A. Ray

L. W. Liou

J. H. Shen

Mechanical Engineering Depratment,
The Pennsylvania State University,
University Park, PA 16802

State Estimation Using Randomly Delayed Measurements[1]

This paper presents a modification of the conventional minimum variance state estimator to accommodate the effects of randomly varying delays in arrival of sensor data at the controller terminal. In this approach, the currently available sensor data is used at each sampling instant to obtain the state estimate which, in turn, can be used to generate the control signal. Recursive relations for the filter dynamics have been derived, and the conditions for uniform asymptotic stability of the filter have been conjectured. Results of simulation experiments using a flight dynamic model of advanced aircraft are presented for performance evaluation of the state estimation filter.

1 Introduction

In diverse control and navigational applications, the state variables are estimated using the on-line sensor data and a model of plant dynamics [1, 2]. It is assumed that the sensor data albeit contaminated with noise, contain information about the current state of the plant. As such, for a linear system with additive white Gaussian noise, a recursive relationship can be formulated to obtain a minimum-variance estimate of the plant states on the basis of the measurement history. In many practical situations such as those in real-time distributed decision-making and control systems, however, the sensor data may be randomly delayed or interrupted so that the available measurements are not up-to-date. An example is the occurrence of random delays in the control law execution due to priority interruption at the controller computer [3]. Another example is the randomly varying delays induced by multiplexed data communication networks in distributed control systems [4–6].

State estimation and control in the presence of delays have been investigated by several researchers, and a brief discussion on this subject was presented by Luck and Ray [7]. While the majority of the reported work deals with constant delays, Nahi [8] and Sawaragi et al. [9] have directly addressed the issue of state estimation under randomly delayed measurements. These estimators are shown to perform better than the conventional minimum-variance estimator on the assumption that, at alternate samples, the sensor data is reduced to the zero-mean noise that is associated with the measurements. The above estimator [8, 9], however, is not realistic for application to control systems because, for a reasonably fast sampling rate, the previous measurements contain significant information about the plant states and therefore should be used in the absence of any new sensor data arrival at a given sample.

We first discuss two concepts for constructing a minimum-variance state estimator when the sensor data arrival at the controller is either timely or delayed by one sampling period. That is, the number of sensor data arrivals during a sampling period is 0 or 1 or 2 as illustrated in Fig. 1. In the first concept, the estimated state is obtained as the predictor output, $\eta_{k|k-1}$, when sensor data is delayed and as the filter output, $\eta_{k|k}$, when the sensor data is not delayed. The algorithm for computing the state error covariance matrix needs to be modified to accommodate the fact that the sensor data could be missing in some sampling periods and there may be two consecutive data in some other sampling periods. Switching between the filter output and predictor output to generate the state estimate could introduce chattering of the control command sequence resulting in performance degradation and possible instability of the closed loop control system. In an earlier publication [5], we have shown, by simulation of the flight control system of an advanced aircraft, that random switching between delayed and non-delayed data indeed causes performance degradation and potential instability. In the second concept, the state estimate is consistently obtained as the predictor output, $\eta_{k/k-1}$, regardless of whether the sensor data is delayed or not. In

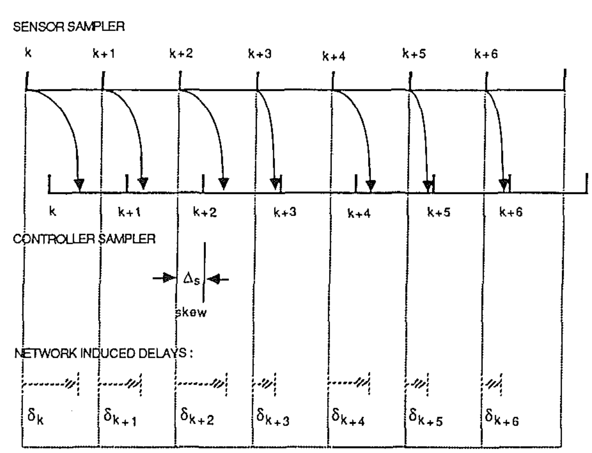

Fig. 1 Illustration of sensor to controller delay characteristics

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effect, this is similar to having the sensor data always delayed by one sample. If the probability of delayed arrival of sensor data is small and the covariance of plant noise is large relative to that of the measurement noise, then the cost of introducing a constant delay of one sampling period may prove to be excessive in view of the control system performance.

To circumvent the above problem we have proposed and developed, in this paper, a state estimator which is a modification of the conventional minimum variance filter and relies on the delay statistics of sensor data arrival at the controller. The proposed filter operates on a single algorithm, and therefore its implementation is less complex relative to one that would switch between filtering and prediction. In this approach, the currently available sensor data is used at each sampling instant to obtain the state estimate which, in turn, could generate the control signal. The usage of currently available data simplifies both hardware and software structure of the state estimator and thereby serves to enhance the control system reliability. In contrast to the predictor algorithm that always operates on the past measurements, the proposed filter is shown (via simulation) to yield smaller state estimation errors although it is more complex to implement.

The paper is organized in five sections including the introduction. Section 2 presents the problem statement along with a description of the delayed system and major assumptions. The filter equations are derived in Section 3. The results of simulation are provided in Section 4 for estimation of flight dynamic states of an advanced aircraft. The paper is summarized and concluded in Section. 5.

2 System Description and Problem Statement

The control system under consideration consists of a continuous-time plant (where some of the states are not directly measurable) and a discrete-time controller (which has an embedded state estimator). The plant is subjected to random disturbances and the sensor data is contaminated with noise. Furthermore, the sensor data set at time k is randomly delayed by an amount δ_k before it arrives at the controller as illustrated in Fig. 1. This delay may occur as a result of multiplexing in communication networks [4,12] or due to priority interruption in time-shared computers [3]. In practice, however, the probability that δ_k exceeds the sampling period T is made to be extremely small [12]; otherwise the control system performance is likely to be significantly degraded. Furthermore, we assume existence of a time skew, $\Delta \in [0, T)$, between the sensor and controller sampling instants because the sensor and controller may have different clocks. (Synchronization of these two clocks and the ramifications of having a non-zero Δ are addressed in our previous publication [5]). This would cause the controller to use either the current sensor data or the past data, whichever is available. That is, the sensor data generated at the $(k-\xi_k)^{\text{th}}$ sample, $\xi_k \in \{0, 1\}$, is used for obtaining an estimate η_k of the state x_k at the kth sample. The objective is to construct a linear state estimator based on the randomly delayed sensor data. In the formulation of the state estimation algorithm, we have set the deterministic part of the plant input to zero with no loss of generality. (Note: The effects of a nonzero deterministic input can be included in the estimator in a way similar to that in (nondelayed) standard minimum-variance filters.)

The continuous-time plant model is discretized relative to a sampling period T, and the resulting discrete-time plant dynamics with delayed sensor data are modeled as:

$$\begin{aligned} x_{k+1} &= F_{k+1,\,k}\, x_k + G_k w_k \\ y_k &= H_k x_k + v_k \\ z_k &= (1-\xi_k) y_k + \xi_k y_{k-1} \end{aligned} \tag{1}$$

where plant state $x \in \mathcal{R}^n$, plant disturbance $w \in \mathcal{R}^r$, plant output $y \in \mathcal{R}^m$, sensor noise $v \in \mathcal{R}^m$, and the delayed sensor data $z \in \mathcal{R}^m$ which is used to obtain the state estimate η. The above matrices are of compatible dimensions, and they are bounded, i.e. there exist positive real numbers γ_1, γ_2, and γ_3 such that, for every k, the following inequalities hold:

$$F_{k+1,\,k}{}^T F_{k+1,\,k} < \gamma_1 I_n;\; G_k{}^T G_k < \gamma_2 I_r;\text{ and } H_k H_k{}^T < \gamma_3 I_m \tag{2}$$

Remark 1. Since the plant model in (1) is a direct consequence of discretization of a continuous-time system, the state transition matrix $F_{k,\,k-1}$ is invertible for every k.

The pertinent assumptions and initial conditions for the stochastic plant model in (1) are stated below:

- Plant noise $\{w_k\}$ is Gaussian with $E[w_k]=0$ and $E[w_k w_l^T] = Q_k \delta_{kl}$ where $Q_k \geq 0 \forall k$;
- Sensor noise $\{v_k\}$ is Gaussian with $E[v_k]=0$ and $E[v_k v_l^T] = R_k \delta_{kl}$ where $R_k > 0 \forall k$;
- Measurement delays $\xi_k \in \{0, 1\}$ are white, i.e., $Pr[\xi_k \xi_l] = Pr[\xi_k] \Pr[\xi_l]$ for $\forall k \neq l$; $Pr[\xi_k = 1] = \alpha_k$, and $Pr[\xi_k = 0] = (1-\alpha_k)$;
- Random sequences $\{w_k\}$, $\{v_k\}$, and $\{\xi_k\}$ are also mutually independent;
- The initial state x_0 is Gaussian with mean μ_0 and covariance matrix $\mathbf{\Pi}_0$, and is statistically independent of other noise and disturbance.

(3)

Remark 2. It follows from (3) that the first and second moments of ξ_k can be derived as follows:

$$E\{\xi_k\} = \alpha_k;\; E\{(\xi_k)^2\} = \alpha_k;\; E\{(1-\xi_k)^2\} = 1-\alpha_k;\; E\{\xi_k(1-\xi_k)\} = 0. \;\blacksquare$$

The information on the measurement history, Z_j, up to the j^{th} instant is available to obtain an estimate of the state x_k for $k \geq j$. Accordingly, we denote the conditional expectation of x_k and the resulting error of estimation based on the measurement history Z_j as:

$$\text{Conditional state estimate: } \eta_{k|j} := E\{x_k | Z_j\} \text{ with } j \leq k \tag{4a}$$

$$\text{State estimation error: } e_{k/j} := (\eta_{k/j} - x_k) \text{ with } j \leq k \tag{4b}$$

Statement of the Problem. Given the linear discrete-time dynamical system in (1) under the assumptions and initial conditions (3), the problem is to find an optimal estimate, $\eta_{k|k}$, of the state x_k that will minimize the quadratic cost functional at each instant k:

$$J_k := E_\xi E_x\{e_{k|k}{}^T e_{k|k}\} \tag{5}$$

where $e_{k|k} := (\eta_{k|k} - x_k)$ is the filter error,
E_ξ indicates the expectation w.r.t. the statistics of (ξ_k), and
E_x indicates the expectation w.r.t. the statistics of $\{w_k\}$ and $\{v_k\}$.
The expectation $E_\xi E_x\{\bullet\}$ w.r.t the statistics of $\{\xi_k\}$, $\{w_k\}$ and $\{v_k\}$ is here after denoted as $E\{\bullet\}$.

3 Development of the State Estimation Filter

Using the concept of a standard (i.e., without measurement delays) state estimator, we propose a linear estimator for randomly delayed measurements, which will minimize the cost functional J_k in (5), to have the following recursive structure:

$$\eta_{k/k} = L_k \eta_{k/k-1} + K_k z_k \tag{6a}$$

$$\eta_{k/k-1} = F_{k,\,k-1} \eta_{k-1/k-1} \tag{6b}$$

where the gain matrices L_k and K_k of the above filter are derived in the sequel. We present the important properties of the state estimation filter as propositions.

Proposition 1. For the linear stochastic filter to be unbiased, i.e., $E\{e_{k/k}\} = E\{\eta_{k|k} - x_k\} = 0 \forall k$, the following relationship must be satisfied:

$$\eta_{k|k}=\eta_{k|k-1}+K_k[z_k-[(1-\alpha_k)H_k+\alpha_k H_{k-1}F_{k,\ k-1}{}^{-1}]\eta_{k|k-1}]$$

Proof of Proposition 1 We need to establish two lemmas for proving the proposition.

Lemma 1 for Proposition 1. If the filter is unbiased, i.e., $E\{e_{k|k}\}=0\ \forall k$, then the predictor is also unbiased, i.e.,

$$E\{e_{k|k-1}\}=0\ \forall k \text{ where } e_{k|k-1}:=(\eta_{k|k-1}-x_k).$$

Proof of Lemma 1. Using (1) in (6*b*) the prediction error $e_{k|k-1}$ can be expressed as:

$$\begin{aligned}e_{k|k-1}&=F_{k,\ k-1}\eta_{k-1|k-1}-(F_{k,\ k-1}x_{k-1}+G_{k-1}w_{k-1})\\&=F_{k,\ k-1}(\eta_{k-1|k-1}-x_{k-1})-G_{k-1}w_{k-1}\\&=F_{k,\ k-1}e_{k-1|k-1}-G_{k-1}w_{k-1}.\end{aligned}$$

Since the filter is given to be unbiased, i.e., $E\{e_{k-l|k-1}\}=0$ and $\{w_k\}$ is a zero mean sequence, the expectation of the right hand side in the above equation is zero. ■

Lemma 2 for Proposition 1. For an unbiased filter, i.e., $E\{e_{k|k}\}=0\ \forall k$, the gain matrix L_k in (6*a*) can be expressed in terms of K_k as:

$$L_k=I_n-K_k[(1-\alpha_k)H_k+\alpha_k H_{k-1}F_{k,\ k-1}{}^{-1}]$$

where I_n is the $(n\times n)$ identity matrix.

Proof of Lemma 2. Using Lemma 1, $E\{e_{k|k}\}=0$ implies $E\{e_{k|k-1}\}=0$. Substituting (6*a*) in the expression for $e_{k|k}$ yields

$$\begin{aligned}e_{k|k}&=\eta_{k|k}-x_k=L_k\ \eta_{k|k-1}+K_kz_k-x_k\\&=L_k(x_k+e_{k|k-1})+K_k[(1-\xi_k)y_k+\xi_k y_k]-x_k\end{aligned}$$

Substituting the relationships for x_k and y_k from (1) into the above equation, we obtain:

$$\begin{aligned}e_{k|k}=&[(1-\xi_k)K_kH_kF_{k,\ k-1}+L_kF_{k,\ k-1}-F_{k,\ k-1}\\&+\xi_kK_kH_{k-1}]x_{k-1}+[(1-\xi_k)K_kH_k+L_k-I_n]G_{k-1}w_{k-1}\\&+L_ke_{k|k-1}+(1-\xi_k)K_kv_k+\xi_kK_kv_{k-1}\end{aligned}$$

Since $\{w_k\}$ and $\{v_k\}$ are zero-mean sequences, taking expectation $E\{\bullet\}$ on both sides yields

$$\begin{aligned}E_\xi\{[(1-\xi_k)K_kH_kF_{k,\ k-1}+L_kF_{k,\ k-1}-F_{k,\ k-1}\\+\xi_kK_kH_{k-1}]\}E_x\{x_{k-1}\}=0\end{aligned}$$

to guarantee the zero mean of $e_{k|k}$. Since $E_x\{x_{k-1}\}\neq 0$ in general, its coefficient matrix must be zero. Noting that the plant state transition matrix $F_{k,\ k-1}$is invertible for $\forall k$ (see Remark 1) and substituting the first moment of ξ_k(see Remark 2) in the above equation, the result follows after some algebraic manipulations. ■

The proof of Proposition 1 is now completed by using Lemma 2 for L_k into (6*a*). ■

Next we proceed to derive the gain matrix K_k of the filter presented in Proposition 1. The objective is to synthesize a sequence of gain matrices, $\{K_j\}$, j = 1, 2, ..., that would minimize the cost functional J_k in (5) for each k. The optimal gain matrix is expressed in terms of the conditional error covariance matrices defined as:

$$\Sigma_{k|j}:=E\{e_{k|j}e_{k|j}{}^T|Z_j\} \text{ with } k\geq j \tag{7}$$

Proposition 2. For the filter structure in Proposition 1, the optimal gain matrix K_k, k = 1, 2, 3, ..., is obtained, by minimizing the cost functional J_k in (5), as:

$$\begin{aligned}K_k=&[(1-\alpha_k)\Sigma_{k|k-1}H_k{}^T+\alpha_kF_{k,\ k-1}\Sigma_{k-1|k-1}H_{k-1}{}^T]\\&\times\{[(1-\alpha_k)H_k+\alpha_kH_{k-1}F_{k,\ k-1}{}^{-1}]\Sigma_{k|k-1}[(1-\alpha_k)H_k\\&+\alpha_kH_{k-1}F_{k,\ k-1}{}^{-1}]^T+\alpha_k(1-\alpha_k)[H_k\\&-H_{k-1}F_{k,\ k-1}{}^{-1}]E(x_kx_k{}^T|Z_k)[H_k-H_{k-1}F_{k,\ k-1}{}^{-1}]^T\\&-\alpha_kH_{k-1}F_{k,\ k-l}{}^{-1}G_{k-1}Q_{k-1}{}^TG_{k-1}{}^TF_{k,\ k-1}{}^{-T}\\&\times H_{k-1}{}^T+\alpha_kR_{k-1}+(1-\alpha_k)R_k\}^{-1}\end{aligned}$$

where the state estimation error covariance matrices are recursively generated as:

$$\begin{aligned}\Sigma_{k|k-1}=&F_{k,\ k-1}\Sigma_{k-1|k-1}F_{k,\ k-1}{}^T+G_{k-1}Q_{k-1}G_{k-1}{}^T,\\ \Sigma_{k|k}=&L_k\Sigma_{k|k-1}L_k{}^T+\alpha_k(1-\alpha_k)K_k[H_k\\&-H_{k-1}F_{k,\ k-1}{}^{-1}]E\{x_kx_k{}^T|Z_k\}\times[H_k-H_{k-1}F_{k,\ k-1}{}^{-1}]^TK_k{}^T\\&-\alpha_kK_kH_{k-1}F_{k,\ k-1}{}^{-1}G_{k-1}Q_{k-1}G_{k-1}{}^TF_{k,\ k-1}{}^{-T}H_{k-1}{}^TK_k{}^T\\&+\alpha_k(G_{k-1}Q_{k-1}G_{k-1}{}^TF_{k,\ k-1}{}^{-T}H_{k-1}^TK_k{}^T\\&+K_kH_{k-1}F_{k,\ k-1}{}^{-1}G_{k-1}Q_{k-1}G_{k-1}{}^T)\\&+(1-\alpha_k)K_kR_kK_k{}^T+\alpha_kK_kR_{k-1}K_k{}^T,\end{aligned}$$

L_k is derived in Lemma 2 for Proposition 1, and the recursive relationship is started from the given initial conditions:

$$\eta_{0|0}:=\mu_0 \text{ and } \Sigma_{0|0}:=\mathbf{\Pi}_0$$

Proof of Proposition 2. Since the cost functional J_k is to be minimized at each instant k based on the measurement history Z_k, the problem is recast as:

$$\min_{K_k} J_k=\min_{K_k}\ \mathrm{Trace}(E_xE_\xi\{e_{k|k}e_{k|k}{}^T|Z_k\}) \tag{8}$$

such that the gain matrix K_k achieves the minimum for the estimator structure laid out in Proposition 1.

We express the filter error $e_{k|k}$ in terms of the prediction error $e_{k|k-1}$ and the measurement z_k by first using (6*a*) and then using (1) for the delayed measurements:

$$\begin{aligned}e_{k|k}=\eta_{k|k}-x_k&=L_k\eta_{k|k-1}+K_kz_k-x_k\\&=L_k\eta_{k|k-1}+K_k[(1-\xi_k)(H_kx_k+v_k)\\&+\xi_k(H_{k-1}x_{k-l}+v_{k-1})]-x_k\end{aligned} \tag{9}$$

Substituting the reverse relationship, $x_{k-1}=F_{k,\ k-1}{}^{-1}\times(x_k-G_{k-1}w_{k-1})$, of the plant model (1) and rearranging the terms in (9) yield:

$$\begin{aligned}e_{k|k}=&L_k(\eta_{k|k-1}x_k)+L_kx_k-[I_n-(1-\xi_k)K_kH_k\\&+\xi_kK_kH_{k-1}F_{k,\ k-1}{}^{-1}]x_k-\xi_kK_kH_{k-1},\ F_{k,\ k-1}{}^{-1}G_{k-1}w_{k-1}\\&+(1-\xi_k)K_kv_k+\xi_kK_kv_{k-1}\\=&L_ke_{k|k-1}+(L_k-\mathfrak{L}_k)x_k\\&-\xi_kK_kH_{k-1}F_{k,\ k-1}{}^{-1}G_{k-1}w_{k-1}+(1-\xi_k)K_kv_k+\xi_kK_kv_{k-1}\end{aligned} \tag{10}$$

where $\mathfrak{L}_k:=I_n-(1-\xi_k)K_kH_k+\xi_kK_kH_{k-1}F_{k,\ k-l}{}^{-1}$.

The following relationships are used for the subsequent derivations:

- Moments of measurement delay statistics given in Remark 2
- $E(w_kw_k{}^T|Z_k)=E_x\{w_kw_k{}^T\}=Q_k$; $E\{v_kv_k{}^T|Z_k\}=E_x\{v_kv_k{}^T\}=R_k$; all other cross terms involving $\{w_k\}$ and $\{v_k\}$ are zero because of their mutual independence and
- $E_\xi\{\mathfrak{L}_k|Z_k\}=L_k$ using the result of Lemma 2 for Proposition 1;

$$\begin{aligned}E\{e_{k|k-1}w_{k-1}{}^T|Z_k\}=&E_x\{(F_{k,\ k-1}\eta_{k-1|k-1}-F_{k,\ k-1}x_{k-1}\\&-G_{k-1}w_{k-1})w_{k-1}{}^T|Z_k\}=-G_k-1Q_{k-1}\\E\{x_kw_{k-1}{}^T|Z_k\}=&E_x\{(F_{k,\ k-1}x_{k-1}+G_{k-1}w_{k-1})w_{k-1}{}^TZ_k\}\\=&G_{k-1}Q_{k-1}\\E_\xi\{\xi_k(L_k-\mathfrak{L}_k)/Z_k\}=&\alpha_kL_k-E_\xi\{\xi_k\mathfrak{L}_k\}\\=&\alpha_k(1-\alpha_k)K_k(H_{k-1}F_{k,\ k-1}{}^{-1}-H_k)\end{aligned}$$

Using (10) and the above relationships, we have after some algebraic manipulations:

$$\Sigma_{k|k}=E\{e_{k|k}e_{k|k}^T|Z_k\}$$
$$=L_kE\{e_{k|k-1}e_{k|k-1}^T|Z_k\}L_k^T+\alpha_k(1-\alpha_k)K_k[H_k - H_{k-1}F_{k,\,k-1}^{-1}]E\{x_kx_k^T|Z_k\}[H_k-H_{k-1}F_{k,\,k-1}^{-1}]^TK_k^T - \alpha_kK_kH_{k-1}F_{k,\,k-1}^{-1}G_{k-1}Q_{k-1}G_{k-1}^TF_{k,\,k-1}^{-T}H_{k-1}^TK_k^T + a_k(G_{k-1}Q_{k-1}G_{k-1}^TF_{k,\,k-1}^{-T}H_{k-1}^TK_k^T + K_kH_{k-1}F_{k,\,k-1}^{-1}G_{k-1}Q_{k-1}G_{k-1}^T) + (1-\alpha_k)K_kR_kK_k^T+\alpha_kK_kR_{k-1}K_k^T \quad (11)$$

and the error covariance for prediction error is obtained as:

$$\Sigma_{k|k-1}=E\{e_{k|k-1}e_{k|k-1}^T|Z_k\}$$
$$=E\{(x_k-\eta_{k|k-1})(x_k-\eta_{k|k-1})^T|Z_k\}$$
$$=E\{(F_{k,\,k-1}x_{k-1}+G_{k-1}w_{k-1}-F_{k,\,k-1}\eta_{k-1|k-1}) \times (F_{k,\,k-1}x_{k-1}+G_{k-1}w_{k-1}-F_{k,\,k-1}\eta_{k-1|k-1})^T|Z_k\}$$
$$=E\{(F_{k,\,k-1}e_{k-1|k-1}+G_{k-1}w_{k-1})(F_{k,\,k-1}e_{k-1|k-1} + G_{k-1}w_{k-1})^T|Z_k\}$$
$$=F_{k,\,k-1}\Sigma_{k-1|k-1}F_{k,\,k-1}^T+G_{k-1}Q_{k-1}G_{k-1}^T \quad (12)$$

Now we can substitute the right hand side of (11) in the cost functional J_k in (8) to find the optimal K_k. To minimize J_k we set the partial derivative of J_k w.r.t. K_k to zero:

$$\partial J_k/\partial K_k=\partial \text{ Trace } (E\{e_{k|k}e_{k|k}^T|Z_k\})/\partial K_k=0 \quad (13)$$

By using the following facts about matrix manipulators:

$\partial[\text{trace}(A(B+B^T)A^T)]/\partial A=2A(B+B^T)$

$\partial[\text{trace}(AB)]/\partial A=B^T,$

the substituted terms of (11) in (13) can be expanded for evaluation of the partial derivatives. Collecting terms containing K_k yields

$$K_k\{[(1-\alpha_k)H_k+\alpha_kH_{k-1}F_{k,\,k-1}]\Sigma_{k|k-1}[(1-\alpha_k)H_k + \alpha_kH_{k-1}F_{k,\,k-l}]^T+\alpha_k(1-\alpha_k)[H_k - H_{k-1}F_{k,\,k-1}^{-1}]E\{x_kx_k^T|Z_k\}[H_k-H_{k-1}F_{k,\,k-1}^{-1}]^T - \alpha_kH_{k-1}F_{k,\,k-1}^{-1}G_{k-1}Q_{k-1}^TG_{k-1}^TF_{k,\,k-1}^{-T}H_{k-1}^T + \alpha_kR_{k-1}+(1-\alpha_k)R_k\}-(1-\alpha_k)\Sigma_{k|k-1}H_k^T-\alpha_k(\Sigma_{k|k-1} - G_{k-1}Q_{k-1}G_{k-1}^T)F_{k,\,k-1}^{-T}H_{k-1}^T=0 \quad (14)$$

The filter gain K_k is obtained by solving (14) and using (12) to substitute for $\Sigma_{k|k-1}$ in the last term on the left hand side of (14). ■

Corollary 1 to Proposition 2. If w_{k-1} is assumed to be independent of z_k, then the conditional covariance of x_k is given as:

$$E(x_kx_k^T|Z_k\}=\eta_{k|k-1}\eta_{k|k-1}^T+\Sigma_{k|k-1} \quad (15)$$

Proof of Corollary Proof follows directly by using the relationship (4*b*) for state estimation error if the conditional expectation $E\{w_{k-1}w_{k-1}^T|Z_k\}$ is approximated to be equal to the unconditional expectation $E\{w_{k-1}w_{k-1}^T\}$. ■

Remark 3 Suppose that a feedback control loop is formed by using the separation principle [2]. That is, the estimated state $\eta_{k|k}$ is used in place of the true state x_k for state feedback control. Then, if $\alpha_k\in(0,1)$, it follows from Proposition 2 that the filter gain K_k involves the conditional autocorrelation $E(x_kx_k^T|Z_k)$ of the plant state x_k. Therefore, the proposed state estimator under randomly delayed measurements may not comply with the principle of certainty equivalence [2] which assures optimal design by separately synthesizing the controller. In this case, the optimal state feedback control law could be different if an equivalent deterministic model is used instead of the stochastic model. Furthermore, the realization of $E\{x_k x_k^T|Z_k\}$ in (15) would yield a suboptimal solution of the optimal filter problem. However, if $\alpha_k = 0\ \forall k$ (i.e., the measurements are not delayed with probability 1), then the notion of certainty equivalence remains valid. ■

3.1 Stability of the State Estimation Filter Under Measurement Delays. The filter equations derived in Propositions 1 and 2 converge to those of a standard minimum variance filter as the measurement delays approach zero, i.e., $\alpha_k\to 0\ \forall k$. Therefore, the filter is uniformly asymptotically stable for $\alpha_k=0$ provided that the system (1) is stochastically controllable and observable [2]. This is true regardless of whether the plant dynamics in (1) are stable or not. However, if the plant is unstable (and is not stabilized by feedback control), then the convergence of the filter is not of practical significance. Therefore, we only consider the case in which the plant state vector is bounded in the mean square sense. That is, $\|E\{x_k x_k^T|Z_k\}\|$ remains bounded as $k\to\infty$. Then, for $\alpha_k>0$, the critical question is whether the state estimation error covariance, $\Sigma_{k|k}$, remains bounded in the mean square sense as $k\to\infty$. We attempt to address this issue in this section.

Since stabilizability of the system (1) is unaffected by the measurements delays, it suffices to determine, for stability of the filter, under what conditions the system is stochastically observable for $\alpha_k\neq 0$. This implies determination of the conditions under which uniform complete observability is affected by the measurement delays. The system (1) is uniformly completely observable [2] if there exist positive real constants α and β, and a positive integer l such that

$$0<\alpha I\le \mathrm{O}(k,\,k-l)\le\beta I \quad \forall k\ge l \quad (16)$$

where the observability matrix is defined as:

$O(k,\,k-l):=\sum_{j=k-l}^{k}[F_{j,\,k}^TH_j^TH_jF_{j,\,k}]$ and I is the identity matrix of appropriate dimension.

After suppressing the term involving w_k (that does not affect observability) for brevity, we augment the state space of the system (1) as follows:

$$X_{k+1}=\Phi_{k+1,\,k}X_k$$
$$z_k \quad =\Xi_kX_k+\nu_k \quad (17)$$

where the augmented state $X_k:=[x_k^Tx_{k-1}^T]^T$;

the augmented state transition matrix

$$\Phi_{k+1,\,k}:=\begin{bmatrix}F_{k+1,\,k} & 0\\ I & 0\end{bmatrix};$$

the modified measurement matrix $\Xi_k=[(1-\xi_k)H_k\ \xi_kH_{k-1}]$; and the modified measurement noise $\nu_k=(1-\xi_k)v_k+\xi_kv_{k-1}$ with the following properties:

$$E_x\{\nu_k\}=0 \text{ and } E_x\{\nu_k\nu_l^T\}=\begin{cases}0 & \text{for } |l-k|\ge 2\\ (1-\xi_k)\xi_{k+1}R_k & \text{for } |l-k|=1\\ [(1-\xi_k)^2R_k+\xi_k^2R_{k-1} & \text{for } l=k\end{cases}$$

The noise sequence $\{\nu_k\}$ is clearly non-white and also non-Markov for $\alpha_k \in (0, 1)$. However, the only possible source of non-whiteness of ν_k is the usage of the same sensor data at two consecutive samples, i.e., whenever the combination of $\xi_k = 0$ and $\xi_{k+1} = 1$ occurs. Since the augmented system (17) cannot be modified into the standard form of having white and mutually independent process and measurement noise, the notion of observability in definition (16) may not be applicable in to the augmented system (17), and a formal proof for uniform asymptotic stability of the filter with delayed measurements is not readily achievable. Therefore, the results of stability analysis of the proposed filter, based on the approximation that the noise sequence $\{\nu_k\}$ is white, are presented as conjectures instead of propositions.

Conjecture 1. If the system (1) is uniformly completely observable under no measurement delays, i.e., $\alpha_k = Pr[\xi_k = 1] = 0$, then it is also uniformly completely observable for $\alpha_k \in [0, 1-\epsilon]$ for any specified $\epsilon > 0$.

Justification for Conjecture 1. Following (16), the observability matrix of the augmented system (17) is defined in a loose sense as:

$$\mathbf{O}(k, k-l) := \sum_{j=k-l}^{k} E_\xi\{\Phi_{j,k}{}^T \Xi_j{}^T \Xi_j \Phi_{j,k}\} \quad (18)$$

where E_ξ indicates expectation relative to the measurement delay statistics. (Note: The above statement requires the modified measurement noise $\{\nu_k\}$ to be white.) Substituting the relationships (17) for the augmented system in the summand of (18), it follows after using the results in Remark 2 and some algebraic manipulations that

$$E_\xi\{\Phi_{j,k}{}^T \Xi_j{}^T \Xi_j \Phi_{j,k}\} = \begin{bmatrix} (1-\alpha_j) F_{j,k}{}^T H_j{}^T H_j F_{j,k} + \alpha_j F_{j-1,k}{}^T H_{j-1}{}^T H_{j-1} F_{j-1,k} & 0 \\ 0 & 0 \end{bmatrix} \quad (19)$$

Since our interest is in the observability of the first part (i.e., x_k) of the partitioned augmented state $X_k := [x_k{}^T x_{k-1}{}^T]^T$, we focus on the top left submatrix [13]

$$\mathbf{O}_{11}(k, k-l) := \sum_{j=k-l}^{k} (1-\alpha_j) F_{j,k}{}^T H_j{}^T H_j F_{j,k} + \sum_{j=k-l}^{k} \alpha_j F_{j-1,k}{}^T H_{j-1}{}^T H_{j-1} F_{j-1,k} \quad (20)$$

of the augmented observability matrix in (18). The bounds of $\mathbf{O}_{11}$ are established as follows.

Since α_j is bounded between 0 and $(1-\epsilon)$, it follows from the inequality (16) that the first part of $\mathbf{O}_{11}$ in (19) is bounded between $\epsilon\alpha I$ and βI. The second part of $\mathbf{O}_{11}$ can be expressed as:

$$\sum_{j=k-l}^{k} \alpha_j F_{j-1,k}{}^T H_{j-1}{}^T H_{j-1} F_{j-1,k} = \left[\sum_{j=k-l}^{k} (\alpha_j F_{j,k}{}^T H_j{}^T H_j F_{j,k})\right] + \alpha_{k-l} F_{k-l-1,k}{}^T H_{k-l-1}{}^T H_{k-l-1} F_{k-l-1,k} - \alpha_k H_k{}^T H_k \quad (21)$$

A lower bound of (21) is clearly 0 because all matrix products are positive semidefinite and $\alpha_j \geq 0 \ \forall j$. An upper bound of (21) is $(1-\epsilon)(\beta + \gamma_1\gamma_3)$ I following the inequalities in (16), (2) and (3). Therefore, by combining the bounds of two parts of (20), we obtain

$$0 < \alpha' I \leq \mathbf{O}_{11}(k, k-l) \leq \beta' I \quad \forall k \geq \ell$$

where l is a positive integer, and $\alpha'(\epsilon) := \epsilon\alpha$ and $\beta'(\epsilon) := (2-\epsilon)\beta + (1-\epsilon)\gamma_1\gamma_3$. This establishes uniform complete observability of the system with measurement delays in (1). ■

Extension to Conjecture 1. Asymptotic stability of the filter for a stationary, non-delayed system suggests asymptotically stability of the filter for the delayed system provided that the measurement delay statistics are stationary.

Justification for Extension to Conjecture 1. Stationarity of the delayed system in (1) implies that $F_{k+1,k} = F$, $H_k = H$, and $\alpha_k = \alpha \ \forall k$. Then, the condition for stochastic observability in (16) is reduced to positive definiteness of the observability matrix $O(l)$ for some positive integer l, i.e.,

$$O(l) := \sum_{j=0}^{l} [(F^{-j})^T H^T H F^{-j}] \quad (22)$$

and the submatrix $\mathbf{O}_{11}$ in (19) also reduces to

$$\mathbf{O}_{11}(l) := (1-\alpha)\left[\sum_{j=0}^{l} (F^{-j})^T H^T H F^{-j}\right] + \alpha\left[\sum_{j=0}^{l-1} (F^{-j-1})^T H^T H F^{-j-1}\right] = [(1-\alpha)O(l) + \alpha F^{-T} O(l) F^{-1}]$$

which is positive definite and bounded for $\alpha \in [0, 1]$ because $O(l)$ is positive definite and F-1 is bounded. ■

Remark 4. It appears, on the basis of extensive simulation runs, that the result of Conjecture 1 is valid under the relaxed condition of $\epsilon = 0$, i.e., $\alpha_k \in [0, 1] \ \forall k$, possibly except for certain contrived cases. Hence, we further conjecture that, for practical applications, the proposed filter is uniformly asymptotically stable provided that the minimum variance filter for no measurement delays is uniformly asymptotically stable. ■

Remark 5. The restriction of having a specified $\epsilon > 0$ as set forth in Conjecture 1 may not be necessary for stationary systems. ■

4 A Simulation Example

The proposed algorithm for state estimation under delayed measurements has been verified by simulation of the longitudinal motion dynamics of an advanced aircraft. The state-variable model of flight dynamics in continuous time is described below.

Plant Variables and Parameters

δ_a = Elevator command, i.e., deterministic input to the actuator (radian)
δ_e = Elevator deflection, i.e., actuator output (radian)
W = Normal component of linear velocity at the center of mass (m/s)
q = Pitch rate about the center of mass (radian/s)

The dimensional stability derivatives [14] for longitudinal motion dynamics were selected as:

$Z_{de} := (\partial Z/\partial\delta_e)/m = -61.655$ m/s^2;
$Z_q := (\partial Z/\partial q)/m = -5.132$ m/s;
$Z_w := (\partial Z/\partial W)/\mathrm{m} = -3.1332$ s^{-1},
$M_{de} := (\partial M/\partial\delta_e)/I_y = -40.465$ s^{-2};
$M_q := (\partial M/\partial q)/I_y = -2.6864$ s^{-1};
$M_w := (\partial M/\partial W)/I_y = -0.04688$ (s$-$m)$^{-1}$);
$m_{wd} := (\partial M/\partial W)/I_y = -0.00377$ m^{-1})

where

M is the pitch moment (m^2 kgm s^{-2});
Z is the normal component of the aerodynamic force (m kgm s^{-2});
m is the lumped mass of the aircraft (kgm); and
I_y is the moment of inertia about the pitching axis (m^2 kgm).

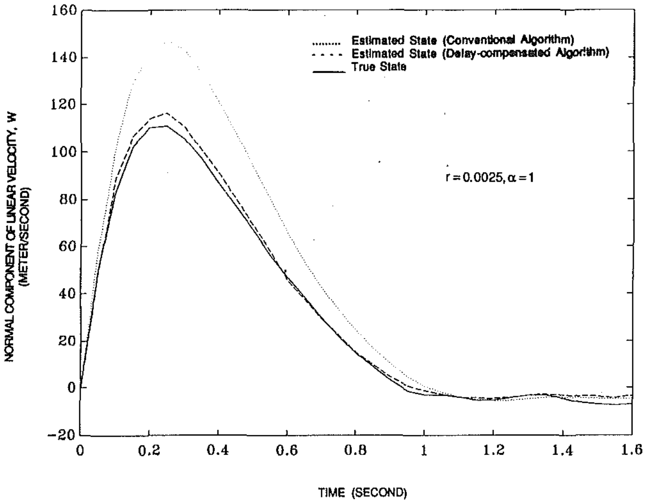

Fig. 2 Performance comparison of the state estimation algorithm [r= 0.0025, α = 1]

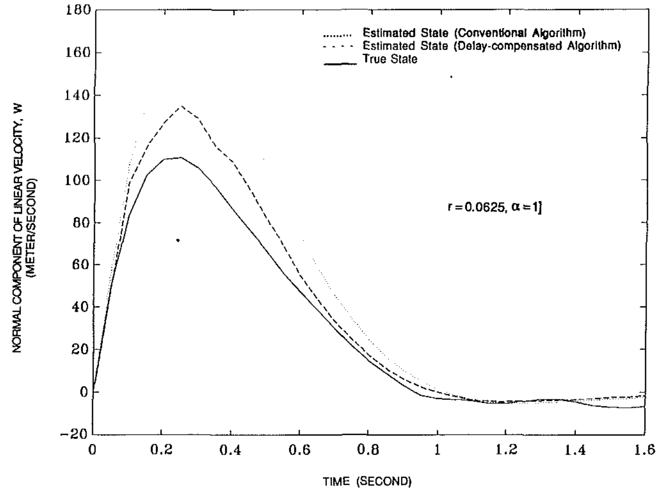

Fig. 4 Performance comparison of the state estimation algorithms [r = 0.0625, α = 1]

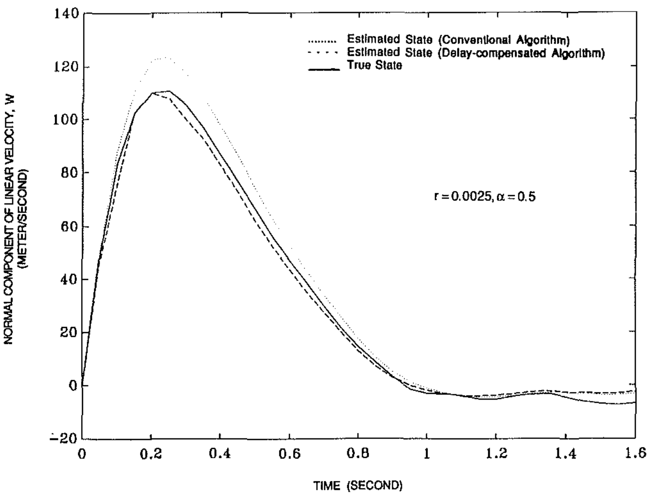

Fig. 3 Performance comparison of the state estimation algorithm [r= 0.0025, α = 0.5]

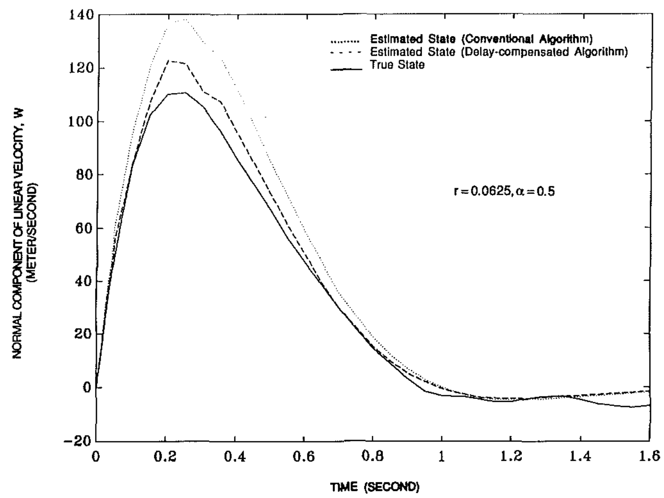

Fig. 5 Performance comparison of the state estimation algorithms [r = 0.0625, α = 0.5]

Other constant parameters were:

τ = Actuator time constant (0.1 s)

U_o = Reference flight speed (306.42 m/s)

Longitudinal Motion Dynamics in the Continuous Time Domain

$$d_x/dt = Fx + Bu + Gw;\ y = Hx + v$$

where $x = [\delta_e\ W\ q]^T$, w is the process noise which is assumed to be white with unit intensity; $u = \delta_a$; $y = q$; and v is the measurement which is assumed to be white.

$$F = \begin{bmatrix} -\tau^{-1} & 0 & 0 \\ Z_{de} & Z_w & S_0 \\ S_1 & S_2 & S_3 \end{bmatrix};\ B = \begin{bmatrix} \tau^{-1} \\ 0 \\ 0 \end{bmatrix};$$

$$G = \begin{bmatrix} 0.00625 \\ 0.25 \\ 0.025 \end{bmatrix};\ H = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T;$$

$$S_0 := (Z_q + U_o);\ S_1 := (M_{de} + M_{wd} Z_{de});$$

$$S_2 := (M_w + M_{wd} Z_w);\ S_3 := [M_q + M_{wd} S_0];\ \text{and}$$

the initial conditions are: $\eta_{0|0} = [0\ \ 0\ \ 1.0]^T$ and $\Sigma_{0|0} = \text{Diag}\ [0.04, 1.0, 0.5]$.

The (deterministic part of) control input, δ_a, was set to zero in the simulation because the specific objective of the simulation was to demonstrate the performance of the estimation aglorithm. The above flight dynamic model was first discretized with a sampling period $T = 0.05s$, and the sequence $\{\xi_k\}$ of measurement delays were taken to be stationary, i.e., $Pr[\xi_k = 1] = \alpha\ \forall k$.(This assumption is in agreement with the fact that the statistics of the network traffic are stationary over a finite window of time.) The following parameters were varied to obtain a series of simulation runs:

- Covariance, r, of the (discretized) measurement noise;
- Probability, α, of the measurement being delayed by one sample.

The performance of the proposed state estimation filter is compared with that of the predictor algorithm of the conventional minimum-variance filter in Figs. 2 to 5 for different values of r and α. The proposed delay-compensated filter uses the currently available sensor data, namely either y_k or y_{k-1}, at time k. That is, if y_k is unavailable at time k, then it uses the stored data y_{k-1} to generate the estimate. If, at the next sampling instant $k+1$, both y_k and y_{k+1} are available, the estimate is based on y_{k+1}. In this case, y_k is not used for state estimation to maintain the strategy of using the latest sensor data that can be easily fetched from the terminal's reciever buffer. In contrast, the predictor algorithm of the conventional filter always uses the delayed data y_{k-1} at the instant k regardless of whether y_k is available or not.

Fig. 2 shows a comparison of the true state, W, with its estimates using the conventional minimum-variance and the proposed algorithms for $r=0.0025$ and $\alpha=1$. (The implication of $\alpha=1$ is that the sensor data is consistently delayed by one sample, and the proposed fitler is identical to the predictor.) The initial condition of the plant state, $x_0 = [0\ 0\ 1.2]^T$, is deliberately made different from that of the plant state estimate, $\eta_{0|0}=[0\ 0\ 1.0]^T$, to demonstrate convergence of the algorithms. The proposed filter is consistently superior to the minimum variance algorithm and they eventually converge to the steady-state value of zero. The true state, however, hovers around this steady-state value due to presence of the zero-mean process noise in the plant model. Figure 3 shows curves under conditions similar to those of Fig. 2 except that $\alpha = 0.5$. Performance of the proposed filter (compare the dotted curves in Figs. 2 and 3) is better for $\alpha = 0.5$ than that for $\alpha = 1$ because, on the average, only a half of the sensor data sequence is delayed while the remaining data arrive at the controller in time. This shows that the filter performs better than the predictor (i.e., $\alpha = 1$) which always relies on the delayed sensor data regardless of whether or not the current sensor data is available.

In Figs. 4 and 5, the measurement noise covariance r is increased to 0.0625 to demonstrate its impact on the performance of both algorithms. An increased r reduces the filter gain implying that the state estimator relies more heavily on the plant model than on the measurement itself. Therefore, the proposed algorithm appears to be slightly less effective in Figs. 4 and 5 relative to their counterparts in Figs. 2 and 3, respectively. Similar curves comparing the performance of the two estimation algorithms for the two other states, namely δ_e and q, were also obtained. These results are not presented in this paper because they convey no significant additional information.

Next we discuss the results on convergence rates of the filter for different values of α. Extensive simulation experiments were conducted to determine convergence of the state estimation filter, and there was no evidence of divergence for any values of r and α. Figure 6 shows a comparison of the convergence of the norm, $\|\Sigma_{k|k}\|$, of the state error covariance matrix for $\alpha = 0, 0.25, 0.5, 0.75$ and 1.00 and the measurement noise covariance $r=0.0625$. While $\|\Sigma_{k|k}\|$ converges in all cases, the smallest value of $\|\Sigma_{k|k}\|$ occurs for $\alpha = 0$ (i.e., no measurement delays) and the largest for $\alpha = 0.5$. The rationale for the latter event is that $\alpha = 0.5$ creates maximum uncertainty regarding arrival of a specific sensor data at the controller. For $\alpha = 0.5$, the probability of using the same sensor data for state estimation during two consecutive samples is also maximized, thereby causing error due to the approximation of $\{\nu_k\}$ to be a white sequence. Although there is no uncertainty regarding arrival of sensor data for $\alpha=1$, the initial overshoot occurs due to a fixed delay of one sampling period. The results for $\alpha = 0.25$ and $\alpha = 0.75$ in Fig. 6 are comparable because of identical uncertainty in sensor data arrival. Figure 7 shows similar results for the measurement noise covariance $r = 0.0025$. In this case, a smaller r causes a larger gain matrix K_k resulting in more pronounced effects of α on the profile of $\|\Sigma_{k|k}\|$ but the convergence for different values of α takes place practically within the same time interval. The conclusions derived on the basis of results of these simulation experiments are summarized below.

- The norm, $\|\Sigma_{k|k}\|$, of the state error covariance matrix is smallest for $\alpha = 0$. This is in agreement with the fact that the filter is essentially optimal at $\alpha = 0$.
- As α is increased from zero, $\|\Sigma_{k|k}\|$ gradually increases for α up to 0.5 where the uncertainty of sensor data arrival is the largest and then gradually decreases until α approaches 1.
- For all values of $r>0$ and $\alpha\in[0, 1]$, the filter response remains bounded, and there is no evidence of divergence.

5 Summary and Conclusions

An optimal stochastic algorithm is presented for state estimation where the sensor data is randomly delayed such that the measurements available at the controller terminal are not up-to-date. This may happen in real-time processes, such as advanced aircraft, if a multiplexed data network is used for communications between the control system components. Another potential situation is the occurrence of random delays in the control law execution due to priority interruption at the controller computer. The random delay in the sensor data arrival at the controller terminal is assumed to be statistically independent of the plant and sensor noise. The delay is also assumed to have binary statistics, i.e., either the sensor data arrives at the controller in time or it is delayed by one sample. Nevertheless the estimation algorithm can be extended for delays more than one sampling period but such conditions are not of much practical significance. The algorithm is derived using a step-by-step procedure, and the main results are given

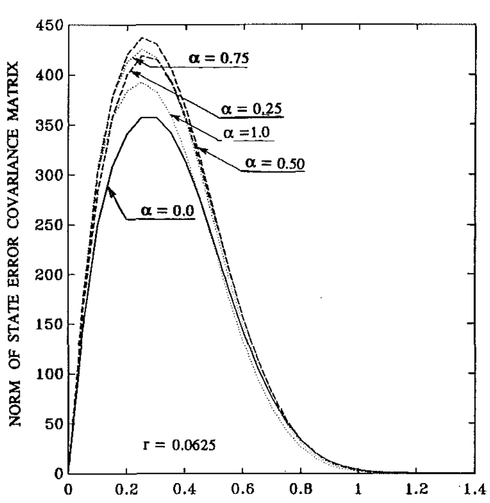

Fig. 6 Convergence of the proposed state estimation filter [$r = 0.0625$]

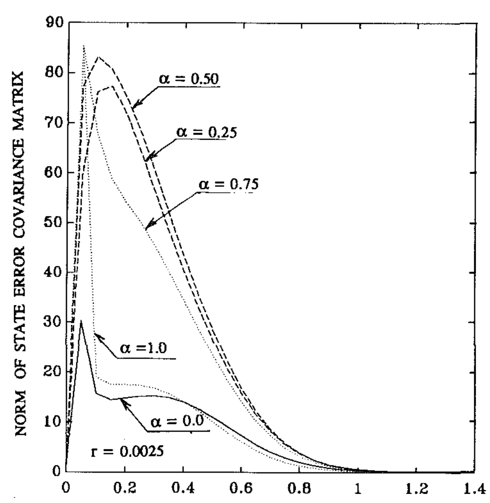

Fig. 7 Convergence of the proposed state estimation filter [$r = 0.0025$]

in the form of two propositions. Convergence of the state estimation filter is conjectured, and this claim is supported on the basis of extensive simulation experiments as there was no evidence of divergence of the filter. However, since criteria for convergence of the proposed filter have not yet been formally established, further research is necessary to identify the conditions under which the filter is uniformly asymptotically stable under measurement delays.

The proposed state estimation filter has been tested via simulation experiments using the flight dynamic model of an advanced aircraft. It appears from the simulation results that the filter, in the presence of randomly delayed measurements, performs better than each of the following two: (i) minimum-variance state estimation algorithm that switches between a predictor (when the data is delayed) and a filter (when the data is not delayed); and (ii) a predictor that always relies on the delayed data regardless of whether the current data is available or not. The proposed state estimation filter is directly applicable to synthesis of large-scale control systems which are subjected to delays due to communications between spatially dispersed components. This state estimation algorithm complements the stochastic control law, recently reported by Liou and Ray [10, 11], for compensation of randomly varying distributed delays.

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