Exact Transport Equation for Turbulent Dissipation Rate

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1. Introduction – Exact Equation for ε

- By proper "manipulation" of the equations of motion, we can generate an *exact* equation for ε.
- The following is taken from Wilcox, D. C. Turbulence Modeling for CFD, second ed., 1998, DCW Industries, Inc., ISBN 0-9636051-5-1, p. 123.

4.3.2 The k-ϵ Model

By far, the most popular two-equation model is the k-ϵ model. The earliest development efforts based on this model were those of Chou (1945), Davidov (1961) and Harlow and Nakayama (1968). The central paper however, is that by Jones and Launder (1972) that, in the turbulence modeling community, has nearly reached the status of the Boussinesq and Reynolds papers. That is, the model is so well known that it is referred to as the **Standard k-ϵ model** and reference to the Jones-Launder paper is often omitted. Actually, Launder and Sharma (1974) "retuned" the model's closure coefficients and most researchers use the form of the model presented in the 1974 paper.

Again, we begin with Equations (4.8) and (4.11). In formulating the k-ϵ model, the idea is to derive the exact equation for ϵ and to find suitable closure approximations for the exact equation governing its behavior. Recall that ϵ is defined by Equation (4.5). The exact equation for ϵ is derived by taking the following moment of the Navier-Stokes equation:

$$2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial}{\partial x_j} [\mathcal{N}(u_i)]} = 0 \tag{4.44}$$

where $\mathcal{N}(u_i)$ is the Navier-Stokes operator defined in Equation (2.26). After a considerable amount of algebra, the following equation for ϵ results.

$$\boxed{\begin{aligned} \frac{\partial \epsilon}{\partial t} + U_j \frac{\partial \epsilon}{\partial x_j} = & -2\nu \left[\overline{u'_{i,k} u'_{j,k}} + \overline{u'_{k,i} u'_{k,j}} \right] \frac{\partial U_i}{\partial x_j} - 2\nu\, \overline{u'_k u'_{i,j}} \frac{\partial^2 U_i}{\partial x_k \partial x_j} \\ & -2\nu\, \overline{u'_{i,k} u'_{i,m} u'_{k,m}} - 2\nu^2 \overline{u'_{i,km} u'_{i,km}} \\ & + \frac{\partial}{\partial x_j} \left[\nu \frac{\partial \epsilon}{\partial x_j} - \nu\, \overline{u'_j u'_{i,m} u'_{i,m}} - 2 \frac{\nu}{\rho} \overline{p'_{,m} u'_{j,m}} \right] \end{aligned}} \tag{4.45}$$

This equation is far more complicated than the turbulence kinetic energy equation and involves several new unknown double and triple correlations of fluctuating velocity, pressure and velocity gradients. The terms on the three lines of the right-hand side of Equation (4.45) are generally regarded as **Production of Dissipation**, **Dissipation of Dissipation**, and the sum of **Molecular Diffusion of Dissipation** and **Turbulent Transport of Dissipation**, respectively. These correlations are essentially impossible to measure with any degree of accuracy so that there is presently little hope of finding reliable guidance from experimentalists regarding suitable closure approximations. Recent DNS studies such as the work of Mansour, Kim and Moin (1988) have helped gain some insight to the exact ϵ transport equation for low-Reynolds-number flows. However, the database for establishing closure approximations similar to those used for the k equation remains very sparse.

2. Modeled Equation for ε

- In class, we will discuss how the above equation is typically modeled in the standard K-ε turbulence model.
- As we will see, there is a lot of "arm-waving" and "shoveling" to get to the modeled equation!