Development of Two-Equation Turbulence Models

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The Standard *K*-*ε* Two-Equation Turbulence Model (High Reynolds number form)

Two-equation turbulence models still employ the eddy viscosity assumption. However, the mixing length is removed since both u' and ℓ_m are specified: $u' = \sqrt{K}$ (same as one-equation models), and $\ell_m \approx \text{constant}\,\frac{u'^3}{\varepsilon} = \text{constant}\,\frac{K^{3/2}}{\varepsilon}$.

Plugging these into the definition of eddy viscosity, $\nu_e \approx \text{constant}\cdot \ell_m u'$ yields $\nu_e = C_\mu \frac{K^2}{\varepsilon}$, where C_μ is a new (combined) constant. Since K and ε are additional unknowns, *two* additional transport equations must be solved: (1) the same K equation we used for one-equation models, and (2) a modeled transport equation for ε,

$$\frac{DK}{Dt} = \frac{\partial K}{\partial t} + U_j \frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\nu_e}{\sigma_k}\frac{\partial K}{\partial x_j}\right) + \nu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \varepsilon \quad (1)$$

Total rate of change = Diffusion (transport) + Production – Dissipation

$$\frac{D\varepsilon}{Dt} = \frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\nu_e}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right) + \frac{C_{\varepsilon_1}\nu_e \varepsilon}{K}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - C_{\varepsilon_2}\frac{\varepsilon^2}{K} \quad (2)$$

Total rate of change = Diffusion (transport) + Production – Dissipation

The standard constants are $\sigma_K = 1.0$, $\sigma_\varepsilon = 1.3$, $C_\mu = 0.09$, $C_{\varepsilon_1} = 1.44$, and $C_{\varepsilon_2} = 1.92$.

Alternate Version of *K*-*ε* Transport Equations: *Note*: Wilcox (and some other authors) define both dissipation and molecular transport differently. With his definitions, the *K*-*ε* transport equations are given as

$$\frac{DK}{Dt} = \frac{\partial K}{\partial t} + U_j \frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_e}{\sigma_k}\right)\frac{\partial K}{\partial x_j}\right) + \nu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \varepsilon, \text{ where } \varepsilon \equiv \nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} \quad (3)$$

$$\frac{D\varepsilon}{Dt} \equiv \frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_e}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right) + \frac{C_{\varepsilon_1}\nu_e \varepsilon}{K}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - C_{\varepsilon_2}\frac{\varepsilon^2}{K} \quad (4)$$

Note: The standard constants remain the same, despite the different definitions of dissipation rate.

The RNG *K*-*ε* Two-Equation Turbulence Model (High Reynolds number form)

RNG stands for ***Renormalization Group***, and is a modification of the standard *K*-*ε* model. Some very high-level, complex mathematics is involved in its derivation, but the final outcome is the same *K* equation, an extra production term in the *ε* equation, and a slight modification of the constants. Using Wilcox's form:

$$\frac{DK}{Dt} = \frac{\partial K}{\partial t} + U_j \frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_e}{\sigma_k}\right)\frac{\partial K}{\partial x_j}\right) + \nu_e\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \varepsilon \quad (5)$$

$$\frac{D\varepsilon}{Dt} = \frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_e}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right) + \frac{C_{\varepsilon_1}\nu_e \varepsilon}{K}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - C_{\varepsilon_2}\frac{\varepsilon^2}{K} - R \quad (6)$$

where $R \equiv f(\varsigma)\cdot P_K \cdot \frac{\varepsilon}{K}$, $P_K \equiv \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_e}{\sigma_k}\right)\frac{\partial K}{\partial x_j}\right)$ = production term in the K equation, $f(\varsigma) \equiv \frac{\varsigma(1-\varsigma/4.38)}{(1+0.012\varsigma^3)}$, and $\varsigma \equiv (2E_{ij}E_{ij})^{\frac{1}{2}}\frac{K}{\varepsilon}$, where $E_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$ as defined previously.

The RNG constants are $\sigma_K = 0.719$, $\sigma_\varepsilon = 0.719$, $C_\mu = 0.085$, $C_{\varepsilon_1} = 1.42$, and $C_{\varepsilon_2} = 1.91$.

Studies have shown that this extra term generally leads to better solutions, especially in flows with swirl.