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# DISCRETE-TIME LOOP TRANSFER RECOVERY WITH MULTISTEP DELAYS

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## SUMMARY

This paper focuses on delay compensation as an extension of the loop transfer recovery (LTR) procedure from one-step prediction to the general case of $p$-step prediction ($p \geqslant 1$). The steady state minimum variance filter gain is shown to be the $H_2$-minimal solution of the relative error between the target sensitivity matrix and the actual sensitivity matrix for $p$-step prediction ($p \geqslant 1$). This result is useful for synthesis of robust delay compensators in multiple-input/multiple-output (MIMO) discrete-time systems.

KEY WORDS Discrete-time $H_2$-optimization LQG/LTR control Delayed systems

## INTRODUCTION

The presence of delay(s) within a multi-input/multi-output (MIMO) feedback system makes the task of controller design significantly more difficult than that without delays. To this effect Luck and Ray[1] proposed a delay compensator to alleviate the detrimental effects of bounded delays by using a multistep predictor. The number $p$ of predicted steps in the compensator is then determined from the upper bound of the delay; that is, at time $k$ the predictor estimates the state using the measurements up to the $(k-p)$th instant. Although Luck and Ray[1] addressed some of the robustness issues of the delay compensator for structured uncertainties, the compensated system used the gain matrices that were originally designed for the non-delayed system. Since the robustness property of linear quadratic optimal regulators (LQRs) is not retained when the state feedback is replaced by state estimate feedback,[2] this problem is likely to become worse with the insertion of a $p$-step predictor for $p \geqslant 1$ because of the additional dynamic errors resulting from plant-modelling uncertainties and disturbances. The objective of this paper is to extend the concept of loop transfer recovery (LTR)[3,4] for multistep delays (i.e., $p \geqslant 1$) in a discrete-time setting.

## REVIEW OF THE LTR CONCEPT FOR ONE-STEP PREDICTION

The concept of loop transfer recovery (LTR) and the existing results for one-step prediction in the discrete-time setting are reviewed in this section. The plant under control is represented

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by a discretized version of a finite-dimensional, linear, time-invariant model in the continuous-time setting. The discretized model is assumed to be minimum phase, stabilizable and detectable:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \tag{1}$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k \tag{2}$$

The full-state feedback control law for the above plant is

$$\mathbf{u}_k = -\mathbf{F}\mathbf{x}_k \tag{3}$$

Following (1) and (2), the plant transfer matrix is given as

$$\mathbf{G}(z) = \mathbf{C}\mathbf{\Phi}(z)\mathbf{B} \tag{4}$$

where $\mathbf{\Phi}(z) = (z\mathbf{I} - \mathbf{A})^{-1}$ is the resolvement matrix. Following (3), the loop transfer matrix at the plant input is

$$\mathbf{H}(z) = \mathbf{F}\mathbf{\Phi}(z)\mathbf{B} \tag{5}$$

and the resulting sensitivity matrix is

$$\mathbf{S}(z) = [\mathbf{I} + \mathbf{H}(z)]^{-1} \tag{6}$$

For the filter observer (i.e., letting $p = 0$) of a stabilizable, detectable and minimum phase plant the loop transfer and sensitivity matrices have been shown by Maciejowski[5] to converge pointwise in frequency to those of the target system as the measurement noise approaches zero. However, this may not be valid for the one-step predictor.[5,6]

In this paper we have assumed that the uncertainties are lumped at the plant input in the form of an input multiplicative term. Therefore, breaking the loop at the plant input, the one-step delay compensator transfer matrix is obtained as

$$\mathbf{G}_1(z) = \mathbf{F}(z\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{F} + \mathbf{L}\mathbf{C})^{-1}\mathbf{L} \tag{7}$$

Then the loop transfer matrix for the one-step delay-compensated system is

$$\begin{aligned}\mathbf{L}_1(z) &= \mathbf{G}_1(z)\mathbf{G}(z) \\ &= \mathbf{F}[\mathbf{I} + \mathbf{\Phi}(z)(\mathbf{B}\mathbf{F} + \mathbf{L}\mathbf{C})]^{-1}\mathbf{\Phi}(z)\mathbf{L}\mathbf{C}\mathbf{\Phi}(z)\mathbf{B}\end{aligned} \tag{8a}$$

which can also be expressed, similarly to the formula proposed by Zhang and Freudenberg,[6] as

$$\mathbf{L}_1(z) = [\mathbf{I} + \mathbf{E}_1(z)]^{-1}[\mathbf{H}(z) - \mathbf{E}_1(z)] \tag{8b}$$

where $\mathbf{E}_1 = \mathbf{F}(z\mathbf{I} - \mathbf{A} + \mathbf{L}\mathbf{C})^{-1}\mathbf{B}$ is the one-step error matrix at the plant input. The resulting one-step sensitivity matrix can be expressed as a function of the error transfer matrix:

$$\begin{aligned}\mathbf{S}_1(z) &= [\mathbf{I} + \mathbf{L}_1(z)]^{-1} \\ &= [\mathbf{I} + \mathbf{H}(z)]^{-1}[\mathbf{I} + \mathbf{E}_1(z)]\end{aligned} \tag{9}$$

It is clear from (6) and (9) that $\mathbf{E}_1(z)$ is essentially the relative error of the sensitivity matrix $\mathbf{S}_1(z)$ of the one-step delay compensator relative to the target sensitivity matrix $\mathbf{S}(z)$, i.e.

$$\mathbf{E}_1(z) = \mathbf{S}(z)^{-1}[\mathbf{S}_1(z) - \mathbf{S}(z)] \tag{10}$$

It is known[5,6] that complete loop recovery, i.e. making $\mathbf{E}_1(z) = 0$ for all $z$, cannot be achieved in general by a constant observer gain $\mathbf{L}$. However, it is possible to identify an $\mathbf{L}$ that minimizes the one-step error transfer matrix $\mathbf{E}_1(z)$ in the $H_2$ sense.[7]

## THE $p$-STEP DELAY COMPENSATOR

If the sum of the distributed delays is represented by a lumped delay of $p$ sampling intervals at the plant output, the sensory information made available to the controller is $\mathbf{y}_{k-p}$ at the $k$th instant. The $p$-step delay compensator (where the plant is completely controllable and observable) proposed by Luck and Ray[1] has the structure

$$\mathbf{u}_k = \mathbf{F}\hat{\mathbf{x}}_{k\,|\,k-p} \tag{11}$$

where $\mathbf{F}$ is the state feedback gain matrix and the state estimate is based on the sensory information up to the $(k-p)$th instant given as

$$\begin{aligned} \hat{\mathbf{x}}_{k\,|\,k-p} &= \mathbf{A}\hat{\mathbf{x}}_{k-1\,|\,k-p} + \mathbf{B}\mathbf{u}_{k-1} \\ &\;\;\vdots \\ \hat{\mathbf{x}}_{k-p+2\,|\,k-p} &= \mathbf{A}\hat{\mathbf{x}}_{k-p+1\,|\,k-p} + \mathbf{B}\mathbf{u}_{k-p+1} \\ \hat{\mathbf{x}}_{k-p+1\,|\,k-p} &= \mathbf{A}\hat{\mathbf{x}}_{k-p\,|\,k-p-1} + \mathbf{B}\mathbf{u}_{k-p} + \mathbf{L}(\mathbf{y}_{k-p} - \mathbf{C}\hat{\mathbf{x}}_{k-p\,|\,k-p-1}) \end{aligned} \tag{12}$$

The key idea of using the LTR approach to the above $p$-step delay compensator is to tune the loop transfer matrix such that the error transfer matrix (i.e. the difference between the actual and target sensitivity matrices) is minimized in a certain sense. Derivation of the loop transfer function of the $p$-step delay compensator is presented below as two propositions.

*Proposition 1*

The transfer matrix of the $p$-step delay compensator ($p \geqslant 1$) from $\mathbf{y}_k$ to $\mathbf{u}_k$ in the equation set (12) is given as

$$\mathbf{G}_p(z) = \mathbf{F}\boldsymbol{\Omega}_p^{-1}(z)\,\frac{\mathbf{A}^{p-1}}{z^{p-1}}\left(z\mathbf{I} - \mathbf{A} + \mathbf{LC} + \mathbf{BF}\boldsymbol{\Omega}_p^{-1}(z)\,\frac{\mathbf{A}^{p-1}}{z^{p-1}}\right)^{-1}\mathbf{L} \tag{13}$$

where

$$\boldsymbol{\Omega}_p(z) = \mathbf{I} + \left(\mathbf{I} - \frac{\mathbf{A}^{p-1}}{z^{p-1}}\right)\boldsymbol{\Phi}(z)\mathbf{BF} \quad \text{for } p > 1, \qquad \boldsymbol{\Omega}_1(z) = \mathbf{I} \tag{14}$$

*Proof.* According to Ishihara,[8] the transfer matrix of the $p$-step compensator from $\mathbf{y}_k$ to $\mathbf{u}_k$ can be expressed as

$$\mathbf{G}_p(z) = -\left(\frac{\mathbf{FA}^{p-1}}{z^{p-1}}(z\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{B} + \mathbf{I} + \sum_{i=0}^{p-2}\frac{\mathbf{FA}^i\mathbf{B}}{z^{i+1}}\right)^{-1}\frac{\mathbf{FA}^{p-1}}{z^{p-1}}(z\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{L} \tag{15}$$

By using the relationship of the resolvement matrix $\boldsymbol{\Phi}(z)$ as $\mathbf{I} = z\boldsymbol{\Phi}(z) - \mathbf{A}\boldsymbol{\Phi}(z)$, the sum in the above equation can be expressed as

$$\begin{aligned} \sum_{i=0}^{p-2}\frac{\mathbf{FA}^i\mathbf{B}}{z^{i+1}} &= \sum_{i=0}^{p-2}\frac{\mathbf{FA}^i z\boldsymbol{\Phi}(z)\mathbf{B}}{z^{i+1}} - \sum_{i=0}^{p-2}\frac{\mathbf{FA}^i\mathbf{A}\boldsymbol{\Phi}(z)\mathbf{B}}{z^{i+1}} \\ &= \mathbf{F}\boldsymbol{\Phi}(z)\mathbf{B} - \mathbf{F}\,\frac{\mathbf{A}^{p-1}}{z^{p-1}}\,\boldsymbol{\Phi}(z)\mathbf{B} \end{aligned} \tag{16}$$

The proof is completed by substituting (16) and (14) into (15) and exercising a few algebraic operations. □

*Proposition 2*

Let the loop transfer matrix of the $p$-step delay-compensated system at the plant input be expressed as

$$\mathbf{L}_p(x) = \mathbf{G}_p(z)\mathbf{G}(z)$$

where $\mathbf{G}_p(z)$ and $\mathbf{G}(z)$ are as defined in (13) and (4) respectively. Then

$$\mathbf{L}_p(z) = [\mathbf{I} + \mathbf{E}_p(z)]^{-1}[\mathbf{H}(z) - \mathbf{E}_p(z)] \tag{17}$$

where

$$\mathbf{E}_p(z) = \mathbf{F}\mathbf{\Phi}(z)\mathbf{B} - \mathbf{F}\,\frac{\mathbf{A}^{p-1}}{z^{p-1}}\,[\mathbf{I} + \mathbf{\Phi}(z)\mathbf{LC}]^{-1}\mathbf{\Phi}(z)\mathbf{LC}\mathbf{\Phi}(z)\mathbf{B} \tag{18}$$

is the $p$-step error transfer matrix and $\mathbf{H}(z)$ is the target loop transfer matrix as defined in (5).

*Proof.* By Proposition 1 the open loop transfer matrix of the $p$-step delay compensator is

$$\begin{aligned}
\mathbf{L}_p(z) &= \mathbf{G}_p(z)\mathbf{G}(z)\\
&= \mathbf{F}\mathbf{\Omega}_p^{-1}(z)\,\frac{\mathbf{A}^{p-1}}{z^{p-1}}\left(z\mathbf{I} - \mathbf{A} + \mathbf{LC} + \mathbf{BF}\mathbf{\Omega}_p^{-1}(z)\,\frac{\mathbf{A}^{p-1}}{z^{p-1}}\right)^{-1}\mathbf{LC}\mathbf{\Phi}(z)\mathbf{B}\\
&= \mathbf{F}\mathbf{\Omega}_p^{-1}(z)\left(\mathbf{I} + \frac{\mathbf{A}^{p-1}}{z^{p-1}}\,[\mathbf{I} + \mathbf{\Phi}(z)\mathbf{LC}]^{-1}\mathbf{\Phi}(z)\mathbf{BF}\mathbf{\Omega}_p^{-1}(z)\right)^{-1}\frac{\mathbf{A}^{p-1}}{z^{p-1}}\\
&\quad\times [\mathbf{I} + \mathbf{\Phi}(z)\mathbf{LC}]^{-1}\mathbf{\Phi}(z)\mathbf{LC}\mathbf{\Phi}(z)\mathbf{B}\\
&= \mathbf{F}\left[\mathbf{I} + \left(\mathbf{I} - \frac{\mathbf{A}^{p-1}}{z^{p-1}}\right)\mathbf{\Phi}(z)\mathbf{BF} + \frac{\mathbf{A}^{p-1}}{z^{p-1}}\,[\mathbf{I} + \mathbf{\Phi}(z)\mathbf{LC}]^{-1}\mathbf{\Phi}(z)\mathbf{BF}\right]^{-1}\frac{\mathbf{A}^{p-1}}{z^{p-1}}\\
&\quad\times [\mathbf{I} + \mathbf{\Phi}(z)\mathbf{LC}]^{-1}\mathbf{\Phi}(z)\mathbf{LC}\mathbf{\Phi}(z)\mathbf{B}\\
&= \left(\mathbf{I} + \mathbf{F}\mathbf{\Phi}(z)\mathbf{B} - \mathbf{F}\,\frac{\mathbf{A}^{p-1}}{z^{p-1}}\,[\mathbf{I} + \mathbf{\Phi}(z)\mathbf{LC}]^{-1}\mathbf{\Phi}(z)\mathbf{LC}\mathbf{\Phi}(z)\mathbf{B}\right)^{-1}\mathbf{F}\,\frac{\mathbf{A}^{p-1}}{z^{p-1}}\\
&\quad\times [\mathbf{I} + \mathbf{\Phi}(z)\mathbf{LC}]^{-1}\mathbf{\Phi}(z)\mathbf{LC}\mathbf{\Phi}(z)\mathbf{B}
\end{aligned}$$

The proof is completed by substituting (18) and (5) in the above equation. □

*Remark 1*

After some algebraic manipulations $\mathbf{E}_p(z)$ can be written as

$$\mathbf{E}_p(z) = \mathbf{F}\mathbf{T}_p(z) \tag{19}$$

where

$$\mathbf{T}_p(z) = \begin{cases} \dfrac{\mathbf{A}^{p-1}}{z^{p-1}}\,(z\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{B} + \displaystyle\sum_{i=0}^{p-2}\frac{\mathbf{A}^{i}}{z^{i+1}}\,\mathbf{B}, & p > 1\\ (z\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{B}, & p = 1 \end{cases} \tag{20}$$

This shows that $\mathbf{E}_p(z)$ can be separated in terms of the full-state feedback gain $\mathbf{F}$ and a function of $\mathbf{L}$ and $p$. □

*Remark 3*

It follows from the expression of $\mathbf{L}_p(z)$ in Proposition 2 that the sensitivity matrix $\mathbf{S}_p(z)$ of the $p$-step delay compensator is

$$\begin{aligned}\mathbf{S}_p(z) &= [\mathbf{I}+\mathbf{L}_p(z)]^{-1}\\ &= [\mathbf{I}+\mathbf{H}(z)]^{-1}[\mathbf{I}+\mathbf{E}_p(z)]\end{aligned} \tag{22}$$

and the difference between the sensitivity matrices of the $p$-step compensated and target systems is

$$\mathbf{S}_p(z)-\mathbf{S}(z) = [\mathbf{I}+\mathbf{H}(z)]^{-1}\mathbf{E}_p(z) \tag{23}$$

where $\mathbf{S}(z)$ is given in (6). This shows that the error transfer matrix $\mathbf{E}_p(z)$ is indeed the error of the sensitivity matrix of the $p$-step delay compensator loop relative to that of the target loop. □

*Remark 4*

If the plant model has an inherent delay of $p_1$ steps and the induced delay in the feedback loop amounts to $p_2$ steps such that the total delay is $p = p_1 + p_2$, then the resulting error transfer matrix satisfies equation (21) as the measurement noise covariance is tuned to zero. This can be easily seen if the plant transfer matrix has the constraints

$$\mathbf{C}\mathbf{A}^i\mathbf{B} = 0, \quad i = 0, 1, \ldots, p_1 - 2, \qquad \det(\mathbf{C}\mathbf{A}^{p_1-1}\mathbf{B}) \neq 0 \tag{24}$$

According to Shaked,[9] the observer gain $\mathbf{L} \rightarrow \mathbf{A}^{p_1}\mathbf{B}(\mathbf{C}\mathbf{A}^{p_1-1}\mathbf{B})^{-1}$ as the measurement noise covariance approaches zero. Applying this observer gain to the loop transfer matrix gives

$$\mathbf{L}_p(z) = \mathbf{G}_p(z)\mathbf{G}(z)$$

and by using (13) in Proposition 1, the error matrix becomes identical to that in (21) with a total delay of $p = p_1 + p_2$.

This shows that any inherent delay in the plant has the same effect on the error matrix as the induced delay in the feedback loop. In other words, we can either consider the delays in the feedback loop outside the plant or as a part of the plant model. In the second case the original plant state space matrices ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$), with det ($\mathbf{CB}$) $\neq 0$, need to be augmented with $p_1$ steps of delay and the new plant state space matrices ($\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$) need to be formed, which must satisfy the following conditions: (i) $\mathbf{C}'(z\mathbf{I}-\mathbf{A}')^{-1}\mathbf{B}' = \mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}z^{-p_1}$; (ii) complete controllability and observability.[10] Therefore we have adopted the first approach of putting the lumped induced delay outside the plant model. □

*Remark 5*

Dual results of Proposition 2, obtained by breaking the loop at the plant output instead of the plant input, yield the loop transfer matrix

$$\begin{aligned}\mathbf{L}_p(z) &= \mathbf{G}(z)\mathbf{G}_p(z)\\ &= [\mathbf{H}(z)-\mathbf{E}_p(z)]\,[\mathbf{I}+\mathbf{E}_p(z)]^{-1}\end{aligned}$$

where the target loop transfer matrix at the plant output and the error transfer matrix are

$$\mathbf{H}(z) = \mathbf{C}\mathbf{\Phi}(z)\mathbf{L}, \qquad \mathbf{E}_p(z) = \mathbf{C}\mathbf{\Phi}(z)\mathbf{L} - \mathbf{C}\mathbf{\Phi}(z)\mathbf{B}\mathbf{F}\mathbf{\Phi}(z)[\mathbf{I}+\mathbf{B}\mathbf{F}\mathbf{\Phi}(z)]^{-1}\frac{\mathbf{A}^{p-1}}{z^{p-1}}\mathbf{L}$$

The resulting loop sensitivity matrix of the $p$-step delay compensator at the plant output is

$$\begin{aligned} \mathbf{S}_p(z) &= [\mathbf{I} + \mathbf{L}_p(z)]^{-1} \\ &= [\mathbf{I} + \mathbf{E}_p(z)]\,[\mathbf{I} + \mathbf{H}(z)]^{-1} \end{aligned}$$

and the difference between the sensitivity matrices of the minimum variance filter and $p$-step delay compensator is

$$\mathbf{S}_p(z) - \mathbf{S}(z) = \mathbf{E}_p(z)\,[\mathbf{I} + \mathbf{H}(z)]^{-1}$$

## $H_2$-MINIMIZATION OF THE $p$-STEP ERROR MATRIX

For the one-step predictor it has been shown by Yen and Horowitz[7] that the steady state minimum variance filter gain with zero measurement noise is obtained by minimizing the $H_2$-norm of the one-step error matrix $\mathbf{E}_1(z)$. Analogously to the case of $p = 1$, we will show that the same filter gain minimizes the $H_2$-norm of the $p$-step error matrix $\mathbf{E}_p(z)$. This result forms the basis for synthesis of robust $p$-step delay compensators ($p > 1$) and is presented in the sequel as two propositions.

For the purpose of tuning the minimum variance gain of the observer, we augment the discrete-time, linear, time-invariant plant model in (1) and (2) with (fictitious) plant noise and measurement noise as

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{w}_k \tag{25}$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{v}_k \tag{26}$$

where $\{\mathbf{w}_k\}$ is a zero-mean white sequence with covariance matrix $E\{\mathbf{w}_k\mathbf{w}_j^{\mathrm{T}}\} = \mathbf{B}\mathbf{B}^{\mathrm{T}}\delta_{kj}$ and $\{\mathbf{v}_k\}$ is a zero-mean white sequence with covariance matrix $E\{\mathbf{v}_k\mathbf{v}_j^{\mathrm{T}}\} = \rho\mathbf{I}\delta_{kj}$. Combining the distributed delays within the control loop as a lumped delay of $p$ sampling intervals at the sensor–controller interface, the state estimate is redefined as

$$\hat{\mathbf{x}}_{k\,|\,k-p} = E\{\mathbf{x}_k\,|\,\mathbf{y}_{k-p}\} \tag{27}$$

where the estimator is described by the set of equations (12).

*Proposition 3*

Let the (zero-mean) state estimation error be defined as

$$\mathbf{e}_{k\,|\,k-p} = \mathbf{x}_k - \hat{\mathbf{x}}_{k\,|\,k-p} \tag{28}$$

Then

$$E\{\mathbf{e}_{k\,|\,k-p}\mathbf{e}_{k\,|\,k-p}^{\mathrm{T}}\} = \sum_{s=0}^{\infty} \mathbf{A}^{p-1}(\mathbf{A} - \mathbf{L}\mathbf{C})^s\mathbf{B}\mathbf{B}^{\mathrm{T}}(\mathbf{A} - \mathbf{L}\mathbf{C})^{s\mathrm{T}}\mathbf{A}^{(p-1)\mathrm{T}} + \sum_{s=0}^{p-2} \mathbf{A}^s\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{s\mathrm{T}} \tag{29}$$

*Proof.* From the plant model in (25) and (26) and the filter equations (12) we can express the estimation error $\mathbf{e}_{k\,|\,k-p}$ in terms of the input sequence $\{\mathbf{w}_s\}$ when the system is initially started at $s = -\infty$:

$$\mathbf{e}_{k\,|\,k-p} = \sum_{s=-\infty}^{k-p} \mathbf{A}^{p-1}(\mathbf{A} - \mathbf{L}\mathbf{C})^{k-p-s}\mathbf{w}_s + \sum_{s=k-p+1}^{k-1} \mathbf{A}^{k-s-1}\mathbf{w}_s \tag{30}$$

Hence the cross-covariance of $\mathbf{e}_{k+s}$ and $\mathbf{w}_k$, defined as $R_{\mathbf{ew}}(s) = E\{\mathbf{e}_{k+s\,|\,k+s-p}\mathbf{w}_k^{\mathrm{T}}\}$, can be expressed in the form

$$R_{\mathrm{ew}}(s)=\sum_{m=p-s}^{\infty}\mathbf{A}^{p-1}(\mathbf{A}-\mathbf{LC})^{s-p-m}\mathbf{BB}^{\mathrm{T}}\delta(m)+\sum_{m=1-s}^{p-s-1}\mathbf{A}^{s-1-m}\mathbf{BB}^{\mathrm{T}}\delta(m)$$

$$=\begin{cases}\mathbf{A}^{p-1}(\mathbf{A}-\mathbf{LC})^{s-p}\mathbf{BB}^{\mathrm{T}}, & s\geqslant p\\ \mathbf{A}^{s-1}\mathbf{BB}^{\mathrm{T}}, & s=1,\ldots,p-1\\ 0, & s=0\end{cases}\tag{31}$$

Similarly, the autocovariance of $\mathbf{e}_k$ is obtained as

$$R_{\mathrm{ee}}(s)=\sum_{m=0}^{\infty}R_{\mathrm{ew}}(s+m+p)(\mathbf{A}-\mathbf{LC})^{m\mathrm{T}}\mathbf{A}^{(p-1)\mathrm{T}}+\sum_{s=0}^{p-2}R_{\mathrm{ew}}(s+m+1)\mathbf{A}^{s\mathrm{T}}\tag{32}$$

The proof is completed by setting $s=0$ and then substituting (31) into (32). □

*Proposition 4*

The $H_2$-norm of the sensitivity error matrix $\mathbf{E}_p(z)$, defined in (18) in Proposition 2, is minimized if the observer gain matrix $\mathbf{L}$ is identically equal to the standard steady state minimum variance filter gain matrix with zero measurement noise.

*Proof.* From the definition of $H_2$-norm[11] and $\mathbf{E}_p(z)$ in (19) of Remark 1 it follows that

$$\|\mathbf{E}_p(z)\|_2^2=\frac{1}{2\pi}\operatorname{trace}\left(\int_0^{2\pi}\mathbf{F}\mathbf{T}_p(\mathrm{e}^{\mathrm{j}\Omega})\mathbf{T}_p^*(\mathrm{e}^{\mathrm{j}\Omega})\mathbf{F}^{\mathrm{T}}\,\mathrm{d}\Omega\right)$$

$$=\frac{1}{2\pi}\operatorname{trace}\left[\int_0^{2\pi}\mathbf{F}\left(\frac{\mathbf{A}^{p-1}}{\mathrm{e}^{\mathrm{j}\Omega(p-1)}}(\mathrm{e}^{\mathrm{j}\Omega}\mathbf{I}-\mathbf{A}+\mathbf{LC})^{-1}\mathbf{B}\right)\right.$$

$$\left.\times\left(\frac{\mathbf{A}^{p-1}}{\mathrm{e}^{-\mathrm{j}\Omega(p-1)}}(\mathrm{e}^{-\mathrm{j}\Omega}\mathbf{I}-\mathbf{A}+\mathbf{LC})^{-1}\mathbf{B}\right)^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\,\mathrm{d}\Omega\right]$$

$$+\frac{1}{2\pi}\operatorname{trace}\left[\int_0^{2\pi}\mathbf{F}\left(\frac{\mathbf{A}^{p-1}}{\mathrm{e}^{\mathrm{j}\Omega(p-1)}}(\mathrm{e}^{\mathrm{j}\Omega}\mathbf{I}-\mathbf{A}+\mathbf{LC})^{-1}\mathbf{B}\right)\left(\sum_{s=0}^{p-2}\frac{\mathbf{A}^s}{\mathrm{e}^{-\mathrm{j}\Omega(s+1)}}\mathbf{B}\right)^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\,\mathrm{d}\Omega\right]$$

$$+\frac{1}{2\pi}\operatorname{trace}\left[\int_0^{2\pi}\mathbf{F}\left(\sum_{s=0}^{p-2}\frac{\mathbf{A}^s}{\mathrm{e}^{\mathrm{j}\Omega(s+1)}}\mathbf{B}\right)\left(\frac{\mathbf{A}^{p-1}}{\mathrm{e}^{-\mathrm{j}\Omega(p-1)}}(\mathrm{e}^{-\mathrm{j}\Omega}\mathbf{I}-\mathbf{A}+\mathbf{LC})^{-1}\mathbf{B}\right)^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\,\mathrm{d}\Omega\right]$$

$$+\frac{1}{2\pi}\operatorname{trace}\left[\int_0^{2\pi}\mathbf{F}\left(\sum_{s=0}^{p-2}\frac{\mathbf{A}^s}{\mathrm{e}^{\mathrm{j}\Omega(s+1)}}\mathbf{B}\right)\left(\sum_{s=0}^{p-2}\frac{\mathbf{A}^s}{\mathrm{e}^{-\mathrm{j}\Omega(s+1)}}\mathbf{B}\right)^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\,\mathrm{d}\Omega\right]\tag{33}$$

Since the sum of the second and third integrals is identically equal to zero and the integrals of the cross-terms in the fourth term also vanish,

$$\|\mathbf{E}_p(z)\|_2^2=\frac{1}{2\pi}\operatorname{trace}\left(\mathbf{F}\mathbf{A}^{p-1}\int_0^{2\pi}[(\mathrm{e}^{\mathrm{j}\Omega}\mathbf{I}-\mathbf{A}+\mathbf{LC})^{-1}\mathbf{BB}^{\mathrm{T}}(\mathrm{e}^{-\mathrm{j}\Omega}\mathbf{I}-\mathbf{A}+\mathbf{LC})^{-\mathrm{T}}\,\mathrm{d}\Omega]\,\mathbf{A}^{(p-1)\mathrm{T}}\mathbf{F}^{\mathrm{T}}\right)$$

$$+\operatorname{trace}\left[\mathbf{F}\left(\sum_{s=0}^{p-2}\mathbf{A}^s\mathbf{BB}^{\mathrm{T}}\mathbf{A}^{s\mathrm{T}}\right)\mathbf{F}^{\mathrm{T}}\right]\tag{34}$$

For given plant model state space matrices $\mathbf{A}$ and $\mathbf{B}$, if the feedback gain matrix $\mathbf{F}$ is fixed, then the observer gain $\mathbf{L}$ is the only adjustable matrix which could change the $H_2$-norm of the error matrix. On the other hand, the covariance of the state estimation error in Proposition

3 can be written, according to the discrete-time Plancherel theorem,[11] as

$$E\{\mathbf{e}_{k|k-p}\mathbf{e}_{k|k-p}^{\mathrm{T}}\} =$$

$$\mathbf{A}^{p-1}\int_{0}^{2\pi}[(\mathrm{e}^{\mathrm{j}\Omega}\mathbf{I}-\mathbf{A}+\mathbf{LC})^{-1}\mathbf{BB}^{\mathrm{T}}(\mathrm{e}^{-\mathrm{j}\Omega}\mathbf{I}-\mathbf{A}+\mathbf{LC})^{-\mathrm{T}}\,\mathrm{d}\Omega]\,\mathbf{A}^{(p-1)\mathrm{T}}+\sum_{s=0}^{p-2}\mathbf{A}^{s}\mathbf{BB}^{\mathrm{T}}\mathbf{A}^{s\mathrm{T}} \quad (35)$$

A comparison of the $H_2$-norm of the error matrix $\mathbf{E}_p(z)$ in (34) with the trace of the error covariance matrix $E\{\mathbf{e}_{k|k-p}\mathbf{e}_{k|k-p}^{\mathrm{T}}\}$ in (35) reveals that minimization of $\|\mathbf{E}_p(z)\|_2$ is equivalent to that of trace $(E\{\mathbf{e}_{k|k-p}\mathbf{e}_{k|k-p}^{\mathrm{T}}\})$ for all $p>0$.

Next we proceed to find an optimal $\mathbf{L}$ that minimizes trace $(E\{\mathbf{e}_{k|k-p}\mathbf{e}_{k|k-p}^{\mathrm{T}}\})$. It follows from Lemma 1 given below that the minimum variance filter gain (with $p=1$) also minimizes trace $(E\{\mathbf{e}_{k|k-p}\mathbf{e}_{k|k-p}^{\mathrm{T}}\})$ while $p>1$. Therefore the optimal observer gain $\mathbf{L}$ that minimizes $\|\mathbf{E}_p(z)\|_2$ is the same $\mathbf{L}$ that minimizes $\|\mathbf{E}_1(z)\|_2$. According to Lemma 2, the steady state minimum variance gain with zero measurement noise is the optimal gain. □

*Lemma 1 for Proposition 4*

For the $p$-step predictor ($p \geqslant 1$), if the estimation error is defined as $\mathbf{e}_{k|k-p}=\mathbf{x}_k-\hat{\mathbf{x}}_{k|k-p}$, then the filter gain $\mathbf{L}$ which minimizes the covariance $E\{\mathbf{e}_{k|k-p}\mathbf{e}_{k|k-p}^{\mathrm{T}}\}$ is identical to the minimum variance filter gain.

*Proof.* The proof follows directly from the derivations in Chap. 5 of Reference 12. □

*Lemma 2 for Proposition 4*

For a fixed $\mathbf{F}$ the $H_2$-optimization of the one-step predictor error matrix given as

$$\min_{\mathbf{L}}\|\mathbf{E}_1(z)\|_2=\min_{\mathbf{L}}\|\mathbf{F}(z\mathbf{I}-\mathbf{A}+\mathbf{LC})^{-1}\mathbf{B}\|_2 \quad (36)$$

is the steady state minimum variance filter gain, where the plant and measurement noise covariance matrices $\mathbf{Q}$ and $\mathbf{R}$ are set to

$$\mathbf{Q}=\mathbf{BB}^{\mathrm{T}}, \qquad \mathbf{R}=\lim_{\rho\to 0}\rho\mathbf{I} \quad (37)$$

*Proof.* The proof follows directly from the dual result of Theorem 3.1 in Reference 7. □

## CONCLUSIONS

Results on robust compensation of induced delays in a multi-input/multi-output discrete-time feedback control system are presented. The delay compensation algorithm formulated in this paper is an extension of the standard loop transfer recovery (LTR) procedure from one-step prediction to the general case of $p$-step prediction ($p \geqslant 1$). The major conclusion is that the concept of the steady state minimum variance filter gain as the $H_2$-minimal solution of the difference between the target sensitivity matrix and the actual sensitivity matrix for one-step prediction does hold for $p$-step prediction ($p>1$). This concept is useful for synthesis of robust delay compensators.

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