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Brief Paper

# Extended Discrete-time LTR Synthesis of Delayed Control Systems*†

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**Key Words**—Discrete systems; optimal control; robustness; delays.

**Abstract**—This paper addresses compensation of dealys within a multi-input–multi-output discrete-time feedback loop for application to real-time distributed control systems. The delay compensation algorithm, formulated, in this paper, is an extension of the standard loop transfer recovery (LTR) procedure from one-step prediction to the general case of $p$-step prediction ($p \geq 1$). It is shown that the steady-state minimum-variance filter gain is the $H_2$-minimization solution of the relative error between the target sensitivity matrix and the actual sensitivity matrix for $p$-step prediction ($p \geq 1$). This concept forms the basis for synthesis of robust $p$-step delay compensators ($p > 1$). The proposed control synthesis procedure for delay compensation is demonstrated via simulation of the flight control system of an advanced aircraft.

## 1. *Introduction*

IN MANY REAL-TIME distributed control systems such as advanced aircraft, spacecraft, and autonomous manufacturing plants, the sensor and control signals within a feedback loop may be delayed or interrupted. An example is the randomly varying delays induced by multiplexed data communication networks in Integrated Communication and Control Systems (ICCS) (Halevi and Ray, 1988; Ray and Halevi, 1988; Liou and Ray, 1990) where delays may be distributed between the sensor and the controller and between the controller and the actuator as illustrated in Fig. 1. Another example is the occurrence of delays in the control law execution due to priority interruption at the control computer (Belle Isle, 1975). In general, the presence of randomly varying distributed delays within a multi-input–multi-output (MIMO) feedback system makes the task of controller design significantly more difficult than that without delays. To this effect Luck and Ray (1990) proposed a delay compensator to alleviate the detrimental effects of randomly varying distributed delays by using a multi-step predictor. The key idea in this multi-step compensator design is to monitor the data when it is generated and to keep track of the delay associated with it. With this knowledge, the problem of varying distributed delays can be alleviated by having a lumped constant delay of multiple sampling intervals as seen by the controller.

The major assumption in the formulation of the above multi-step compensation algorithm (Luck and Ray, 1990) is that the randomly varying delays are bounded. This assumption is justified in view of the fact that an unbounded delay would render the closed loop open. Using a specified confidence interval, an upper bound can be assigned to each of the randomly varying distributed delays. The number ($p$) of predicted steps in the compensator is then determined from these bounds. That is, at time $k$, the predictor estimates the state using the measurements up to the $(k-p)$th instant. Although Luck and Ray (1990) addressed some of the robustness issues of the delay compensator for structured uncertainties, the compensated system used the gain matrices that were originally designed for the non-delayed system. Since the robustness property of linear quadratic optimal regulators (LQR) is not retained when the state feedback is replaced by state estimate feedback (Doyle and Stein, 1979), this problem is likely to become worse with the insertion of a $p$-step predictor for $p \geq 1$ because of the additional dynamic errors resulting from plant modeling uncertainties and disturbances.

The objective here is to develop a procedure for synthesis of the $p$-step delay compensator with a trade-off between performance and stability robustness. To achieve this goal we propose to extend the concept of loop transfer recovery (LTR) (Doyle and Stein, 1981; Stein and Athans, 1987) which is a well-established procedure for synthesis of robust controllers. In continuous-time systems, the key step in LTR design is to select an observer gain so that the full-state feedback loop transfer property can be recovered asymptotically. For the discrete-time LTR, Maciejowski (1985) has shown that although the target sensitivity matrix can be completely recovered with *a posteriori* state estimation (i.e. $p = 0$ in the $p$-step predictor) a predictive state estimator (i.e. $p = 1$) is not capable of full recovery. Along this line Zhang and Freudenberg (1991) have analysed the loop transfer recovery error for predictive state estimation. We have adopted an approach, following the multi-step prediction method of Luck and Ray (1990) to synthesize the control system for delay compensation. This approach minimizes the loop recovery error where the gain of the $p$-step observer is tuned to a prescribed value. An alternative approach is to incorporate the delay in the plant model and then synthesize the controller and observer gains that must accommodate the effects of loss of robustness margins due to the delay and uncertainties. This approach is also discussed in this paper and it has been shown that the above two approaches yield identical relative sensitivity errors. However, if the plant model is simply augmented to accommodate the induced delays, the state-space realization may not be minimal as pointed out by Kinnaert and Peng (1990).

The paper is organized in five sections including the introduction. Section 2 summarizes the pertinent properties of LTR for one-step prediction as reported in the existing literature. Main results including the structure and properties of the $p$-step delay compensator ($p \geq 1$) are generated in Section 3. A general description of the $p$-step compensator is first presented. Then, the loop transfer matrix of the compensator is derived along with the error of the sensitivity matrix relative to that of the target loop. Finally, it is shown

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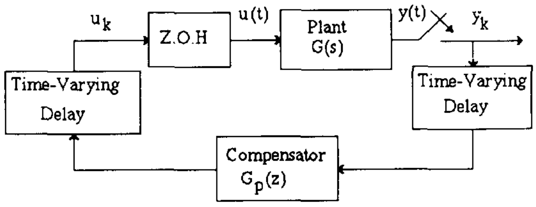


FIG. 1. Random distributed delays in the control system.

that the minimum variance filter gain minimizes the error of sensitivity matrices. A procedure to synthesize the observer and controller gains for the $p$-step compensator is presented in Section 4. The proposed procedure is tested by the simulation of the flight control system of an aircraft which is subjected to a lumped delay of two sampling periods. The paper is summarized and concluded in Section 5.

2. *Review of the LTR concept for one-step prediction*

The concept of loop transfer recovery (LTR) and the existing results for one-step prediction in the discrete-time setting are presented in this section. The plant under control is represented by a discretized version of a finite-dimensional, linear, time-invariant model in the continuous-time setting. The discretized model is assumed to be minimum-phase, stabilizable and detectable.

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k. \tag{1}$$

Following (1) the plant transfer matrix is given as:

$$G(z) = C\Phi(z)B, \tag{2}$$

where $\Phi(z) = (zI - A)^{-1}$ is the resolvement matrix. The target loop transfer matrix at the plant input is:

$$H(z) = F\Phi(z)B, \tag{3}$$

and the resulting target sensitivity matrix is:

$$S(z) = [I + H(z)]^{-1}. \tag{4}$$

The full-state feedback control law for the above plant is:

$$u_k = -Fx_k. \tag{5}$$

In this paper, we have assumed that the uncertainties are unstructured and lumped at the plant input in the form of an input multiplicative term:

$$\tilde{G}(z) = G(z)[I + \Delta(z)], \tag{6}$$

with given bound:

$$\bar{\sigma}[\Delta(e^{j\omega})] < l_m(\omega), \quad \forall\omega \geq 0. \tag{7}$$

Usually, unstructured uncertainties include high-frequency dynamics that are not modeled in the plant dynamics. The bound $l_m(\omega)$ finally restricts the system design specifications for stability robustness in terms of the closed loop complementary sensitivity matrix (Doyle and Stein, 1981). The task of control synthesis via the standard LQG/LTR approach is focused on shaping the loop sensitivity matrices for required performance and stability robustness, and can be carried out in two stages as discussed in Doyle and Stein (1979).

For the filter observer (i.e. $p = 0$) of a stabilizable, detectable and minimum phase plant, the loop transfer and sensitivity matrices have been shown by Maciejowski (1985) to converge pointwise in frequency to those of the target system as the measurement noise approaches zero. However, this may not be valid for the one-step predictor (Maciejowski, 1985; Zhang and Freudenberg, 1991; Yen and Horowitz, 1989). Breaking the loop at the plant input, as shown in Fig. 2, the one-step delay-compensator transfer matrix is:

$$G_1(z) = F(zI - A + BF + LC)^{-1}L. \tag{8a}$$

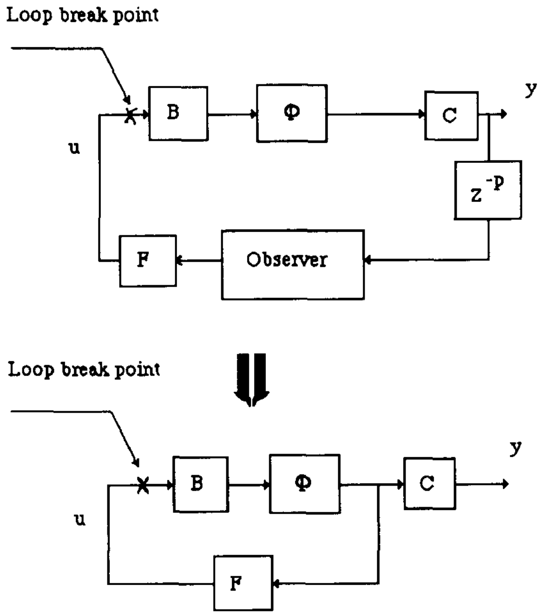


FIG. 2. Loop transfer recovery breaking the loop at plant input.

An alternative form is:

$$G_1(z) = F[I + \Phi(z)(BF + LC)]^{-1}\Phi(z)L. \tag{8b}$$

Then the loop transfer matrix for the one-step delay-compensated system is:

$$L_1(z) = G_1(z)G(z) = F[I + \Phi(z)(BF + LC)]^{-1}\Phi(z)LC\Phi(z)B, \tag{9a}$$

which can also be expressed, similar to the formula proposed by Zhang and Freudenberg (1991), as:

$$L_1(z) = [I + E_1(z)]^{-1}[H(z) - E_1(z)], \tag{9b}$$

where $E_1(z) = F[zI - A + LC]^{-1}B$ is the one-step error matrix at the plant input. The resulting one-step sensitivity matrix can be expressed as a function of the error transfer matrix:

$$\begin{aligned} S_1(z) &= [I + L_1(z)]^{-1} \\ &= [I + H(z)]^{-1}[I + E_1(z)]. \end{aligned} \tag{10}$$

It is clear from (6) and (9) that $E_1(z)$ is essentially the relative error of the sensitivity matrix, $S_1(z)$, of one-step delay compensator relative to the target sensitivity matrix, $S(z)$.

$$E_1(z) = S(z)^{-1}[S_1(z) - S(z)]. \tag{11}$$

It is known (Zhang and Freudenberg, 1991; Yen and Horowitz, 1989) that complete loop recovery, i.e. making $E_1(z) = 0$ for all $z$, cannot be achieved in general by a constant observer gain $L$. However, it is possible to identify an $L$ that minimizes the one-step error transfer matrix $E_1(z)$ in the $H_2$ sense.

3. *The p-step delay compensator*

Figure 1 illustrates a feedback control system where both sensor and control signals are delayed. (This might happen in a network-based integrated control system of advanced aircraft where sensor, controller and actuator are not collocated.) If the sum of the upper bounds of distributed delays are represented by a lumped delay of $p$ sampling intervals at the plant output, the sensory information available at the controller is $y_{k-p}$ at the $k$th instant. The $p$-step delay compensator (where the plant is completely controllable and observable), proposed by Luck and Ray

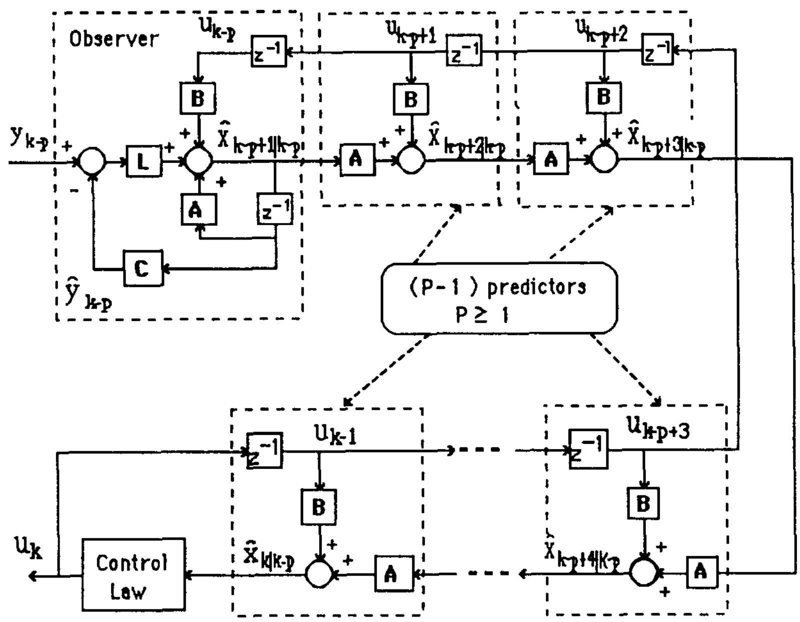


FIG. 3. Structure of $p$-step delay compensator.

(1990) and illustrated in Fig. 3, has the following structure:

$$u_k = -F\hat{x}_{k|k-p}, \tag{12}$$

where $F$ is the state feeback gain matrix and the state estimate is based on the sensory information up to the $(k-p)$th instant given as:

$$\begin{aligned}\hat{x}_{k|k-p} &= A\hat{x}_{k-1|k-p} + Bu_{k-1}\\ &\vdots\\ \hat{x}_{k-p+2|k-p} &= A\hat{x}_{k-p+1|k-p} + Bu_{k-p+1}\\ \hat{x}_{k-p+1|k-p} &= A\hat{x}_{k-p|k-p-1} + Bu_{k-p} + L(y_{k-p} - C\hat{x}_{k-p|k-p-1}).\end{aligned} \tag{13}$$

The key idea of applying the LTR approach to the above $p$-step delay compensator is to tune the loop transfer matrix such that the error transfer matrix (i.e. the different between the actual and target sensitivity matrices) is minimized in a certain sense. Derivation of the loop transfer function of the $p$-step delay compensator is presented below as two propositions.

*Proposition* 1. The transfer matrix of the $p$-step delay compensator ($p \geq 1$) from $y_k$ to $u_k$ in the equation set (13) is given as:

$$G_p(z) = F\Omega_p^{-1}(z)\frac{A^{p-1}}{z^{p-1}}\left[zI - A + LC + BF\Omega_p^{-1}(z)\frac{A^{p-1}}{z^{p-1}}\right]^{-1}L, \tag{14}$$

where

$$\Omega_p(z) = I + \left(I - \frac{A^{p-1}}{z^{p-1}}\right)\Phi(z)BF \quad \text{for } p > 1, \quad \text{and } \Omega_1(z) = I. \tag{15}$$

*Proof of Proposition* 1. The transfer matrix of the $p$-step ($p > 1$) compensator from $y_k$ to $u_k$ is obtained by substituting (13) into (12):

$$u_k = -FA^{p-1}\hat{x}_{k-p+1|k-p} - F\sum_{i=0}^{p-2} A^iBu_{k-i-1},$$

where the summation on the right hand side reduces to zero for $p = 1$. From the above equation, it follows that

$$u_k = -FA^{p-1}(A - LC)\hat{x}_{k-p|k-p-1} - FA^{p-1}Ly_{k-p} - F\sum_{i=0}^{p-1} A^iBu_{k-i-1}. \tag{16}$$

The $Z$-transform of (16) is:

$$U(z) = -FA^{p-1}(A - LC)\hat{X}(z)z^{-p} - FA^{p-1}LY(z)z^{-p} - F\sum_{i=0}^{p-1} A^iBU(z)z^{-i-1}, \tag{17}$$

and $Z$-transform of the last equation in (13) yields:

$$\hat{X}(z) = (zI - A + LC)^{-1}[BU(z) + LY(z)], \tag{18}$$

where $\hat{X}(z) = Z[\hat{x}_{k|k-1}]$, $Y(z) = Z[y_k]$, and $U(z) = Z[u_k]$.
Substituting (18) into (17) yields:

$$\begin{aligned}U(z) = &-FA^{p-1}(A - LC)(zI - A + LC)^{-1}[BU(z) + LY(z)]z^{-p}\\ &-FA^{p-1}LY(z)z^{-p},\\ &-F\sum_{i=0}^{p-2} A^iBU(z)z^{-i-1},\end{aligned}$$

which in turn can be simplified as:

$$\begin{aligned}U(z) = &-F\left[z^{p-1}I + \sum_{i=0}^{p-2} A^iBz^{p-i-2}F\right.\\ &\left.+A^{p-1}(zI - A + LC)^{-1}BF\right]^{-1},\\ &\times A^{p-1}(zI - A + LC)^{-1}LY(z).\end{aligned} \tag{19}$$

Using the relationship $\Phi(z) = (zI - A)^{-1}$ and equating like powers of $A$, equation (15) can be expressed as:

$$\Omega_p(z) = \begin{cases} I + \sum_{i=0}^{p-2} A^iBz^{-i-1}F & p > 1,\\ I & p = 1.\end{cases} \tag{20}$$

The proof is completed by substituting (20) into (19) and exercising a few algebraic operations.

An alternative proof of Proposition 1 can be formulated following (Ishihara, 1988) by expressing the transfer matrix of the $p$-step compensator from $y_k$ to $u_k$ as:

$$G_p(z) = -\left[\frac{FA^{p-1}}{z^{p-1}}(zI - A + LC)^{-1}B + I + \sum_{i=0}^{p-2}\frac{FA^iB}{z^{i+1}}\right]^{-1} \times \frac{FA^{p-1}}{z^{p-1}}(zI - A + LC)^{-1}L. \tag{21}$$

*Proposition* 2. Let the loop transfer matrix of the $p$-step delay compensated system at the plant input be expressed as:

$$L_p(z) = G_p(z)G(z),$$

where $G_p(z)$ and $G(z)$ are defined in (14) and (2), respectively. Then,

$$L_p(z) = [I + E_p(z)]^{-1}[H(z) - E_p(z)], \tag{22}$$

where

$$E_p(z) = F\Phi(z)B - F\frac{A^{p-1}}{z^{p-1}}[I + \Phi(z)LC]^{-1}\Phi(z)LC\Phi(z)B, \tag{23}$$

is the $p$-step error transfer matrix; and $H(z)$ is the target loop transfer matrix as defined in (3).

*Proof of Proposition* 2. By Proposition 1, the open loop transfer matrix of the $p$-step delay compensator is:

$$\begin{aligned} L_p(z) &= G_p(z)G(z), \\ &= F\Omega_p^{-1}(z)\frac{A^{p-1}}{z^{p-1}}\left[zI - A + LC + BF\Omega_p^{-1}(z)\frac{A^{p-1}}{z^{p-1}}\right]^{-1} \\ &\quad \times LC\Phi(z)B \\ &= F\Omega_p^{-1}(z)\left[I + \frac{A^{p-1}}{z^{p-1}}(I + \Phi(z)LC)^{-1}\Phi(z)\right. \\ &\quad \left. \times BF\Omega_p^{-1}(z)\right]^{-1}\frac{A^{p-1}}{z^{p-1}} \\ &\quad \times (I + \Phi(z)LC)^{-1}\Phi(z)LC\Phi(z)B \\ &= F\left[I + \left(I - \frac{A^{p-1}}{z^{p-1}}\right)\Phi(z)BF + \frac{A^{p-1}}{z^{p-1}}\right. \\ &\quad \left. \times (I + \Phi(z)LC)^{-1}\Phi(z)BF\right]^{-1}\frac{A^{p-1}}{z^{p-1}} \\ &\quad \times (I + \Phi(z)LC)^{-1}\Phi(z)LC\Phi(z)B \\ &= \left[I + F\Phi(z)B - F\frac{A^{p-1}}{z^{p-1}}\right. \\ &\quad \left. \times (I + \Phi(z)LC)^{-1}\Phi(z)LC\Phi(z)B\right]^{-1} \\ &\quad \times F\frac{A^{p-1}}{z^{p-1}}(I + \Phi(z)LC)^{-1}\Phi(z)LC\Phi(z)B. \end{aligned}$$

The proof is completed by substituting (24) and (3) in the above equation.

*Remark* 1. After some algebraic manipulations $E_p(z)$ can be written as:

$$E_p(z) = FT_p(z), \tag{24}$$

where

$$T_p(z) = \begin{cases} \frac{A^{p-1}}{z^{p-1}}(zI - A + LC)^{-1} + \sum_{i=0}^{p-2} A^iBz^{-i-1}, & p \geq 2 \\ (zI - A + LC)^{-1}B, & p = 1. \end{cases} \tag{25}$$

This shows that $E_p(z)$ can be separated in terms of the full-state feedback gain $F$ and a function of $L$ and $p$.

*Remark* 2. For a minimum-phase plant $(A, B, C)$ and with $\det(CB) \neq 0$, as the measurement noise convariance matrix $R$ approaches to zero and consequently $L \to AB(CB)^{-1}$, following Shaked (1985) the error matrix $E_p(z)$ in (23) in Proposition 2 simplifies to:

$$E_p(z) = F\left(I - \frac{A^p}{z^p}\right)\Phi(z)B. \tag{26}$$

For $p = 1$, the error matrix $E_p(x)$ in (26) is identically equal to $E_1(z)$ in (11).

*Remark* 3. It follows from the expression of $L_p(z)$ in Proposition 2 that the sensitivity matrix, $S_p(z)$, of the $p$-step delay compensator is:

$$S_p(z) = [I + L_p(z)]^{-1}, \tag{27}$$

and the difference between the sensitivity matrices of the $p$-step compensated and target systems is:

$$S_p(z) - S(z) = [I + H(z)]^{-1}E_p(z), \tag{28}$$

where $S(z)$ is given in (4). This shows that the error transfer matrix $E_p(z)$ is indeed the error of the sensitivity matrix of the $p$-step delay compensator loop relative to that of the target loop.

*Remark* 4. If the plant model has an inherent delay of $p_1$ steps and the induced delay in the feedback loop amounts to $p_2$ steps such that the total delay is $p = p_1 + p_2$, then the resulting error transfer matrix satisfies equation (26) as the measurement noise covariance is tuned to zero. This can be easily seen if the plant transfer matrix satisfies the following constraints:

$$CA^iB = 0, \quad i = 0, 1, \ldots, p_1 - 2 \text{ and } \det(CA^{p_1-1}B) \neq 0, \tag{29}$$

i.e.

$$\begin{aligned} G(z) &= C\Phi(z)B \\ &= C\sum_{i=0}^{\infty}\frac{A^i}{z^{i+1}}B, \\ &= C\frac{A^{p_1-1}}{z^{p_1-1}}\Phi(z)B. \end{aligned} \tag{30}$$

According to Shaked (1985), the observer gain $L$ approaches $A^{p_1}B(CA^{p_1-1}B)^{-1}$, as the measurement noise covariance approaches zero. Applying this observer gain to the loop transfer matrix:

$$L_p(z) = G_p(z)G(z),$$

and by using (14) in Proposition 1, the error matrix becomes identical to that in (26) with a total delay of $p = p_1 + p_2$.

This shows that any inherent delay in the plant has the same effect on the error matrix as the induced delay in the feedback loop. In other words, we can either consider the delays in the feedback loop outside the plant or as a part of the plant model. In the second case, the original plant state-space matrices $(A, B, C)$, with $\det(CB) \neq 0$, need to be augmented with $p_1$ steps of delay and the new plant state-space matrices $(A', B', C')$ need to be formed, which must satisfy the following conditions: (i) $C'(zI - A')^{-1}B' = C(zI - A)^{-1}Bz^{-p_1}$; and (ii) complete controllability and observability. Therefore, we have adopted the first approach of putting the lumped induced delay outside the plant model.

*Remark* 5. Dual results of Proposition 2, obtained by breaking the loop at the plant output instead of plant input, yield the loop transfer matrix

$$\begin{aligned} L_p(z) &= G(z)G_p(z) \\ &= [H(z) - E_p(z)][I + E_p(z)]^{-1}, \end{aligned}$$

where the target loop transfer matrix at the plant output (i.e. the transfer matrix of the minimum variance filter) is:

$$H(z) = C\Phi(z)L,$$

and

$$E_p(z) = C\Phi(z)L - C\Phi(z)BF\Phi(z)[I + BF\Phi(z)]^{-1}\frac{A^{p-1}}{z^{p-1}}L.$$

The resulting loop sensitivity matrix of the $p$-step delay compensator at the plant output is:

$$S_p(z) = [I + L_p(z)]^{-1} = [I + E_p(z)][I + H(z)]^{-1},$$

and the difference between the sensitivity matrices of the minimum-variance filter and $p$-step delay compensator is:

$$S_p(z) - S(z) = E_p(z)[I + H(z)]^{-1}.$$

*$H_2$-minimization of the* p*-step error matrix*

For the one-step predictor, it has been shown (Zhang and Freudenberg, 1991; Yen and Horowitz, 1989) that the steady-state minimum-variance filter gain with zero measurement noise is obtained by minimizing the $H_2$ norm of the one-step error matrix $E_1(z)$. Analogous to the case of $p = 1$, we will show that the same filter gain minimizes the $H_2$ norm of the $p$-error matrix $E_p(z)$. This result forms the basis for synthesis of robust $p$-step delay compensators ($p > 1$) and is presented below as two propositions. Then, the procedure for synthesizing the observer gain matrix is outlined.

For the purpose of tuning the minimum variance gain of the observer, we augment the discrete-time, linear, time-invariant plant model in (1) and (2) with (fictitious) plant and measurement noises as:

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{31}$$

$$y_k = Cx_k + v_k, \tag{32}$$

where $\{w_k\}$ is a zero-mean white sequence with covariance matrix: $E\{w_k w_j^T\} = BB^T\delta_{kj}$, and $\{v_k\}$ is a zero-mean white sequence with covariance matrix: $E\{v_k v_j^T\} = \rho I \delta_{kj}$, combining the distributed delays within the control loop as a lumped delay of $p$ sampling intervals at the sensor-controller interface, the state estimate is redefined as:

$$\hat{x}_{k|k-p} = E\{x_k \mid y_{k-p}\}, \tag{33}$$

where the estimator is described by the set of equations (13).

*Proposition* 3. Let the (zero-mean) state estimation error be defined as:

$$e_{k|k-p} = x_k - \hat{x}_{k|k-p}. \tag{34}$$

Then,

$$E\{e_{k|k-p}e_{k|k-p}^T\} = \begin{cases} \sum_{s=0}^{+\infty} A^{p-1}(A-LC)^s BB^T (A-LC)^{sT} A^{(p-1)T} + \sum_{s=0}^{p-2} A^s BB^T A^{sT}, & p \ge 2 \\ \sum_{s=0}^{+\infty} A^{p-1}(A-LC)^s BB^T (A-LC)^{sT} A^{(p-1)T}, & p = 1. \end{cases} \tag{35}$$

*Proof of Proposition* 3. From the plant model in (31) and (32), and the filter equations (13), we can express the estimation error $e_{k|k-p}$ in terms of the input sequence $\{w_s\}$, when the system is initially started at $s = -\infty$:

$$e_{k|k-p} = \sum_{s=-\infty}^{k-p} A^{p-1}(A-LC)^{k-p-s} w_s + \sum_{s=k-p+1}^{k-1} A^{k-s-1} w_s. \tag{36}$$

Hence the cross-covariance of $e_{k+s}$ and $w_k$, defined as $R_{ew}(s) = E\{e_{k+s|k+s-p} w_k^T\}$, can be expressed in the following form:

$$\begin{aligned} R_{ew}(s) &= \sum_{m=p-s}^{+\infty} A^{p-1}(A-LC)^{s-p-m} BB^t \delta(m) + \sum_{m=1-s}^{p-s-1} A^{s-1-m} BB^T \delta(m) \\ &= \begin{cases} A^{p-1}(A-LC)^{s-p} BB^T & (s \ge p) \\ A^{s-1} BB^T & (s = 1, \ldots, p-1) \\ 0 & (s = 0). \end{cases} \end{aligned} \tag{37}$$

Similarly, the autocovariance of $e_k$ is obtained as:

$$R_{ee}(s) = \begin{cases} \sum_{m=0}^{+\infty} R_{ew}(s+m+p)(A-LC)^{mT} A^{(p-1)T} + \sum_{s=0}^{p-2} R_{ew}(s+m+1) A^{sT}, & p \ge 2 \\ \sum_{m=0}^{+\infty} R_{ew}(s+m+p)(A-LC)^{mT} A^{(p-1)T}, & p = 1. \end{cases} \tag{38}$$

The proof is completed by setting $s = 0$, and then substituting (37) into (38).

*Proposition* 4. The $H_2$ norm of the sensitivity error matrix $E_p(z)$, defined in (23) in Proposition 2, is minimized if the observer gain matrix $L$ is identically equal to the standard steady-state minimum-variance filter gain matrix with no measurement noise.

*Proof of Proposition* 4. From the definition of $H_2$ norm (Francis, 1987) and $E_p(z)$ in (24) of Remark 1, it follows that:

$$\begin{aligned} \|E_p(z)\|_2^2 &= \frac{1}{2\pi}\,\mathrm{trace}\left(\int_0^{2\pi} FT_p(e^{j\Omega})T_p^*(e^{j\Omega})F^T\,d\Omega\right) \\ &= \frac{1}{2\pi}\,\mathrm{trace}\left(\int_0^{2\pi} F\left[\frac{A^{p-1}}{e^{j\Omega(p-1)}}(e^{j\Omega}I - A + LC)^{-1}B\right] \times \left[\frac{A^{p-1}}{e^{-j\Omega(p-1)}}(e^{-j\Omega}I - A + LC)^{-1}B\right]^T F^T\,d\Omega\right) \\ &+ \frac{1}{2\pi}\,\mathrm{trace}\left(\int_0^{2\pi} F\left[\frac{A^{p-1}}{e^{j\Omega(p-1)}}(e^{j\Omega}I - A + LC)^{-1}B\right] \times \left[\sum_{s=0}^{p-2}\frac{A^s}{e^{-j\Omega(s+1)}}B\right]^T F^T\,d\Omega\right) \\ &+ \frac{1}{2\pi}\,\mathrm{trace}\left(\int_0^{2\pi} F\left[\sum_{s=0}^{p-2}\frac{A^s}{e^{j\Omega(s+1)}}\right] \times \left[\frac{A^{p-1}}{e^{-j\Omega(p-1)}}(e^{-j\Omega}I - A + LC)^{-1}B\right]^T F^T\,d\Omega\right) \\ &+ M(p), \end{aligned} \tag{39}$$

where

$$M(p) = \begin{cases} \frac{1}{2\pi}\,\mathrm{trace}\left(\int_0^{2\pi} F\left[\sum_{s=0}^{p-2}\frac{A^s}{e^{j\Omega(s+1)}}B\right] \times \left[\sum_{s=0}^{p-2}\frac{A^s}{e^{-j\Omega(s+1)}}B\right]^T F^T\,d\Omega\right), & p \ge 2 \\ 0 & p = 1. \end{cases}$$

Since the sum of the second integral and the third integral is identically equal to zero and the integrals of the cross terms in the fourth term also vanish,

$$\|E_p(z)\|_2^2 = \frac{1}{2\pi}\,\mathrm{trace}\left\{FA^{p-1}\int_0^{2\pi}[(e^{j\Omega}I - A + LC)^{-1}BB^T \times (e^{-j\Omega}I - A + LC)^{-T}\,d\Omega]A^{(p-1)T}F^T\right\} + N(p), \tag{40}$$

where

$$N(p) = \begin{cases} \mathrm{trace}\left[F\left(\sum_{s=0}^{p-2} A^s BB^T A^{sT}\right)F^T\right], & p \ge 2 \\ 0 & p = 1. \end{cases}$$

For given plant model state-space matrices $A$ and $B$, if the feedback gain matrix $F$ is fixed, then the observer gain $L$ is the only adjustable matrix which could change the $H_2$ norm of the error matrix. On the other hand, the covariance of the state estimation error in Proposition 3 can be written, according to the discrete-time Plancherel theorem (Francis, 1987), as follows:

$$E\{e_{k|k-p}e_{k|k-p}^T\} = A^{p-1}\int_0^{2\pi}[(e^{j\Omega}I - A + LC)^{-1}BB^T \times (e^{-j\Omega}I - A + LC)^{-T}\,d\Omega]A^{(p-1)T} + S(p)$$

where

$$S(p) = \begin{cases} \sum_{s=0}^{p-2} A^s BB^T A^{sT}, & p \ge 2 \\ 0, & p = 1. \end{cases} \tag{41}$$

A comparison of the $H_2$ norm of the error matrix $E_p(z)$ in (40) with the trace of error covariance matrix $E\{e_{k|k-p}e_{k|k-p}^T\}$ in (41) reveals that any adjustment of $L$ can only change the first term of both equations. Therefore, minimization of $\|E_p(z)\|_2$ is equivalent to that of $\mathrm{trace}[E\{e_{k|k-p}e_{k|k-p}^T\}]$ for $\forall p > 0$.

Next we proceed to find an optimal $L$ that minimizes the trace $[E\{e_{k|k-p}e_{k|k-p}^T\}]$. It follows from Lemma 1, given below, that the minimum variance filter gain (with $p = 1$) also minimizes the $\mathrm{trace}[E\{e_{k|k-p}e_{k|k-p}^T\}]$ while $p > 1$. Therefore, the optimal observer gain $L$ that minimizes $\|E_p(z)\|_2$ is the same $L$ that minimizes $\|E_1(z)\|_2$. According to Lemma 2, the steady-state minimum variance gain with no measurement noise is the optimal gain.

*Lemma* 1 *for Proposition* 4. For the $p$-step predictor ($p \geq 1$), if the estimation error is defined as $e_{k|k-p} = x_k - \hat{x}_{k|k-p}$, then the filter gain $L$ which minimizes the covariance $E\{e_{k|k-p}e^T_{k|k-p}\}$ is identical to the minimum variance filter gain.

*Proof of Lemma* 1. The proof directly follows the derivations in Chapter 5 of Maybeck (1979).

*Lemma* 2 *for Proposition* 4. For a fixed $F$, the $H_2$ optimization of one-step predictor error matrix given as:

$$\min_L \|E_1(z)\|_2 = \min_L \|F(zI - A + LC)^{-1}B\|_2, \quad (42)$$

is the steady-state minimum variance filter gain where the plant and measurement noise covariance matrices, $Q$ and $R$, are set to:

$$Q = BB^T, \qquad R = \lim_{\rho \to 0} \rho I, \quad (43)$$

*Proof of Lemma* 2. The proof directly follows from the dual result of Theorem 3.1 in Yen and Horowitz (1989).

4. *Synthesis of the* p-*step delay compensator*

We propose a procedure for robust synthesis of the $p$-step delay compensator on the basis of the analytical results derived in Section 3. This two-stage procedure is structurally similar to the conventional LQR/LTR, and is described below.

- In the first stage, the target loop is designed assuming no delay (i.e. $p = 0$) and full-state feedback. The controller gain is optimized relative to a specified performance index. This is accomplished by shaping the target loop transfer matrix and the sensitivity matrix.
- In the second stage, according to Proposition 4, the observer gain $L$ is made identically equal to the steady-state minimum variance filter gain. This gain is calculated by solving the steady-state Riccati equation for a fictitious measurement noise convariance matrix $R$ which is set to $\rho I$ where $\rho$ is a tunable scalar parameter and $I$ is the identity matrix. The plant noise covariance matrix $Q$ is set equal to $BB^T$. For a given $\rho$, the loop transfer matrix and the sensitivity matrix of the $p$-step delay compensator, derived in Proposition 2 and Remark 4, are computed. Finally, $\rho$ is tuned to achieve the specified performance and robustness requirements. The similar procedure can be found in Maciejowski (1985) and Yen and Horowitz (1989).

The above synthesis procedure for $p$-step delay compensation has been verified by simulation of the flight control system of a fighter aircraft. The lumped delay of $p$ sampling intervals is a representation of the sensor-to-controller and controller-to-actuator delays (that could result from a data communication network interconnecting specially distributed components of the flight control system). The (continuous-time) plant model, linearized at the operating condition of 7.62 k and 0.9 Mach, is given below (Safonov *et al.*, 1981).

$$A = \begin{bmatrix} -0.0226 & -36.6170 & -18.8970 & -32.0900 & 3.2509 & -0.7626 \\ 0.0001 & -1.8997 & 0.9831 & -0.0007 & -0.1708 & -0.0050 \\ 0.0123 & 11.7200 & -2.6316 & -0.0009 & -31.6040 & 22.3960 \\ 0 & 0 & 1.000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30.0000 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 & 30 & 0 \end{bmatrix}^T, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

where the six plant state variables are forward speed, angle of attack, pitch rate, attitude angle, elevon actuator position, and canard actuator position; the two control inputs are elevon and canard signals; and the two output variables are angle of attack and attitude angle. The plant model was discretized at a sampling frequency of 1000 Hz which is sufficiently high relative to the desired operating frequency range of the closed loop system. On the basis of the discretized plant model, and a given set of performance and robustness specifications which can be found in Safonov *et al.* (1981), the control matrices for the target and delay-compensated systems were synthesized using a standard commerically available toolbox on a personal computer.

The full-state feedback regulator was designed using the standard LQR procedure with $Q_c = I_6$ and $R_c = 10^{-2}I_2$. The resulting optimal state feedback gain, $F$, is given below:

$$F = \begin{bmatrix} 7.3570 & -19.395 & -9.9957 & -15.975 & 8.8425 & -0.70813 \\ -4.2423 & 11.685 & 6.6904 & 10.395 & -0.70716 & 8.2537 \end{bmatrix}.$$

For tuning the filter gain in the LTR design procedure, the plant noise covariance was set as $Q = BB^T$, and the (fictitious) measurement noise covariance was set to $R = \rho I_2$ where $\rho$ was tuned in the range of $10^{-7}$–$10^{-12}$. Initially, a series of simulation experiments were conducted with no delays, i.e. $p = 0$, to verify that the system performance is degraded as $p$ is increased and the error recovery becomes impossible. The LTR procedure yielded good results for recovering the full-state feedback robustness properties at $p = 0$ as expected. The system performance improved as the measurement noise (i.e. $\rho$) was reduced, and the loop sensitivity matrix converged to the target sensitivity matrix as $\rho$ was made to approach zero.

In the simulation experiments, the scalar parameter $\rho$ was tuned to adjust the observer gain $L$ for both one-step and two-step compensators such that the stability robustness for each case (i.e. $p = 1$ and $p = 2$) bears a desired safe margin relative to the target system while the state feedback gain is retained at the optimal value $F$ for the target system. The robustness margin in the frequency range of 10–1000 Hz was set to 10 dB in terms of the maximum singular value of the loop transfer matrix to overcome the detrimental effects of loss of phase margin resulting from delays. The parameter $\rho$ in the design procedure of the delay compensator was adjusted to satisfy this requirement for robustness. The respective values of $\rho$ and the resulting filter gain $L$ that were used for one-step and two-step delay compensators are given in Table 1.

Figure 4 shows a comparison of the maximum and minimum singular values of the loop transfer matrices for the one-step delay compensated (i.e. $p = 1$) and two-step delay-compensated (i.e, $p = 2$) systems. (Note: The target loop with full state feedback and no delay merely serves as a reference for the synthesis procedure.) The minimum singular values of the loop transfer matrices of the individual systems represent the lower bounds of their respective performance, and their maximum singular values represent the upper bounds of stability robustness. As stated earlier, the observer gain matrices of both compensators are adjusted by tuning $\rho$ such that their maximum singular values are about 10 dB lower than that of the target system in the range of 10–1000 Hz. The minimum singular values for both compensators, as seen in Fig. 4, are significantly lower than that for the target system. This is expected in view of the reduction in observer gain resulting in decreased loop transfer gain. However, the two-step compensator consistently exhibits a lower performance (of about 12 dB) relative to the one-step compensator because of lower observer gain due to a higher value of the parameter $\rho$.

One major criterion in the synthesis procedure is to reduce the difference between the sensitivity matrices of the target loop and the delay compensated loop (in the $H_2$ sense) as

TABLE 1. OBSERVER GAINS AND ADJUSTING PARAMETERS OF THE $p$-STEP DELAY COMPENSATOR

One-step delay compensator $\rho = 1 \times 10^{-12}$

$$L = \begin{pmatrix} -0.42\times10^2 & 0.2\times10^1 & 0.33\times10^3 & 0.000 & -0.6252\times10^4 & -0.341\times10^3 \\ -0.41\times10^2 & 0.000 & 0.772\times10^3 & 0.1\times10^3 & 0.2703\times10^4 & 1.217\times10^4 \end{pmatrix}^T$$

Two-step delay compensator $\rho = 5 = 10^{-8}$

$$L = \begin{pmatrix} -6.6239 & 3.4811\times10^{-1} & 3.7994\times10 & 1.2893\times10^{-1} & -1.3923\times10^2 & 1.5987\times10 \\ -4.1893 & 1.1793\times10^{-1} & 3.0325\times10 & 2.2847\times10^{-1} & 1.9327\times10 & 1.2122\times10^2 \end{pmatrix}^T$$

much as possible. From this perspective, Fig. 5 shows comparisons of the maximum and minimum singular values of the sensitivity matrices for the target, one-step delay compensated (i.e. $p = 1$) and two-step delay-compensated (i.e. $p = 2$) systems. As shown in Table 1, the observer gains of the delay compensators are tuned to their respective values such that the specification for stability robustness is satisfied. This renders the minimum singular values of the sensitivity matrices of the two compensators remain close to each other and above that of the target system except in the high frequency region where all of them move towards 0 dB. Therefore, the sensitivity of the delay compensators reduces at high frequency, which is a very desirable feature from the point of view of stability robustness. On the other hand, the performance of the delay compensators (in the low-frequency range) degrades as $p$ is increased. It follows from Fig. 5 that $\bar{\sigma}(E_1(e^{j\omega})) = 5.5$ dB and $\bar{\sigma}(E_2(e^{j\omega})) = 31.8$ dB in the frequency range of $10^{-2}$–10 Hz.

## 5. *Conclusions*

In many real-time distributed control systems such as advanced aircraft, spacecraft and autonomous manufacturing plants, the sensor and control signals within a feedback loop are subjected to delays induced by multiplexed data communication networks or due to priority interruption at the control computer. From this perspective, a procedure has been developed for robust compensation of delays in multi-input–multi-output discrete-time control systems. This control synthesis procedure is an extension of the standard loop transfer recovery (LTR) from one-step prediction to the general case of $p$-step prediction ($p \geq 1$), and is carried out in two steps; (1) evaluation of the state feedback gain assuming full state feedback and no induced delays; and (2) tuning of the filter gain by varying a scalar parameter representing the (fictitious) measurement noise covariance matrix. The delay-compensated system, albeit inferior in performance relative to the non-delayed full state feedback system, can be synthesized for a given value of $p$. The synthesis procedure is demonstrated via simulation of the flight control system of a fighter aircraft.

The major conclusion derived from the analytical work reported in this paper is as follows. The concept of the steady-state minimum-variance filter gain as the $H_2$-minimization solution of the difference between the target sensitivity matrix and the actual sensitivity matrix for one-step prediction does not hold for $p$-step prediction ($p > 1$). The conclusions from the perspective of control synthesis are; (i) it is impossible to tune the observer gain for a delayed system (i.e. $p \geq 1$) to fully recover the target loop characteristics; and (ii) if the delay-compensated system is designed to satisfy a specified requirement of stability robustness, then its performance decreases as $p$ increases. The results and conclusions are also applicable if the plant model has inherent delays which have the same effects on the loop recovery error.

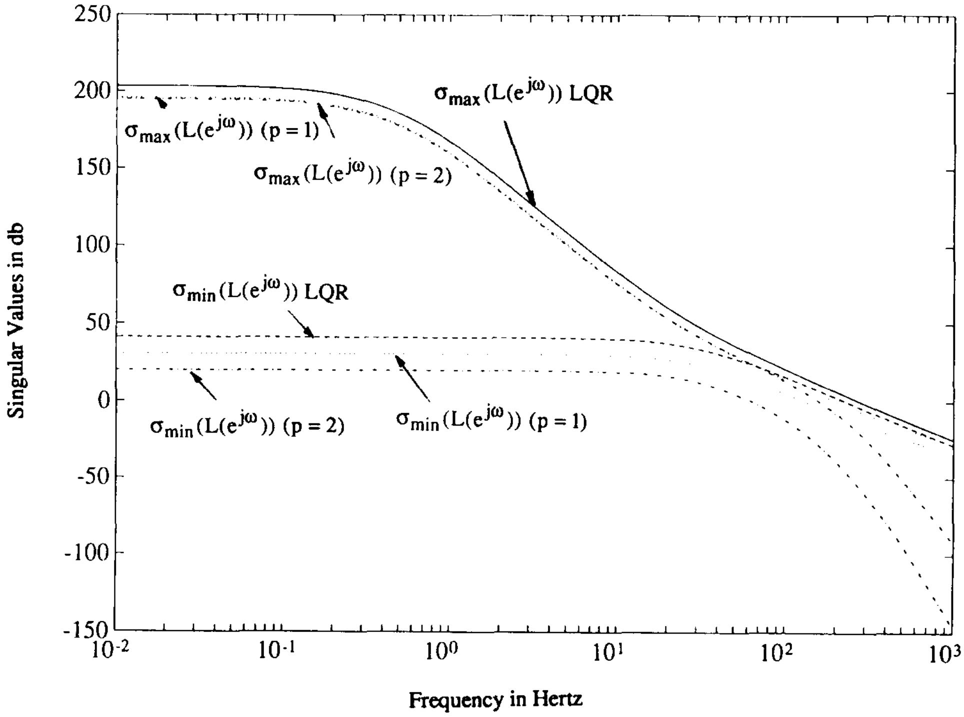


FIG. 4. Comparison loop transfer matrices.

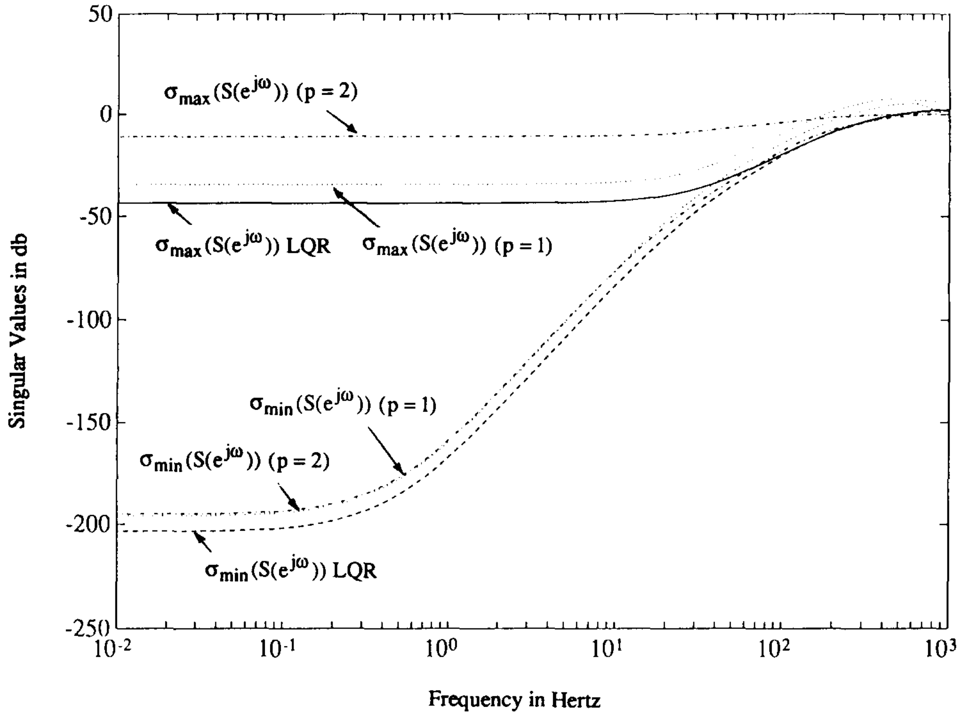


FIG. 5. Comparison of sensitivity matrices.

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