are nonlinear in y, but their sum ends up being linear in y

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We can integrate again

(8a) $\div \mu$ & rearrange $\quad \frac{dU}{dy} = \frac{1}{\nu}\overline{uv} + \frac{1}{\mu}\frac{\partial P}{\partial x} y$

$\left[\nu = \frac{\mu}{\rho}\right]$

const w.r.t. y

Int. from $y = -b$ to $y = y$

$$U(y) - \cancel{U(-b)}^{\,0} = \frac{1}{\nu}\int_{y=-b}^{b} \overline{uv}\, dy + \frac{1}{2\mu}\frac{\partial P}{\partial x}\left(y^2 - b^2\right)$$

$$\boxed{U(y) = \underbrace{\frac{1}{\nu}\int_{y=-b}^{y} \overline{uv}\, dy}_{\text{turbulent contribution}} + \underbrace{\frac{1}{2\mu}\frac{\partial P}{\partial x}\left(y^2 - b^2\right)}_{\text{parabolic laminar-type contribution}}} \qquad (9)$$

This is as far as we can go analytically → need to model $\overline{uv}$

Can solve for $U(y)$ if you model $\overline{uv}$ with a turbulence model

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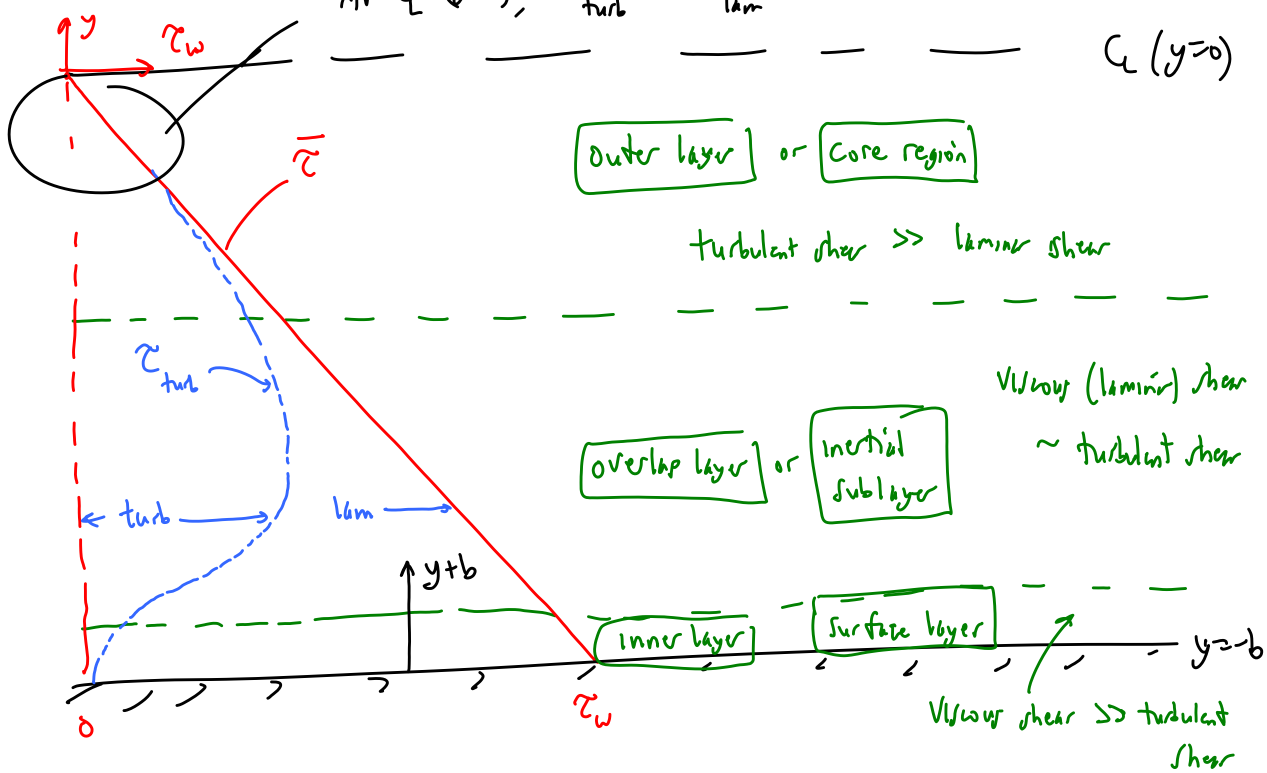
b. Approximate Analysis using the Layered Approach

Split the lower half of the channel into 3 layers

At ℄ $(y=0)$, $\tau_{turb} \gg \tau_{lam}$



At the wall, <u>all</u> of $\bar{\tau}$ is <u>laminar</u> $(\overline{uv}=0)$

<u>Inner layer</u>: expect $U = U(\mu, \rho, \tau_w, (y+b))$ (distance from lower wall)

[note: the flow in the inner layer, very close to the wall does not "know" or "care" about $U_{℄}$ $(U @ y=0)$]

<u>Dimensional Analysis</u>:

$$\boxed{\Pi_1 = \frac{U}{\sqrt{\tau_w/\rho}}} \qquad \boxed{\Pi_2 = \frac{(y+b)\sqrt{\tau_w/\rho}\,\rho}{\mu}}$$

Define <u>Friction Velocity</u> $= \boxed{u^* = \sqrt{\frac{\tau_w}{\rho}}}$

a kind of Reynolds #

$$\boxed{\Pi_2 = \frac{(y+b)\,u^*}{\nu}}$$

[<u>Notation</u>: T&L i.e. Kundu use u^*
White uses v^*
Wilcox " u_τ]

Define "Inner coordinates"

"Inner variables"

Let $\boxed{\Pi_1 = \dfrac{U}{u^*} = u^+ \text{ = Inner velocity variable}}$

Let $\boxed{\Pi_2 = \dfrac{(y+b)u^*}{\nu} = y^+ \text{ = Inner coordinate or wall coordinate}}$

Dim. anal. $\Pi_1 = \text{fnc}(\Pi_2)$ or vice-versa.

☆ $\boxed{u^+ = f(y^+) \text{ in the inner layer}}$

☆ "LAW OF THE WALL" → "universal" for any turbulent flow near a smooth solid wall

Eq (8b), recall, $\mu \dfrac{dU}{dy} - \rho\overline{uv} = -\tau_w \dfrac{y}{b}$ (8b)

In terms of inner variables:

we get

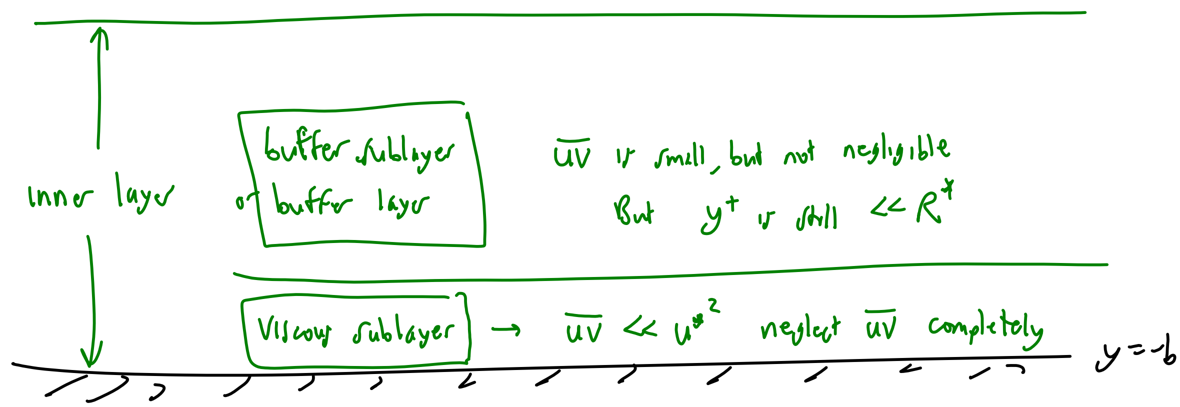
$$\boxed{\frac{du^+}{dy^+} - \frac{\overline{uv}}{u^{*2}} = 1 - \frac{y^+}{R^*}} \quad (11)$$

where $R^* = \dfrac{u^* b}{\nu} = y^+$ @ centerline of the channel

In the inner layer, $y^+ \ll R^*$

In inner layer, (11) becomes $\boxed{\dfrac{du^+}{dy^+} - \dfrac{\overline{uv}}{u^{*2}} \simeq 1}$ (11 i)

Let's split the inner layer itself into two layers:



In viscous sublayer, (11 i) becomes $\dfrac{du^+}{dy^+} - \cancel{\dfrac{\overline{uv}}{u^{*2}}}_{\approx 0} \approx 1 \Rightarrow du^+ = dy^+$

Integrate : $u^+ = y^+ + \text{const}$

↳ @ $y=-b$ (at wall), $y^+=0$, $u^+=0$, const $=0$

$$\boxed{u^+ = y^+ \text{ in the viscous sublayer of the inner layer}}$$

In physical variables, [U varies linearly with y]

also called the linear sublayer

or laminar sublayer

Experimentally, the viscous sublayer extends to approx. 5 wall units from the wall

$$\boxed{u^+ \approx y^+ \quad \text{for} \quad 0 \le y^+ \lesssim 5}$$