Example of Scale Factors for 3-D Boundary Layer Equations – Flow over a Cone

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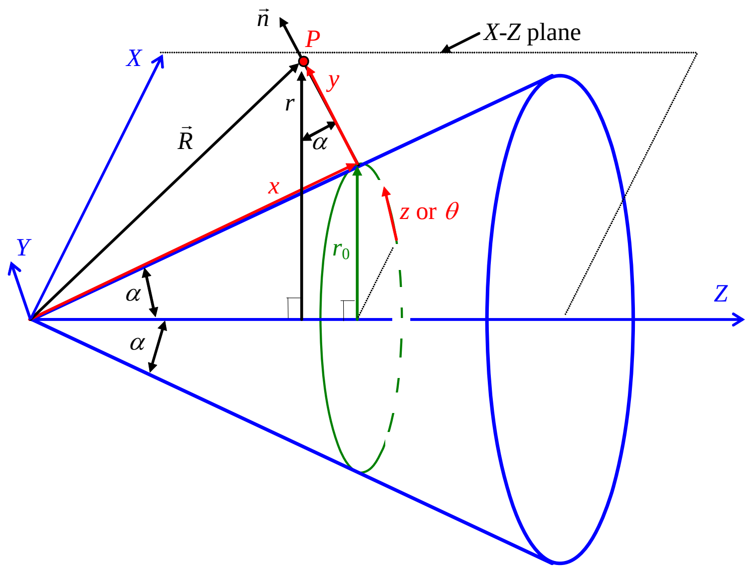
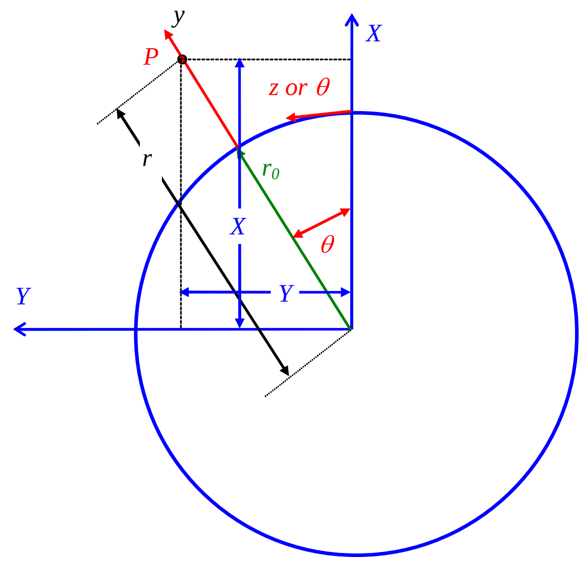
Geometry



3-D view, in *x*-*y* plane (*X*-*Z* plane is tilted back as shown)



View from the rear (*X*-*Y* plane)

Definition of scale factors or stretching factors

Recall, $h_x \equiv \left|\frac{d\vec{R}}{dx}\right|$, $h_y \equiv \left|\frac{d\vec{R}}{dy}\right|$, and $h_z \equiv \left|\frac{d\vec{R}}{dz}\right|$, where $\vec{R} \equiv (X, Y, Z)$ is the distance from a fixed origin to a point *P* inside the boundary layer. Consider a simple cone of half-angle α as an example (axisymmetric about the *Z*-axis). Let the fixed origin be the apex (tip) of the cone. Here, let r_0 be the perpendicular distance from the *Z*-axis to the body surface. Let *r* be the perpendicular distance from the *Z*-axis to point *P*. (Note that this is a different *r* than what we defined previously for a general 3-D coordinate system. It is the *same r*, however, that we defined previously when we discussed axisymmetric boundary layers.)

The boundary layer coordinates are (*x*,*y*,*z*) where *x* is a straight line along the body surface from the origin (a ray), *y* is normal to the body surface, and *z* is the angle along the body in the θ-direction, measured from the *X*-axis. (In fact, we can let $z = \theta$.) From trig, we see that $X = r\cdot\cos(\theta)$, $Y = r\cdot\sin(\theta)$, and $Z = x\cdot\cos(\alpha) - y\cdot\sin(\alpha)$. Also, we see that $r = x\cdot\sin(\alpha) + y\cdot\cos(\alpha)$.

Now, by definition, $h_x \equiv \left|\frac{d\vec{R}}{dx}\right| = \sqrt{\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial x}\right)^2}$, and similarly for the other scale factors, i.e.,

$h_y \equiv \left|\frac{d\vec{R}}{dy}\right| = \sqrt{\left(\frac{\partial X}{\partial y}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2}$ and $h_z \equiv \left|\frac{d\vec{R}}{dz}\right| = \sqrt{\left(\frac{\partial X}{\partial z}\right)^2 + \left(\frac{\partial Y}{\partial z}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2}$.

In class we will solve for these scale factors for this example. These can then be plugged into the 3-D boundary layer equations.

Note: If we assume that the boundary layer is thin with respect to r_0, and that there is no swirl, these equations should reduce to the Mangler equations for axisymmetric flow!