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Fault Detection Using a Linear Algebraic Approach

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Abstract: The paper presents the concept of a redundancy management procedure for fault detection and isolation (FDI) from a linear algebraic point of view. The procedure can be used for developing intelligent instrumentation in strategic processes like spacecraft, aircraft, and nuclear power plants where redundant measurements are usually available for individual critical variables.

The redundancy management procedure is independent of the fault detection strategy and measurement noise statistics, and can be adopted for real-time applications using commercially available microcomputers. Its efficacy has been verified by on-line fault detection in operating nuclear reactors.

1. Introduction

Control systems for strategic processes such as aircraft, spacecraft, and hazardous chemical and nuclear power plants require intelligent instrumentation for coordination of plant monitoring, fault diagnostics, and decision making. Specifically, safety and reliability of such complex processes can be improved by systematic failure detection procedures [1-11]. Along this line the intelligent instrumentation system can be designed to accommodate, for each essential process variable, redundant measurements that may comprise both sensor outputs (possibly of different accuracies) as well as analytically derived measurement(s).

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The analytic measurements are synthesized from physical relationships between measurements of other process variables and/or from known characteristics of the process [3,9,10,11].

Some of the known fault detection procedures [1,2,7,8,9] are based on certain assumptions such as a known probability density function (or its characteristics) for nominal and failed conditions, and existence of a process model for constructing a filter. For many industrial applications, measurement noise statistics are not usually completely known nor do they follow a specific pattern such as Gaussian distribution, and the available process models are too complex or of insufficient accuracy to be used in filters. In the absence of relevent statistical information, the noise model may be approximated as amplitude-limited, i.e., the maximum noise amplitude that does not exceed a specified error bound is interpreted to be acceptable under nominal operating conditions. The information on sensor and plant equipment that is routinely available from the manufacturers is usually sufficient to quantify such requisite error bounds. However, if sufficient information on noise statistics is available, the afore-said restriction should be removed by assigning appropriate probabilities, instead of 1 and 0, to the noise amplitude being within the error bound for nominal and failed conditions, respectively; and established procedures such as sequential tests of Wald [12], Shiryayev [13] and Chien [14] can be used for algorithm development. Along this line we propose a fault detection technique using multiply-redundant measurements of a process variable which could be a vector quantity (such as the velocity or acceleration of an object in space) or a scalar (such as the thermal power or coolant temperature of a nuclear reactor).

The fault detection procedure presented in this paper is an extension of our earlier work reported in [6] with the following major improvements.

- The nonlinear and time-varying effects of measurement errors are considered, i.e., the errors are no longer assumed to be purely additive and time-independent.
- The fault detection procedure is partitioned into two modules: (1) redundancy management, (2) detection and isolation of faulty measurements via either single sample or sequential tests.
- A concept of measurement estimation which is equivalent to the midvalue selection technique of Potter and Suman [16,17] is introduced in the setting of the fault detection procedure.

The goal of this paper is to present the concept of the redundancy management procedure from both an algebraic and a geometric point of view, that could be used in the development of a fault detection and isolation methodology

for intelligent instrumentation. The redundancy management procedure makes use of all available measurements -- sensor outputs and analytic redundancy, and is essentially independent of the failure detection strategy, the measurement noise statistics and the associated assumptions such as amplitude-limited noise. An outline of the procedure is given below.

Failure decisions should be made by concurrent checking of "consistency" or "inconsistency" of individual measurements at each time sample. (Precise definitions of the terms in quotation and their physical significance are provided in the next section.) Checking of consistency or inconsistency of each redundant measurement can be carried out by one of the two options: (1) use of the allowable error bounds of respective measurements, that are specified a priori and can be time-varying; or (2) use of individual measurement's noise statistics provided that the available information is adequate for developing a decision algorithm. Both options can be implemented either using single sample tests or via sequential tests based on the time history of current and past observations [11,14,15].

The paper is organized in four sections and two appendices. Section 2 discusses the theoretical development of the redundancy management procedure, its physical interpretation, and how to apply this procedure. Sections 3 and 4 address the real-time application and experimental verification of this procedure to nuclear power plants, and summary and conclusions, respectively. Appendix A contains the three supporting theorems. A sketch of a sequential test algorithm that can be used in conjunction with the redundancy management algorithm is presented in Appendix B.

2. Theory of the Redundancy Management Procedure

The redundant measurements of a plant variable are modelled at the time sample t as

$$z(t) = [H(t) + \delta H(t)]\, x(t) + b(t) + \epsilon(t) \qquad (1)$$

where

z is (qx1) vector of known measurements,
H is (qxn) a priori known measurement matrix with q>n such that any n rows of H are linearly independent,
δH is (qxn) matrix for unknown scale factor errors,
x is (nx1) unknown value of the measured variable,
b is (qx1) unknown array of bias errors, and
ϵ is (qx1) vector of measurement noise with $E(\epsilon) = 0$.

In a previous publication [18] we reported the development of a calibration and estimation filter where, along with estimation of the measured variable, the combined effect of the scale factor and bias errors $c(t) = \delta H(t)x(t) + b(t)$ is also estimated for calibration of the measurements z(t). The calibrated measurements are defined as

$$m(t) = z(t) - \hat{c}(t) = H(t)\, x(t) + e(t) \qquad (2)$$

where

$\hat{c}(t)$ is the estimate of $c(t)$ and

$e(t) \triangleq (c(t) - \hat{c}(t)) + \epsilon(t)$ is the additive noise and errors associated with the calibrated measurements.

The calibration technique [18] is particularly suitable when scale factor errors occur slowly with respect to the sampling frequency, which is usually the case for the majority of industrial processes. However, if this is not the case as it is in inertial navigational instruments, the calibration process may require scale factor and bias errors to be separately estimated using nonlinear filtering techniques. (For example, see [19].)

If failure decisions are made by observing the calibrated measurements, then occurrence of alarms due to gradual degradation is significantly reduced and abrupt failures that are large in comparison to the allowable bounds are easily detected. However, the calibration correction $c(t)$ may track the failure in the event of a gradual degradation. Therefore, additional steps must be taken in the failure detection system design to guard against such a situation. This feature has been addressed and experimentally verified in our earlier work [18] and will not be repeated in this paper. In the rest of this section, we present the theoretical development of a redundancy management procedure.

For brevity, we are dropping the time dependence notation in subsequent mathematical equations. Thus, (2) is rewritten as

$$m = Hx + e \tag{3}$$

A measure of relative consistencies between redundant measurements is given by the the projection of the measurement vector m onto the left null space of the measurement matrix H such that the variations in the underlying variable Hx in (3) are eliminated and only the effects of the noise vector are observed. A $((q-n)xq)$ matrix V is chosen such that its $(q-n)$ rows form an orthonormal basis for the left null space of H, i.e.,

$$VH = 0 \quad \text{and} \quad VV^T = I_{q-n} \tag{4}$$

The column space of V is known as the parity space of H and the projection of m onto the parity space as the parity vector [16] which is given as

$$p = Vm = Ve \tag{5}$$

From (4), it follows that

$$V^T V = I_q - H[H^T H]^{-1} H^T \tag{6}$$

Because of the idempotent property of $V^T V$, the norm of the projection $V^T Vm$ of m onto the left null space of H is iden-

tically equal to $\|p\|$. The columns, $v_1, v_2, \ldots, v_q$, of V that are projections of the measurement directions (in R^q) onto the parity space are called failure directions since the failure of the i-th measurement m_i implies the growth of the parity vector p in (4) in the direction of v_i. For nominally unfailed operations, $\|p\|$ remains small. If a failure occurs, p may grow in magnitude along the failure subspace, i.e., the subspace spanned by the specific column vectors associated with the failed measurements; and if the fault is time-varying, then the failure directions (and hence the failure subspace) may also be time-dependent. The increase in the magnitude of the parity vector signifies abnormality of one or more measurements and its direction can be used for identification of abnormal measurement(s).

The redundancy management strategy was presented in our earlier publication [6] as determining the largest subset of unfailed measurements or alternatively as determining a failure subspace of smallest dimension such that the projection of the parity vector onto the orthogonal complement of the failure subspace does not indicate the presence of a failure. This observation follows from the Theorem 1 of Appendix A.

The uniqueness of a failure subspace of smallest dimension depends upon the number of failed measurements relative to the total number of redundant measurements. It is shown in Theorem 2 of Appendix A that the number of failures occurring at a given time that can be uniquely isolated, is less than or equal to $[(q-n)/2]$, i.e., the integer part of $(q-n)/2$. However, if the number of failures is greater than $[(q-n)/2]$ but less than or equal to $(q-n-1)$, the non-uniqueness needs to be addressed explicitly for possible identification of a consistent subset of unfailed measurements. (The upper bound of $(q-n-1)$ arises because at least $(n+1)$ measurements are required to ascertain consistency between the remaining unfailed measurements at a given time.)

A comprehensive procedure is adopted for identifying an appropriate failure subspace where relative orientation of the parity vector is determined with respect to all failure subspaces of dimension $(q-n-1)$ (which is one less than that of the parity space). Equivalently, such information can be generated by checking the consistency of individual $(n+1)$-tuplets of measurements, that involve only the computation of magnitudes of associated one-dimensional parity vectors. The procedure for evaluating consistencies of $(n+1)$-tuplets can be explained from the geometry of the bounded region in the $(q-n)$-dimensional parity space, where the parity vector lies if the set of q measurements is consistent. The consistency region in the parity space is the projection of the error cell in R^q, where the error vector e should be contained in the absence of any malfunctions, and is a polyhedron in the parity space bounded by hyperplanes belonging to failure subspaces of dimension $(q-n-1)$. Each hyperplane is the projection of a boundary of the error cell in R^q.

The presence of a malfunction can be tested by checking proximity of the parity vector to each of the hyperplanes bounding the polyhedral consistency region. The task of checking proximity involves determination of the components of the parity vector orthogonal to the bounding hyperplanes; equivalently, consistency of each (n+1)-tuplet is tested.

We introduce several definitions to delineate the concepts of consistency and inconsistency of a set S of q measurements (q>n) of an n-dimensional variable relative to all (n+1)-tuplets, i.e., distinct subsets of cardinality (n+1) denoted as $s_1, s_2, \ldots, s_r$

$$\text{where} \quad r = \frac{q!}{(n+1)!(q-n-1)!} \tag{7}$$

The magnitude of the parity vector p_i of dimension one, generated from (n+1) measurements in the (n+1)-tuplet s_i, i=1,2, ...r, is a measure of inconsistency of the (n+1) measurements in s_i.

Definition 1: The inconsistency index $\sigma(t)[s_i]$ of the (n+1)-tuplet s_i at sample time t is a real number directly related to $|p_i(t)|$. Thus, after dropping the time dependence notation for brevity, the inconsistency index can be expressed as

$$\sigma: s_i \dashrightarrow [0,\infty), \quad i = 1,2,\ldots\ldots,r \tag{8}$$

Furthermore, σ is appropriately scaled such that, under nominal conditions, $\sigma[s_i] \leq 1$ for every i. The exact structure of σ is dependent on the noise statistics as well as on the detection algorithm.

An example of how σ can be selected in a single-sample test assuming amplitude-limited noise is illustrated in [6]. For a sequential procedure, σ can be obtained via a recursive relationship which relies on the information derived from the past and current observations. For each (n+1)-tuplet the one-dimensional parity vector is generated at every sample time to compute the inconsistency index. As an example, an algorithm for obtaining σ is presented in Appendix B using Chien's modified sequential probability ratio test [14]. In Chien's methodology the noise associated with each parity vector (which is a linear combination of all measurements in the (n+1)-tuplet) is assumed to be Gaussian. We reiterate that the redundancy management procedure is independent of the noise statistics and the fault detection strategy, and is not restricted to Gaussian noise.

Definition 2: An (n+1)-tuplet s_i, i=1,2, ...,r is defined to be internally consistent if its inconsistency index is less than or equal to unity, i.e., $\sigma[s_i] \leq 1$.

Definition 3: A set of measurements is defined to be consistent if each of its (n+1)-tuplets is internally consistent.

Definition 4: Two disjoint subsets S_1 and S_2 of a measurement set S is defined to be relatively inconsistent if there exists no internally consistent (n+1)-tuplet having at least one element from each of S_1 and S_2.

Definition 5: A set of measurements that is not consistent is defined to be inconsistent [moderately consistent] if the set can [cannot] be split into two or more relatively inconsistent subsets.

The concept of moderate consistency is germane to the situation when errors in some of the measurements are in the vicinity of their rspective error bounds such that the measurements are contiguously dispersed and no measurement appears to be clearly malfunctioning.

The above definitions and associated separation of measurements serve the objective of isolation of inconsistent measurements. At this stage an appropriate subset of the remaining consistent or moderately consistent measurements should be identified for estimation of the measured variable. The estimate can be obtained via one of the two alternatives: (1) a weighted average of all measurements in the consistent or moderately consistent subset where the weights are adaptively updated on the basis of individual measurement's a posteriori probability of failure [18]; or (2) a selected n-tuplet using the midvalue selection concept of Potter and Suman [16,17].

Definition 6: Degree of inconsistency D(S) of a measurement set S is the largest of the inconsistency indices associated with each of its (n+1)-tuplets, i.e., $D(S) = \operatorname{Max}_i \sigma[s_i]$.

Remark: S is consistent iff $D(S) \leq 1$.

Remark: The degree of inconsistency is a single scalar measure of the consistency of a measurement set and can be geometrically interpreted as the inverse of the amount by which the parity vector should be multiplied so that it lies on the surface of the polyhedral consistency region.

Definition 7: Given a nonempty subset T of S, the relative degree of inconsistency D(T;S) of T with respect to S is defined to be the largest of inconsistency indices of all (n+1)-tuplets of S, that contains at least u elements of T where u = Min(#T,n) and #T denotes cardinality of T.

Remark: D(T;S) is a measure of relative consistency between the subsets T and (S-T); and D(T;S)>1 if T and S-T are relatively inconsistent. The inconsistency indices of the (n+1)-plets [if any] of T are taken into account for determination of D(T;S) whereas those of (S-T) have no direct bearing on D(T;S).

Definition 8: A nonempty subset T of S is defined to be a most relatively consistent subset of S if T has the least relative degree of inconsistency amongst all subsets of S.

Remark: Theorem 3 in Appendix A states that there exists an n-plet which is a most relatively consistent subset of S. Determination of a most relative consistent n-tuplet is equivalent to the midvalue selection process in the thresholdless redundancy management procedure proposed by Potter and Suman [16,17].

3. Real-time Experimental Verification in a Nuclear Reactor

The procedure described above was experimentally evaluated by on-line testing of sensor failures in the 5 MWt nuclear reactor MITR-II operated by the Massachusetts Institute of Technology. A description of the system configuration and instrumentation of the 5-MWt fission reactor is given in the MITR-II Reactor Systems Manual [20]. The nuclear instrumentation for the research reported in this paper consists of three neutron flux sensors and a gamma-ray sensor that correlates neutron power with the radioactivity (N-16) of the primary coolant. Four independent measurements of primary coolant flow are obtained from pressure differences across orifices. Primary coolant temperatures are measured as follows: two sensors for the hot leg, two sensors for the cold leg, and one sensor for the temperature difference between the legs. The noise and staistical characteristics of the MITR-II's flow, temperature, and neutron flux instrumentation are similar to those in commercial reactors. These sensors are connected to a portable LSI-11/23 microcomputer system through appropriate isolators, signal conditioners, and A/D converters.

In line with the rationale and definitions presented in the previous section, a real-time computer code was developed for concurrent checking of consistencies between all redundant measurements of a (scalar) process variable such as power, flow and temperature. Built-in tests such as limit checks and rate checks were routinely incorporated within the sequential test procedure. The execution time of the code on the LSI-11/23 processor is about 150 ms per cycle including the time required for data acquisition and signal processing.

Fault detection and isolation capabilities of the reported procedure were tested for both natural and injected failures of sensors. For continuous operations extended over a period of six months, two natural failures in temperature and flow sensors were automatically detected and isolated, and no false alarms were reported. The efficacy of the procedure was also tested for abnormal operations by injecting faults as described below.

Faulty sensor calibration. In a given sensor, a bias was introduced in excess of the error bound. The sensor was

isolated as faulty and the estimate was generated from the remaining measurements.

Gradual drift. Drift was intrduced in a given sensor. The sensor was isolated after the accumulated drift exceeded the permissible error bound.

Failed Sensors. Some of the sensors were disconnected from the prototype device and the corresponding ports of the the data acquisition system was short-circuited. The respective sensors were identified as faulty.

The procedure was also tested for transient operations. During a reactor shutdown process, the power estimate generated by the prototype device followed closely the true power until the calibration of neutron flux sensors (at a power level below 1 MWt) was no longer accurate making the measurements inconsistent.

4. Summary and Conclusions

The paper presents the concept (from both a geometric and an algebraic point of view) of a redundancy management procedure for fault detection and isolation to be used in intelligent instrumentation where the measured variables may be time-dependent vector or scalar quantities. For the process variable under consideration, there should be redundant measurements which could be sensor outputs and/or analytic measurements. The redundancy management procedure is independent of the fault detection strategy and measurement noise statistics.

The procedure has been verified for real-time detection and isolation of faulty sensors and plant equipment using commercially available microcomputers in nuclear reactors by experimentation.

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Appendix A: Theorems

Theorem 1: Let $v_1, v_2, \ldots, v_q$ represent the failure directions in the parity space corresponding to the set of q redundant measurements $\{m_1, m_2, \ldots, m_q\}$ of an n-dimensional variable. The norm of the projection of the (q-n)-dimensional parity vector (generated from the q measurements) onto the (k-n)-dimensional subspace that is orthogonal to $v_{k+1}, v_{k+2}, \ldots, v_q$, is identically equal to the norm of the (k-n)-dimensional parity vector which is generated by the set of measurements $\{m_1, m_2, \ldots, m_k\}$ where $n<k<q$.

Proof: The measurement vector m, mesurement matrix H, and projection matrix V in (3) are compatibly partitioned as follows:

$$m^T=[\mu_1^T \,|\, \mu_2^T],\ H^T=[H_1^T \,|\, H_2^T],\ \text{and}\ V=[V_1 \,|\, V_2] \qquad \text{(A-1-1)}$$

where

μ_1 is the (kx1) vector comprising k measurements and
μ_2 is the ((q-k)x1) vector comprising the remaining measurements.

From (3) it follows that

$$\mu_1 = H_1 x + e_1 \qquad \text{(A-1-2)}$$

where any n rows of the (kxn) matrix H_1 are linearly independent. The (k-n)-dimensional parity vector $\tilde{p}$ for the set of k measurements is given by (5) as

$$\tilde{p} = \tilde{V}\mu_1 \qquad \text{(A-1-3)}$$

where $\tilde{V}$ has the properties given by (4):

$$\tilde{V}H_1 = 0 \quad \text{and} \quad \tilde{V}\tilde{V}^T = I_{k-n} \qquad \text{(A-1-4)}$$

Let C be a (k-n)x(q-n) matrix such that its rows form an orthonormal basis for the left null space of V_2, i.e., the subspace orthogonal to the directions $v_{k+1}, v_{k+2}, \ldots, v_q$. Thus C has the properties given by (4):

$$CV_2 = 0 \quad \text{and} \quad CC^T = I_{k-n} \qquad \text{(A-1-5)}$$

The projection matrix $\tilde{V}$ in (A-1-3) can be chosen to be identical to CV_1 because CV_1 satisfies the properties in (A-1-4). Then, the projection of the parity vector p onto the column space of C is

$$Cp = CVm = C[V_1 \mid V_2][\mu_1^T \mid \mu_2^T]^T$$

$$= CV_1\mu_1 = \tilde{V}\mu_1 = \tilde{p} \qquad \text{(A-1-6)}$$

The proof is completed by arguing that the norm of the projection of p onto the left null space of $\tilde{V}$ is identically equal to $\|Cp\|$.

Theorem 2: Given q redundant measurements of an n-dimensional variable at a sample time, the number g of failed measurements that can be uniquely isolated is given as $2g \le (q-n)$, i.e., $g \le [(q-n)/2]$ where [*] indicates the integer part of *.

Proof: For unique isolation of g failures in a set of q redundant measurements, the subset of the remaining (q-g) measurements must be consistent. Since no more than (q-g-n) failure directions are needed to span the (q-g-n)-dimensional parity space that is generated from the subset of (q-g) measurements, such a subset containing failed measurements may exhibit consistency if more than (q-g-n) failures (that are mutually consistent) occur. For unique isolation of g failures, there must be one and only one largest consistent subset containing (q-g) unfailed measurements. This implies that $g \le (q-g-n)$ or $2g \le (q-n)$.

Theorem 3: Given a set S of q measurements of an n-dimensional variable such that q>n, there exists an n-tuplet which is a most relatively consistent subset of S. (See Definition 8.)

Proof: Let a j-tuplet ($1 \le j < q$) be denoted as $S^j \subset S$. With respect to a given S^j, let an i-tuplet be denoted as ${}_jS^i$ if ${}_jS^i \subset S^j$ or as ${}_j\tilde{S}^i$ if ${}_j\tilde{S}^i \subset (S - S^j)$. From Definitions 1, 6 and 7, it follows that

$$D(S^j;S) = \max_{{}_jS^k} \max_{{}_k\tilde{S}^{n+1-k} \subset (S - {}_jS^k)} \sigma[{}_jS^k \cup {}_k\tilde{S}^{n+1-k}] \qquad \text{(A-3-1)}$$

where $k = \text{Min}(j,n)$.

By Definition 8, a most relatively consistent subset satisfies the following condition

$$D(\Omega;S) = \min_{1 \le j \le q} \min_{S^j} D(S^j;S) \qquad \text{(A-3-2)}$$

Two cases are considered to show that there exists an n-tuplet which satisfies (A-3-2).

Case 1: $1 \le j \le n$, i.e., $k=\min(j,n)=j$ in Definition 7.

Let $X(S^j)$ be the collection of all $(n+1)$-tuplets $S^j \cup {}_j\tilde{S}^{n+1-j}$ generated from a given j-plet S^j. Substituting j for k in (A-3-1), it follows that

$$D(S^j;S) = \operatorname{Max}_{\phi \in X(S^j)} \sigma[\phi] \qquad \text{(A-3-3)}$$

By use of Lemma 1 in (A-3-3), it follows that

$$D(S^{j+1};S) \le D(S^j;S) \text{ if } S^j \subset S^{j+1} \text{ and } 1 \le j < n. \qquad \text{(A-3-4)}$$

From (A-3-4) it suffices to observe that

$$\operatorname{Min}_{S^n} D(S^n;S) \le \operatorname{Min}_{S^j} D(S^j;S) \text{ for } 1 \le j < n \qquad \text{(A-3-5)}$$

Case 2: $n \le j \le q$, i.e., $k = \min(j,n) = n$ in Definition 7.

Let $Y(S^j)$ be the collection of all $(n+1)$-tuplets $({}_jS^n \cup {}_n\tilde{S}^1)$. $({}_n\tilde{S}^1$ has a cardinality 1 and is given as ${}_n\tilde{S}^1 \subset (S - {}_jS^n).)$ Substituting n for k in (A-3-1), it follows that

$$D(S^j;S) = \operatorname{Max}_{\phi \in Y(S^j)} \sigma[\phi] \qquad \text{(A-3-6)}$$

By use of Lemma 2 in (A-3-6), it follows that

$$D(S^j;S) \le D(S^{j+1};S) \text{ if } S^j \subset S^{j+1} \text{ and } n \le j < q. \qquad \text{(A-3-7)}$$

From (A-3-7) it suffices to observe that

$$\operatorname{Min}_{S^n} D(S^n;S) \le \operatorname{Min}_{S^j} D(S^j;S) \text{ for } n \le j \le q. \qquad \text{(A-3-8)}$$

Combining (A-3-5) and (A-3-8), it follows that

$$\operatorname{Min}_{S^n} D(S^n;S) \le \operatorname{Min}_{S^j} D(S^j;S) \text{ for } 1 \le j \le q. \qquad \text{(A-3-9)}$$

Thus, there exists an n-tuplet which satisfies (A-3-2).

Lemma 1 [of Theorem 3]: If $S^j \subset S^{j+1}$ and $1 \le j < n$, then $X(S^{j+1}) \subset X(S^j)$.

Proof: Let $u \notin S^j$ and $S^{j+1} = S^j \cup \{u\}$. This implies that $u \notin (S - S^{j+1})$ and $(S - S^j) = (S - S^{j+1}) \cup \{u\}$. For every ${}_{j+1}\tilde{S}^{n-j} \subset (S - S^{j+1})$, there exists an ${}_j\tilde{S}^{n+1-j} \subset (S - S^j)$ such that ${}_j\tilde{S}^{n+1-j} = ({}_{j+1}\tilde{S}^{n-j} \cup \{u\})$. Thus, every $(n+1)$-tuplet $(S^{j+1} \cup {}_{j+1}\tilde{S}^{n-j}) \in X(S^{j+1})$ is contained in $X(S^j)$.

Lemma 2 [of Theorem 3]: If $S^j \subset S^{j+1}$ and $n < j < q$, then $Y(S^j) \subset Y(S^{j+1})$.

Proof: Since $S_j \subset S_{j+1}$, for every ${}_jS^n$ there exists an ${}_{j+1}S^n$ that ${}_{j+1}S^n = {}_jS^n$. Thus, every (n+1)-tuplet $(S^n \cup_n \tilde{S}^i) \in Y(S_j)$ is contained in $Y(S_{j+1})$.

Appendix B: A Sequential Test Algorithm

Given redundant measurements for an n-dimensional process variable, the one-dimensional parity vector $p_i(t)$ for the i-th (n+1)-tuplet s_i at the sampling inst-ant t is given as

$$p_i(t) = V_i \mu_i(t) \quad \text{for } i=1,2,\ldots,r \tag{B-1}$$

where V_i is the 1x(n+1) projection matrix associated with μ_i which is the (n+1)x1 vector representing measurements in s_i. The probability distribution of p_i is assumed to be Gaussian on the justification that p_i is a linear combination of the measurements in s_i, which are usually uncorrelated or weakly correlated. Using the a priori information on the covariace matrix of the noise in measurements in s_i, $p_i(t)$ is scaled to $z_i(t)$ such that the variance of $z_i(t)$ is unity. For nominally unfailed conditions $E[z_i(t)]=0$ for every i and t.

In the sequential tests a decision is made between the no-failure hypothesis, and one or more failure hypotheses, on the basis of the information processed at consecutive samples. If M distinct modes of failures are considered, then (M+1) distinct modes of operations should be designated by (M+1) mutually exclusive and exhaustive hypotheses such that each hypothesis can be treated as a Markov state. The recursive relations for a posteriori probalities in multiple hypotheses are derived in our earlier publication [18]. However, only one hypothesis which represents all abnormal modes including high and low failures is considered in this example.

The no-failure and failure hypotheses, H_0 and H_1 respectively, are defined below.

H_0: $z_i(t)$ is Gaussian with zero mean and unit variance at every sample instant t for all i.

H_1: $z_i(t)$ is Gaussian with mean $\pm\theta_i$ and unit variance at every sample instant t and for each (n+1)-tuplet i. The mean is positive or negative signifying high or low failures, respectively.

The log likelihood ratio at the t-th sample is defined as

$$\phi_i(t) = -\ell n \frac{p[z_i(t)|H_1)}{p[z_i(t)|H_0)}, \quad i=1,2,\ldots,r \tag{B-2}$$

If the measurement noise is stationary, the log likelihood ratio $\Phi_i(k)$ for k consecutive conditionally independent samples is given by

$$\Phi_i(k) = -\ell n \frac{p[z_i(1), z_i(2), \ldots, z_i(k) \mid H_1]}{p[z_i(1), z_i(2), \ldots, z_i(k) \mid H_0]} = \sum_{t=1}^{k} \phi_i(t) \tag{B-3}$$

which yields the following recursive relations for positive and negative values of the mean in hypothesis H_1.

$$\Phi_i^+(k) = \Phi_i^+(k-1) + \theta_i(\theta_i/2 - z_i(k))$$
$$\Phi_i^-(k) = \Phi_i^-(k-1) + \theta_i(\theta_i/2 + z_i(k)) \tag{B-4}$$

Following Chien's sequential test procedure [14], the above algorithm is formulated for $i=1,2,\ldots,r$ as follows.

Initialization:

$$\Phi_i^+(0) = \Phi_i^-(0) = 0.$$

Lower limit setting:

$$\Phi_i^+(k) = \text{Max}[\Phi_i^+(k), \epsilon_i] \quad \text{for all } k>0$$
$$\Phi_i^-(k) = \text{Max}[\Phi_i^-(k), \epsilon_i]$$

Consistency of the i-th (n+1)-tuplet:

$$\Phi_i^+(k) \lesssim \delta_i \text{ and } \Phi_i^-(k) \lesssim \delta_i \text{ for all } k>0$$

Inconsistency of the i-th (n+1)-tuplet:

$$\Phi_i^+(k) > \delta_i \text{ or } \Phi_i^-(k) > \delta_i \text{ for all } k>0$$

Upper limit setting:

$$\Phi_i^+(k) = \text{Min}[\Phi_i^+(k), \delta_i] \quad \text{for all } k>0$$
$$\Phi_i^-(k) = \text{Min}[\Phi_i^-(k), \delta_i]$$

where $\delta_i = \ell n[N(\theta_i)^2/2]$ is the detection threshold, N being the allowable mean time, i.e. the number of samples, between false alarms ($N>>1$), and the lower limit setting ϵ_i is the probability of failure of any measurement in the i-th (n+1)-tuplet. The parameter for upper limit setting is set to be equal to δ_i to enhance recovery from a failure condition after the faulty measurement has been reinstated. The inconsistency index σ in Definition 1 of Section 2 can be interpreted as follows.

$$\sigma(k)[s_i] = \text{Max}[\Phi_i^+(k), \Phi_i^-(k)]/\delta_i \tag{B-5}$$

The magnitude θ_i of the mean can be chosen as a function of the error bounds of the measurements in the i-th (n+1)-tuplet. Error bounds can be computed by measuring sensor noise statistics or from manufacturer's specifications.