Incompressible 3-D Laminar Flow Boundary Layer Equations

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Assumptions/Approximations

- x and z coordinates lie along the body surface; y coordinate is normal to the body surface.
- The fluid is either an *incompressible liquid* or a *nearly incompressible ideal gas at very low Mach number*.
- Gravity is neglected.

Scale Factors or Stretching Factors

Define $h_x \equiv \left|\frac{d\vec{R}}{dx}\right|$, $h_y \equiv \left|\frac{d\vec{R}}{dy}\right|$, and $h_z \equiv \left|\frac{d\vec{R}}{dz}\right|$, where $\vec{R} \equiv \vec{r}(x,z) + y\vec{n}(x,z)$ is the distance from the fixed origin to a point inside the boundary layer, $\vec{r}$ is the distance from the fixed origin to the body surface, and $\vec{n}$ is the unit outward normal (in the y-direction away from the body surface).

Note: h_y is always unity when y is normal to the body surface, as assumed above.

General 3-D Boundary Layer Equations

Continuity:

$$\frac{1}{h_x h_z}\left[\frac{\partial}{\partial x}\left(h_z u\right) + \frac{\partial}{\partial z}\left(h_x w\right)\right] + \frac{\partial v}{\partial y} = 0,$$

***x*-momentum**:

$$\frac{u}{h_x}\frac{\partial u}{\partial x} + \frac{w}{h_z}\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial y} + \frac{uw}{h_x h_z}\frac{\partial h_x}{\partial z} - \frac{w^2}{h_x h_z}\frac{\partial h_z}{\partial x} = -\frac{1}{\rho h_x}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2},$$

***y*-momentum**:

$$\frac{\partial p}{\partial y} = 0$$

(same as 2-D B. L., as long as $\delta <<$ any local radius of curvature of the body),

***z*-momentum**:

$$\frac{u}{h_x}\frac{\partial w}{\partial x} + \frac{w}{h_z}\frac{\partial w}{\partial z} + v\frac{\partial w}{\partial y} - \frac{u^2}{h_x h_z}\frac{\partial h_x}{\partial z} + \frac{uw}{h_x h_z}\frac{\partial h_z}{\partial x} = -\frac{1}{\rho h_z}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial y^2}.$$

Euler Equations for the Irrotational Outer Flow (*U* and *W* along the body surface)

Note: These expressions can replace the pressure gradient terms in the x- and z-momentum equations above, so that the boundary layer equations can be written in terms of the known outer flow velocity field at the wall, $U(x,z)$ and $W(x,z)$.

***x*-momentum**:

$$\frac{U}{h_x}\frac{\partial U}{\partial x} + \frac{W}{h_z}\frac{\partial U}{\partial z} + \frac{UW}{h_x h_z}\frac{\partial h_x}{\partial z} - \frac{W^2}{h_x h_z}\frac{\partial h_z}{\partial x} = -\frac{1}{\rho h_x}\frac{\partial p}{\partial x},$$

***z*-momentum**:

$$\frac{U}{h_x}\frac{\partial W}{\partial x} + \frac{W}{h_z}\frac{\partial W}{\partial z} - \frac{U^2}{h_x h_z}\frac{\partial h_x}{\partial z} + \frac{UW}{h_x h_z}\frac{\partial h_z}{\partial x} = -\frac{1}{\rho h_z}\frac{\partial p}{\partial z}.$$

Boundary Conditions

- No slip conditions: $u = v = w = 0$ at $y = 0$ (at the surface of the body) for all x and z.
- Edge conditions: $u \to U$ and $w \to W$ as $y \to \infty$ (outside the BL) for all x and z.
- Starting conditions: Must specify u and w profiles at some x and z locations (starting profiles) to begin the calculations.