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Output Feedback Control Under Randomly Varying Distributed Delays

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An output feedback control law has been formulated in a stochastic setting, based on the principles of minimum variance filtering and dynamic programming, for application to processes that are subjected to randomly varying distributed delays. The proposed estimation and control law for delay compensation is built on the concept of the conventional linear quadratic Gaussian (LQG), called delay compensated linear quadratic Gaussian (DCLQG). Although the certainty equivalence property of LQG does not hold for DCLQG in general, the combined state estimation and state feedback approach of DCLQG offers a suboptimal solution to the control problem under randomly varying distributed delays. DCLQG is potentially applicable to analysis and synthesis of control systems for vehicle management of future generation aircraft where a computer network is employed for distributed processing and on-line information exchange between diverse control and decision-making functions. Results of simulation experiments are presented to demonstrate efficacy of the proposed DCLQG algorithm for flight control of an advanced aircraft.

Introduction

THE vehicle management system in future generation aircraft would require highly integrated control and decision functions that will have direct flight criticality implications. For example, the integrated flight-propulsion controller must take into account the effects of a strong coupling between the propulsion and aerodynamics to take advantage of propulsive moments and forces for increased flight maneuverability and flight envelope.[1] These functions, combined with new strategies such as self-repairing and reconfigurable flight control systems, management of actuator failures and surface damage, control surface reconfiguration, and applications of artificial intelligence techniques to decision support systems would generate distributed computation and communications requirements. A data communication network is needed for information processing between the onboard spatially dispersed computers, intelligent terminals, sensors, and actuators to implement the stated requirements.

Asynchronous time-division-multiplexed networks, such as those using the Society of Automotive Engineers (SAE) token bus[2] and Fiber Distributed Data Interface (FDDI) token ring[3] protocols, are expected to meet the data rate, data latency, and reliability requirements for network-based control systems[4] in future generation aircraft. Although time-division-multiplexed networking has distinct advantages over conventional point-to-point connections in terms of reduced wiring, flexibility of operations, and evolutionary design, the randomly varying distributed delays induced by the network could degrade the control system performance and are a source of potential instability. The key issue is that filters and controllers designed for nonnetworked systems may not satisfy the performance and stability requirements in the delayed environment of network-based systems such as those in high performance aircraft. From this perspective, the established methods of state estimation and state feedback such as Linear Quadratic Gaussian (LQG)[5,6] need to be modified or reformulated.

A finite-dimensional discrete-time model of the control system with randomly varying distributed delays has been reported earlier[7,8] where the delays are represented as stochastic parameters in the matrices of the state-space model. The randomly delayed measurements render the task of control systems analysis and synthesis more complex than those involving nondelayed systems with stochastic parameters as reported by several investigators.[9–11] Along this line Liou and Ray[12,13] have proposed a linear stochastic regulator for compensation of randomly varying delays as an alternative to the deterministic approach of multistep prediction reported by Luck and Ray[14] and Shen and Ray.[15] This stochastic control law is formulated on the principles of dynamic programming and optimality[6] where the effects of randomly varying sensor-to-controller and controller-to-actuator delays, as explained in Refs. 7 and 8, are taken into account. Following the structure of linear quadratic regulator (LQR), the control law is formulated in the presence of randomly varying delays from the controller to actuator under full state feedback and no plant and measurement noise. Conditions for stochastic stability of the closed-loop system are established in the mean square sense.

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To augment the stochastic linear regulator for output feedback control, Ray et al.[16] have formulated a state estimation filter to account for random delays in the measurements with the stipulation that the control law can be obtained as a function of the estimated state vector. This filter is a modification of the conventional minimum variance filter[5] and is based on the statistics of the delay in sensor data arrival at the controller in addition to the statistics of plant and measurement noise. In this approach, the most up-to-date sensor data is used at each sampling instant to obtain the state estimate. We propose an output feedback control law where the full state feedback stochastic regulator is complemented with the minimum-variance state estimator under delayed measurements.

The objective of this paper is to present a unified estimation and control methodology for compensation of randomly varying distributed delays in the stochastic setting, hereafter called the delay compensated linear quadratic Gaussian (DCLQG) control. The proposed concept of DCLQG is essentially a combination of the optimal full state feedback stochastic regulator[12,13] and the minimum-variance state estimator[16] which compensate for the randomly varying delays distributed between the controller and actuator and the sensor and controller, respectively. The certainty equivalence property[6] of LQG does not hold, in general, for DCLQG because of the presence of multiplicative uncertainties in the system matrices.[9] Although DCLQG is a suboptimal output feedback compensator, it is potentially a tool for control systems analysis and synthesis in the presence of randomly varying distributed delays in general, and particularly for future generation aircraft that are equipped with computer networks to serve the vehicle management system. However, further work in the area of performance and stability robustness is needed to this effect.

Delay Compensated Linear Quadratic Gaussian Control

The control system under consideration consists of a continuous-time plant (where some of the states may not be directly measurable) and a discrete-time controller (having an embedded state estimator) which share a data communication network with other subscribers.[4] Therefore, the sensor and controller data are subjected to randomly varying delays induced by the network before they arrive at their respective destinations as shown schematically in Fig. 1. A finite-dimensional discrete-time model of the delayed control system has been reported in our previous publications[7,8] where the effects of random delays are realized in the form of multiplicative uncertainties. Furthermore, the plant is subjected to random disturbances and the sensor data is contaminated with noise. Based on this model structure, the proposed delay compensated linear quadratic Gaussian control system is partitioned into: 1) a stochastic regulator based on full state feedback and induced delays

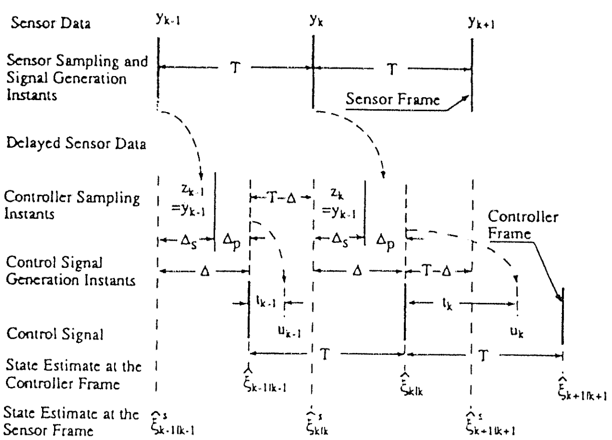

Fig. 1 Illustration of induced delay characteristics.

between the controller and the actuator but no plant and measurement noise; and 2) a minimum-variance state estimator based on the model of plant dynamics, plant disturbances, and noisy sensor data along with the statistics of induced delays between the sensor and the controller. The main results, including the pertinent assumptions and their implications, are presented in this section.

Assumptions

The pertinent assumptions for constructing the stochastic regulator and minimum-variance filter of DCLQG are delineated. The underlying justifications are laid out in the paragraphs following each of these assumptions.

1) The sensor and controller have the sampling period T with a skew Δ_S, between the sensor and controller sampling instants. Δ_S is a slowly varying parameter to be periodically reset and is treated as a constant parameter.

Since the clock rates of individual crystals are very close but they are never identical, the skew Δ_S varies slowly with time.

2) There is no delay in the process of sensor signal generation, i.e., the instant of sensor sampling is identically equal to the instant of sensor signal generation.

The actual delay in the signal generation process is on the order of microseconds whereas the sampling period T is on the order of tens of milliseconds.

3) The delay Δ_p in the processing of the control signal is constant. Therefore, the skew between the instants of sensor and control signal generation, which is equal to the sum $\Delta = \Delta_S + \Delta_p$, is also a constant. The skew Δ is bounded above by T, i.e., $\Delta \le T$ with probability 1.

The processing delay Δ_p can be made small by appropriate selection of the control computer, and Δ_s can be periodically reset to a prescribed value. Nevertheless this assumption is not critical and can be waived at the expense of additional computations.

4) Network-induced data latencies between the sensor and controller δ_k^{sc} and between the controller and actuator δ_k^{ca} are bounded between 0 and T with probability 1.

This follows the standard practice of network design[4] in which the maximum data latency is constrained not to exceed the sampling interval.

5) Network-induced data latencies δ_k^{sc} and δ_k^{ca} are mutually independent, white sequences with identical and a priori known statistics. The sequence $\{\zeta_k\}$ of measurement delays in units of the sampling period T to which the sensor data is subjected has the following statistics for every sampling instant k: $Pr[\zeta_k = 0] = \alpha_k$, i.e., $Pr[\delta_k^{sc} < \Delta_S] = \alpha_k$; and $Pr[\zeta_k = 1] = 1 - \alpha_k$, i.e., $Pr[\delta_k^{sc} \ge \Delta_S] = 1 - \alpha_k$.

Independence and whiteness of δ_k^{sc} and δ_k^{ca} represent the situation of a large number of subscribers on the network with random traffic. It is an approximation for an onboard network of an aircraft. If this assumption is not made, the task of control systems analysis become mathematically untractable because the σ-algebra associated with the stochastic process will grow indefinitely with time.

6) The sampler is ideal, and the digital-to-analog conversion is implemented via a zero-order hold (ZOH). The actuator operates as a continuous-time device, i.e., the control input acts on the plant immediately after its arrival at the actuator terminal.

7) The second assumption is justified in view of the fact that the actuator is equipped with a dedicated microprocessor which is sampled much faster (ten times or more, for example) than the controller computer.

8) The probability of data loss, due to noise in the communication medium and protocol malfunctions, is zero.

The data communication network in future generation aircraft is expected to be fiber-optic based in which the bit error is extremely small (10^{-12} or less, for example).

9) Plant noise $\{w_k\}$ is Gaussian with $E[w_k] = 0$ and $E[w_k w_\ell^T] = Q_k \delta_{k\ell}$ where $Q_k \ge 0\ \forall k$, sensor noise $\{v_k\}$ is Gaussian with $E[v_k] = 0$ and $E[v_k v_\ell^T] = R_k \delta_{k\ell}$ where $R_k > 0\ \forall k$, and random sequences $\{w_k\}$, $\{v_k\}$, and $\{\zeta_k\}$ are mutually indepen-

dent. The plant state at the initial time is Gaussian with known expected value and covariance matrix and is statistically independent of other noise and disturbance.

These assumptions are in line with the standard LQG problem and represent an approximation of the real situation for mathematical tractability of control systems analysis.

10) The discretized version of the stochastic plant model is both uniformly completely reachable and observable.

This assumption is in line with the standard LQG problem to ensure stochastic stability of the closed-loop control system in the mean-square sense.

Remark 1: Referring to the sensor frame in Fig. 1, assumptions 3 and 4 imply that the number of new sensor signals arriving at the controller terminal is exactly 1 and that the number of control signals arriving at the actuator terminal is 0 or 1 or 2 during a sampling period. Similarly, referring to the controller frame, the number of new sensor signals arriving at the controller terminal is 0 or 1 or 2, and the number of control signals arriving at the actuator terminal is exactly 1 during a sampling period. □

Delay Compensated Linear Quadratic Regulator

Setting the plant noise to zero, the plant dynamics for a full state feedback system are represented by a finite-dimensional, linear, continuous-time model in a deterministic setting

$$\frac{d\xi}{dt} = a(t)\xi(t) + b(t)u(t); \qquad \xi(0) = \xi_0 \tag{1}$$

where $\xi \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$. The problem is to formulate a state-feedback control law in the discrete-time setting for compensation of the randomly varying delays between the controller and the actuator. Because of the varying (but bounded) controller-actuator delay, the input $u(t)$ to the plant is piecewise constant during a sampling interval $[kT, (k+1)T)$ in the controller frame where the changes in $u(t)$ occur at the random instants $kT + t^k$ as shown in Fig. 1. On this basis, the continuous-time plant model in Eq. (1) is discretized in the controller frame to yield

$$\xi_{k+1} = \Phi((k+1)T, kT)\xi_k + \sum_{i=0}^{1} b_k^i u_{k-i} \tag{2}$$

where $\Phi[(k+1)T, kT]$ is the state transition matrix from the kth to $(k+1)$th sampling instant

$$b_k^0 \equiv \int_{kT+t^k}^{(k+1)T} \Phi[(k+1)T, \tau]\, d\tau$$

$$b_k^1 \equiv \int_{kT}^{(k+1)T} \Phi[(k+1)T, \tau]b(\tau)\, d\tau - b_k^0$$

Extending the modeling methodology of Halevi and Ray,[7] the discretized model (2) is augmented in the controller frame to take into account the effects of controller-to-actuator delays. The augmented plant model is

$$x_{k+1} = A_{k+1,k}x_k + B_k u_k \tag{3}$$

where $u_k \in \mathcal{R}^m$ represents $u(t)$ at discrete instants of time,

$$A_{k+1,k} \equiv \begin{bmatrix} \Phi((k+1)T, kT) & b_k^1 \\ 0 & 0 \end{bmatrix}; \qquad B_k \equiv \begin{bmatrix} b_k^0 \\ I_m \end{bmatrix} \tag{4}$$

and

$$x_k \equiv [\xi_k^T \quad u_{k-1}^T]^T \in \mathcal{R}^{n+m} \tag{5}$$

is the augmented state vector.

Remark 2: During the kth sampling period, u_{k-1} may affect the plant state ξ_{k+1} in addition to u_k. Since u_{k-1} is already generated by the controller, it is known at the kth sampling instant. □

Remark 3: The elements b_k^i in the matrices, $A_{k+1,k}$ and B_k, of the augmented state-space model (3) are stochastic processes because the time epochs $\{t^k\}$ that form the limits of the integration in Eq. (2) are random. Therefore, $A_{k+1,k} = A_{k+1,k}(\omega)$, $B_k = B_k(\omega)$, $x_k = x_k(\omega)$, and $u_k = u_k(\omega)$ where ω is a sample point of the random sample space Ω of the induced delays. Moreover, $A_{k+1,k}$ and B_k are not statistically independent in view of Eqs. (2) and (4). □

Similar to the standard linear quadratic regulator, an optimal control law for delay compensation was formulated by Liou and Ray[12,13] on the assumption of availability of the augmented state vector x_k. The recursive relationship for optimal state feedback control $\{u_k^*, k = N-1, N-2, \ldots\}$, was derived by minimizing the following performance cost functional over a finite-time horizon of N sampling intervals:

$$J_k(x_k, u_k) = \tfrac{1}{2}[x_k^T Q_k x_k + u_k^T R_k u_k] + E\{J_{k+1}^*(x_{k+1}|x_k)\} \tag{6}$$

if $k < N$, where $J_k^*(x_k) := J_k(x_k, u_k^*)$ and u_k^* is the optimal state feedback law at the kth sample, i.e., in the interval $[kT, (k+1)T)$. For $k = N$, the terminal state is reached, and there is no need for any control. Therefore,

$$J_N^*(x_N) \equiv J_N(x_N, u_N^*) = \tfrac{1}{2} x_N^T P_N x_N \tag{7}$$

where $P_N = S$ is given.

The $(m \times m)$ control penalty matrix R_k is symmetric positive definite. The $(n+m) \times (n+m)$ state penalty matrix Q_k and the final state penalty matrix S have the following structures:

$$Q_k \equiv \begin{bmatrix} \tilde{Q}_k & 0 \\ 0 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} \tilde{S} & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

where $\tilde{Q}_k \in \mathcal{R}^{n\times n}$ is the state penalty matrix which is constrained to be positive definite, and $\tilde{S} \in \mathcal{R}^{n\times n}$ is the final state penalty matrix for the discretized plant in Eq. (2). The optimal state feedback control law[12] is given via a recursive relationship in the following proposition.

Proposition 1: If the stochastic matrices $A_{k+1,k}$ and B_k are independent of $\{A_{j+1,j}\}$ and $\{B_j\}$ for $j = k-1, k-2, \ldots$, respectively, and the statistics of the network-induced delays are given, then the optimal control law at the kth stage is

$$u_k^*(x_k) = -F_k x_k \tag{9}$$

for $k < N$ and the resulting minimum performance cost is

$$J_k^*(x_k) = \tfrac{1}{2} x_k^T P_k x_k \tag{10}$$

where

$$F_k = [R_k + E\{B_k^T P_{k+1} B_k\}]^{-1} E\{B_k^T P_{k+1} A_{k+1,k}\} \tag{11}$$

$$P_k \equiv Q_k + E\{A_{k+1,k}P_{k+1}(A_{k+1,k} - B_k F_k)\} \tag{12}$$

with $P_N = S$; and each equation is evaluated backward starting from $N-1$.

Proof: The proof of proposition 1 is given in the Appendix. □

Remark 4: The control law in Eqs. (11) and (12) is computed off-line to numerically obtain a steady-state value of the gain matrix F on a finite-time horizon.[13] The expected values are numerically generated based on the a priori known probability distribution of the controller to actuator delay δ_k^{ca}. □

Joshi[11] formulated a recursive algorithm, similar to that in Proposition 1, as a solution to the optimal regulator problem in the presence of multiplicative uncertainty. However, convergence of the recursive algorithm and stability of the closed-loop control system were not established. For the stochastic plant model similar to the model in Eq. (3) and the state feedback control law structurally identical to that in Proposition 1, Yaz[10] established exponential stability of the resulting closed-loop control system in both mean square sense and almost sure sense. The state feedback control algorithm was generated in the settings of both infinite-time horizon and finite-time horizon. The sufficiency conditions for stability were shown to be $Q_k > 0$ and the plant model being mean square stabilizable. For the specific structure of the system matrices $A_{k+1,k}$ and B_k in Eq. (4), the first condition of $Q_k > 0$ can be relaxed to that specified in Eq. (8). Therefore, the recursive relationship for the control law given in Eqs. (9–12) yields exponential stability of the closed-loop system in the mean square sense and almost sure sense provided that the pair $(A_{k+1,k}, B_k)$ is mean square stabilizable and that the number N of iterations is sufficiently large. Now we establish convergence of the recursive algorithm for the special case where the plant is time invariant.

Proposition 2: Let the cost matrices Q_k and R_k be constant, and the plant be time invariant, i.e., the matrices $a(t)$ and $b(t)$ in Eq. (1) are constant. Then the matrix P_0 in Proposition 1 converges to the limit matrix P as $N \rightarrow \infty$ if the augmented plant model $x_{k+1} = A_{k+1,k}x_k + B_k u_k$ in Eq. (3) is mean square stabilizable.

Proof: The proof of Proposition 2 is given in the Appendix. □

State Estimator for Delayed Measurements with Plant and Sensor Noise

The control law derived in Eq. (9) is a function of the augmented state vector x_k defined in Eq. (5). The first n elements of x_k are the plant states ξ_k, and the remaining m elements of x_k; namely, u_{k-1}, are already computed and stored at the controller buffer. If all plant states are not measurable or if the sensor noise level is unacceptable, then a filter is necessary to provide an estimate, $\hat{\xi}_{k|k}$, of ξ_k using the measurement history; and this estimate may replace the actual plant state part of the augmented state vector in the formulation of the optimal control u_k^*. The design of such an on-line state estimator, which must account for the delayed sensor arrival at the controller and also delayed control inputs to the plant, is addressed in this section.

The continuous-time plant model in Eq. (1) is now supplemented with the unit intensity white noise vector $w(t)$ and the noise input gain matrix $g(t)$. The resulting discrete-time plant dynamics at the instants of controller sampling are obtained by including the noise term in Eq. (2) as

$$\xi_{k+1} = \Phi[(k+1)T, kT]\xi_k + w_k^c + \sum_{i=0}^{1} b_k^i u_{k-i} \quad (13)$$

where

$$w_k^c \equiv \int_{kT}^{(k+1)T} \Phi[(k+1)T, \tau]g(\tau)w(\tau)\, d\tau \quad (14)$$

However, for plant state estimation based on the sensor data, we first set the plant model in the frame of sensor sampling instants (see Fig. 1), and then predict the state estimate at the controller sampling instants. Considering the constant skew Δ between the instants of sensor and control signal generation (see assumption 3 at the beginning of this section), the discretized model in the sensor frame is given as

$$\xi_{k+1}^s = \Phi[(k+1)T - \Delta, kT]\xi_k + w_k^1 + \sum_{i=0}^{1} \varphi_k^i u_{k-i} \quad (15)$$

$$\xi_{k+1} = \Phi((k+1)T, (k+1)T - \Delta)\xi_{k+1}^s + w_k^0 + \sum_{i=0}^{1} \psi_k^i u_{k-i} \quad (16)$$

where ξ_k^s is the plant state at the instant of kth sensor signal generation, and ξ_k is the plant state at the instant of kth control signal generation following Fig. 1,

$$w_k^0 \equiv \int_{(k+1)T-\Delta}^{(k+1)T} \Phi[(k+1)T, \tau]g(\tau)w(\tau)\, d\tau$$

$$w_k^1 \equiv \int_{kT}^{(k+1)T-\Delta} \Phi[(k+1)T - \Delta, \tau]g(\tau)w(\tau)\, d\tau \quad (17)$$

$$\varphi_k^0 \equiv \begin{cases} \int_{kT+t^k}^{(k+1)T-\Delta} \Phi[(k+1)T - \Delta, \tau]b(\tau)\, d\tau & \text{if } t^k < (T-\Delta) \\ 0 & \text{if } t^k \geq (T-\Delta) \end{cases} \quad (18)$$

$$\varphi_k^1 \equiv \int_{kT}^{(k+1)T-\Delta} \Phi[(k+1)T - \Delta, \tau]b(\tau)\, d\tau - \varphi_k^0$$

$$\psi_k^0 \equiv \begin{cases} \int_{(k+1)T-\Delta}^{kT+t^k} \Phi((k+1)T - \Delta, \tau)b(\tau)\, d\tau & \text{if } t^k \geq (T-\Delta) \\ 0 & \text{if } t^k < (T-\Delta) \end{cases} \quad (19)$$

$$\psi_k^1 \equiv \int_{(k+1)T-\Delta}^{(k+1)T} \Phi((k+1)T, \tau)b(\tau)\, d\tau - \psi_k^0$$

Then, the plant state and measurement equations in the sensor frame (see Fig. 1 and refer to Remark 1) are given as

$$\xi_{k+1}^s = \Phi_{k+1,k}\xi_k^s + w_k + \sum_{i=0}^{2} \beta_k^i u_{k-i} \quad (20)$$

$$y_k = H_k \xi_k^s + v_k \quad (21)$$

$$z_k = (1 - \zeta_k)y_k + \zeta_k y_{k-1} \quad (22)$$

where $\Phi_{k+1,k} \equiv \Phi[(k+1)T - \Delta, kT - \Delta]$ is the plant state transition matrix from the kth to the $(k+1)$th sampling instant in the sensor frame

$$w_k \equiv \int_{kT-\Delta}^{(k+1)T-\Delta} \Phi[(k+1)T - \Delta, \tau]g(\tau)w(\tau)\, d\tau \quad (23)$$

$$\beta_k^0 = \varphi_k^0; \qquad \beta_k^1 = \varphi_k^1 + \psi_{k-1}^1; \qquad \beta_k^2 = \psi_{k-1}^0 \quad (24)$$

Remark 5: It follows from Fig. 1 that the delayed sensor data z_k in Eq. (22) is either y_k if the data arrives at the controller before the kth sampling instant or y_{k-1} if the arrival takes place after the kth sampling instant. □

Remark 6: It follows from Eq. (24) that

$$\sum_{i=0}^{2} \beta_k^i = \int_{kT}^{(k+1)T-\Delta} \Phi((k+1)T - \Delta, \tau)b(\tau)\, d\tau + \int_{kT-\Delta}^{kT} \Phi(kT, \tau)b(\tau)\, d\tau$$

Remark 7: Since the plant model (20) is a direct discretization of the continuous-time model (1), the state transition matrix, $\Phi_{k+1,k}$, is invertible. □

The information on the measurement history, $Z_j = \{z_0, z_1, \ldots, z_j\}$, up to the jth instant is available to obtain an estimate of the state x_k for $k \geq j$. Accordingly, we denote the conditional expectation of ξ_k and the resulting error of estimation based on the measurement history Z_j as follows.

Conditional state estimate:

$$\hat{\xi}_{k|j} = E\{\xi_k | Z_j\} \qquad \text{for } j \leq k \tag{25}$$

State estimation error:

$$e_{k|j} \equiv (\hat{\xi}_{k|j} - \xi_k) \qquad \text{for } j \leq k \tag{26}$$

Conditional error covariance:

$$\Sigma_{k|j} \equiv E\{e_{k|j} e_{k|j}^T | Z_j\} \qquad \text{with } k \geq j \tag{27}$$

where $E\{\cdot\}$ indicates the expectation with respect to the statistics of one or more of the three independent sequences of white noise $\{w_k\}$, $\{v_k\}$, and $\{\zeta_k\}$. (See assumption 9 at the beginning of this section.) It follows from assumption 5 that the first and second moments of ζ_k can be derived as follows:

$$E\{\zeta_k\} = 1 - \alpha_k; \qquad E\{(\zeta_k)^2\} = 1 - \alpha_k$$

$$E\{(1 - \zeta_k)^2\} = \alpha_k; \qquad E\{\zeta_k(1 - \zeta_k)\} = 0$$

Remark 8: In the formulation of the state estimation filter reported by Ray et al.,[16] u_k was set to zero with no loss of generality but, in this paper, u_k is included in the state equation as needed for closed-loop control. Since the task of state estimation is executed on-line, the past control signals and their arrival instants $\{t^k\}$ at the actuator terminal are recorded at the controller terminal. Consequently, the matrices $\{b_k^i\}$, $\{\beta_k^i\}$, and $\{\psi_k^i\}$ are known, and the term $\Sigma_{i=0}^2 \beta_k^i u_{k-i}$ in Eq. (20) is deterministic so that it can be linearly combined with the zero-input state estimate to obtain the actual state estimate $\hat{\xi}_{k|k}$ and the conditional covariance $\Sigma_{k|j}$ in Eq. (27) is unchanged. □

The problem of state estimation under randomly delayed measurements is stated as follows: Given the linear discrete-time dynamic system in Eqs. (20–22) under the assumptions and initial conditions stated earlier, the problem is to find an optimal estimate, $\hat{\xi}_{k|k}$, of the state ξ_k that minimizes the quadratic cost functional at each sensor sampling instant k

$$J_k \equiv E\{e_{k|k}^T e_{k|k}\} = \mathrm{tr}[E\{e_{k|k} e_{k|k}^T\}] \tag{28}$$

where $e_{k|k} = (\hat{\xi}_{k|k} - \xi_k)$ is the filter error. The objective is to synthesize a sequence of filter gain matrices, $\{K_k\}$, $k = 1, 2, \ldots$, that would minimize the cost functional J_k in Eq. (28) for each k. Using the concept of a standard (i.e., without measurement delays) state estimator, we propose a linear estimator for randomly delayed measurements.

Proposition 3: The unbiased linear filter that minimizes the cost functional J_k in Eq. (28) has the following recursive structure:

$$\hat{\xi}_{k|k} = \{I_n - K_k[\alpha_k H_k + (1 - \alpha_k)H_{k-1}\Phi_{k,k-1}^{-1}]\} \times \hat{\xi}_{k|k-1} + K_k z_k \tag{29}$$

$$\hat{\xi}_{k|k-1} = \Phi_{k,k-1}\hat{\xi}_{k-1|k-1} + \sum_{i=0}^{2} \beta_{k-1}^i u_{k-i-1} \tag{30}$$

where I_n is the $(n \times n)$ identity matrix, and the optimal gain matrix K_k, $k = 1, 2, 3, \ldots$, is given as

$$\begin{aligned} K_k = {} & [\alpha_k \Sigma_{k|k-1} H_k^T \\ & + (1 - \alpha_k)\Phi_{k,k-1}\Sigma_{k-1|k-1}H_{k-1}^T] \\ & \times [\alpha_k H_k + (1 - \alpha_k)H_{k-1}\Phi_{k,k-1}^{-1}] \\ & \times \Sigma_{k|k-1}[\alpha_k H_k + (1 - \alpha_k)H_{k-1}\Phi_{k,k-1}^{-1}]^T \\ & + \alpha_k(1 - \alpha_k)[H_k - H_{k-1}\Phi_{k,k-1}^{-1}] \\ & \times E(\xi_k \xi_k^T | Z_k)[H_k - H_{k-1}\Phi_{k,k-1}^{-1}]^T \\ & - (1 - \alpha_k)H_{k-1}\Phi_{k,k-1}^{-1}Q_{k-1}\Phi_{k,k-1}^{-T}H_{k-1}^T \\ & + (1 - \alpha_k)R_{k-1} + \alpha_k R_k\}^{-1} \end{aligned} \tag{31}$$

and the state estimation error covariance matrices are recursively generated as

$$\Sigma_{k|k-1} = \Phi_{k,k-1}\Sigma_{k-1|k-1}\Phi_{k,k-1}^T + Q_{k-1} \tag{32}$$

$$\begin{aligned} \Sigma_{k|k} = {} & L_k \Sigma_{k|k-1} L_k^T \\ & + \alpha_k(1 - \alpha_k)K_k[H_k - H_{k-1}\Phi_{k,k-1}^{-1}] \\ & \times E(\xi_k \xi_k^T | Z_k)[H_k - H_{k-1}\Phi_{k,k-1}^{-1}]^T \\ & \times K_k^T - (1 - \alpha_k)K_k H_{k-1}\Phi_{k,k-1}^{-1}Q_{k-1}\Phi_{k,k-1}^{-T}H_{k-1}^T K^T \\ & + (1 - \alpha_k) \times (Q_{k-1}\Phi_{k,k-1}^{-1}H_{k-1}^T K_k^T \\ & + K_k H_{k-1}\Phi_{k,k-1}^{-1}Q_{k-1}) + \alpha_k K_k R_k K_k^T \\ & + (1 - \alpha_k)K_k R_{k-1} K_k^T \end{aligned} \tag{33}$$

via the recursive relationship starting from the given initial conditions

$$\hat{\xi}_{0|0} = \mu_0 \quad \text{and} \quad \Sigma_{0|0} = \Pi_0 \tag{34}$$

Proof: The proof of Proposition 3 is given in the Appendix. □

For implementation of the filter, the term $E[\xi_k \xi_k^T | Z_k]$ in Eqs. (31) and (33) is realized as

$$E\{\xi_k \xi_k^T | Z_k\} \approx \hat{\xi}_{k|k-1}\hat{\xi}_{k|k-1}^T + \Sigma_{k|k-1} \tag{35}$$

Remark 9: The relation in Eq. (35) follows directly from the plant model (20) and the assumptions of white and mutually uncorrelated plant and sensor noise statistics except for the following approximation:

$$\begin{aligned} E[w_{k-1}w_{k-1}^T | Z_k] &= E[w_{k-1}w_{k-1}^T | z_k] \\ &\approx E[w_{k-1}w_{k-1}^T] \\ &= Q_{k-1} \end{aligned} \tag{36}$$

It follows from Eq. (22) that if $z_k = y_k$, then w_{k-1} is not conditionally independent of z_k. Therefore, this realization of the optimal filter in Proposition 3 would provide a suboptimal solution unless α_k is either 0 or 1 as seen in Eqs. (31) and (33). □

Integration of the State Estimator and the State Feedback Controller

The results of the delay compensated state feedback and state estimation algorithms under randomly varying delays, given in Propositions 1 and 3, are now combined to generate the control sequence $\{u_k\}$ for the closed-loop system. The estimate $\hat{\xi}_{k|k}$ of the plant state at the kth sensor sampling instant is now extrapolated (see Remark 8 and Fig. 1) to yield the estimate, $\hat{\xi}_{k|k}$, at the kth controller sampling instant by using the deterministic part of Eq. (16) as

$$\hat{\xi}_{k|k} = \Phi(kT, kT - \Delta)\hat{\xi}_{k|k} + \sum_{i=0}^{1} \psi_{k-1}^i u_{k-i-1} \tag{37}$$

The combined state estimation and state feedback control law is now obtained by changing a part of the augmented state vector in the full state feedback control law. Since $x_k = [\xi_k^T u_{k-1}^T]^T$ is as defined in Eq. (5), and the control action u_{k-1}

is already computed, the conditional expectation of the augmented state is

$$E[x_k|Z_k] = [\hat{\xi}^T_{k|k} u^T_{k-1}]^T \tag{38}$$

where Z_k is the history of the delayed measurements used for generating the control signal at the sampling instant k in the controller frame. Using the conditional expectation (38) in the full state feedback control law (9), the DCLQG control law is defined as

$$u_k(Z_k) = -F_k[\hat{\xi}^T_{k|k} u^T_{k-1}]^T \quad \text{for } k < N \tag{39}$$

where F_k is the state feedback control gain given in Eqs. (11) and (12), and N is the final time when no control action needs to be computed. If the control law is synthesized for a large N, then F_k in Eq. (39) can be replaced by the limiting gain matrix F provided that the plant is time invariant (see Proposition 2). Hence, for a time-invariant plant, the DCLQG control law is

$$u_k(Z_k) = -F[\hat{\xi}^T_{k|k} u^T_{k-1}]^T \tag{40}$$

Equation (40) implies that the DCLQG loop is formed via integration of state estimation and state feedback control on the assumption that the state feedback control can be separately formulated from the state estimation. However, if $\alpha_k \in (0, 1)$, it follows from Eq. (31) that the filter gain K_k involves the conditional expectation $E\{\xi_k^c \xi_k^{cT} | Z_k\}$ which is dependent on the plant state ξ_k^c. Therefore, the DCLQG concept does not comply with the principle of certainty equivalence. That is, the optimal control law could be different if an equivalent deterministic model is used instead of the stochastic model. Therefore, unlike the conventional LQG, the proposed DCLQG control may not be optimal in the environment of delayed sensor signals.

Remark 10: Recently De Koning[9] has proposed an optimal compensation algorithm for nondelayed systems with white parameters in which the problems of synthesizing the controller and estimator gain matrices are simultaneously handled. Apparently, in the environment of delayed measurements, De Koning's approach is not applicable because the measurement noise cannot be modeled as a white sequence as explained by Ray et al.[16] □

Simulation of a Flight Control System Under Delay Compensated Linear Quadratic Gaussian

The proposed DCLQG control law was tested by simulation of the longitudinal motion dynamics of an advanced aircraft. A (continuous-time) plant model of the aircraft,[17] linearized at the operating condition of 7.62-km altitude and 0.9 Mach, is given as

$$a = \begin{bmatrix} -0.0226 & -36.6170 & -18.8970 & -32.0900 & 3.2509 & -0.7626 \\ 0.0001 & -1.8997 & 0.9831 & -0.0007 & -0.1708 & -0.0050 \\ 0.0123 & 11.7200 & -2.6316 & 0.0009 & -31.6040 & 22.3960 \\ 0 & 0 & 1.000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30.0000 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 & 30 & 0 \end{bmatrix}^T, \qquad h = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

where the six plant state variables are 1) forward speed, 2) angle of attack, 3) pitch rate, 4) attitude angle, 5) elevon actuator position, and 6) canard actuator position; the two control inputs are elevon and canard signals; and the two output variables are angle of attack and attitude angle. In this example, the linearized plant model is unstable in the open loop.

On the basis of the discretized plant model, the control matrices of the proposed DCLQG were synthesized using a standard commercially available toolbox on a personal computer under the following conditions: sampling time was chosen as $T = 0.025$ s, the processing delay Δ_p was assumed to be 0, and the plant initial condition was set to $\xi_0 = [0.0\ 0.02\ 0.0\ 0.02\ 0.0\ 0.0]^T$. Statistics of the network-induced data latencies (between the sensor and controller δ_k^{sc} and between the controller and actuator δ_k^{ca}) were assumed to be independent white sequences with Rayleigh distribution truncated at T.

$$F_\delta(\theta) = [1 - \exp(-8 \quad \theta^2/T^2)]/[1 - \exp(-8)]$$
$$\text{for } 0 \le \theta \le T$$

For the stochastic regulator, the penalty matrices for the performance cost was chosen to be $\tilde{Q}_k = I_6$ and $R_k = 10^{-2} I_2$ for every k; and the state feedback gain matrix was synthesized to be

$$\begin{bmatrix} 0.8589 & -3.4484 & -1.6247 & -2.6987 & 1.5314 & -0.3659 \\ -0.4106 & 1.9130 & 1.0092 & 1.4771 & -0.3319 & 1.2174 \end{bmatrix}$$
$$\begin{matrix} 0.3129 & 0.0483 \\ -0.0428 & 0.2711 \end{matrix}\Bigg]$$

For the state estimator, the initial conditions are $\hat{\xi}^c_{0|0} = \xi_0$; and $\Sigma_{0|0} = 10^{-6} I_6$.

The plant noise covariance matrix is

$$\begin{bmatrix} 0.1435 & -0.1321 & -0.1366 & -0.1355 & -0.0941 & -0.0941 \\ -0.1321 & 0.1216 & 0.1258 & 0.1248 & 0.0867 & 0.0867 \\ -0.1366 & 0.1258 & 0.1301 & 0.1290 & 0.0896 & 0.0896 \\ -0.1355 & 0.1248 & 0.1290 & 0.1280 & 0.0889 & 0.0889 \\ -0.0941 & 0.0867 & 0.0896 & 0.0889 & 0.0618 & 0.0618 \\ -0.0941 & 0.0867 & 0.0896 & 0.0889 & 0.0618 & 0.0618 \end{bmatrix} \times 10^{-7}$$

and the measurement noise covariance is $10^{-6} I_2$.

Figures 2–5 present typical results of the closed-loop system simulation to show a comparison of the performances of the DCLQR and DCLQG. Figures 2 and 3 show the transient responses of the two output variables: angle of attack and attitude angle, which correspond to the state variables 2 and 4, respectively. Similarly, Figs. 4 and 5 present the transient responses of the state variables 5 and 6, which are the elevon and canard actuator positions, respectively. As discussed earlier, the DCLQR stochastic regulator operates under the assumption of full state feedback and no plant and measurement noise, but the presence of controller-to-actuator delay δ_k^{ca} is considered. The DCLQR responses are plotted as a benchmark for comparison with those in DCLQG for different values of the normalized skew (Δ/T) which have been chosen to be 0.0, 0.5, and 1.0 in Figs. 2 and 3 whereas Figs. 4 and 5 show comparisons of the actuator state responses under DCLQR and DCLQG along with the estimates of these states for $\Delta/T = 0.5$.

The results in Figs. 2 and 3 show that, compared to DCLQG, the system performance of DCLQR is always better because the plant noise and sensor noise are absent, all states are available, and the delay effects of the skew Δ are exactly compen-

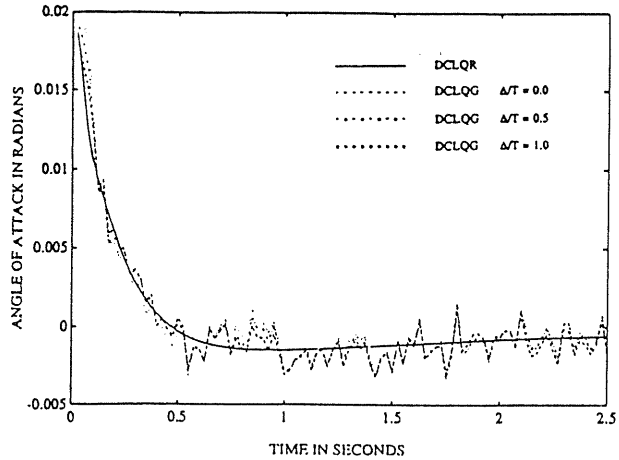

Fig. 2 Transients of angle of attack.

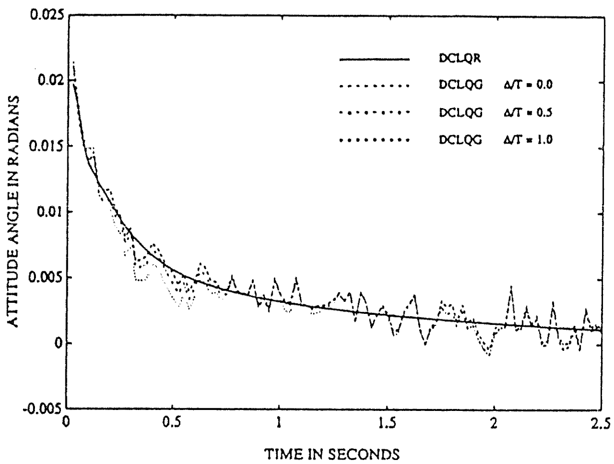

Fig. 3 Transients of attitude angle.

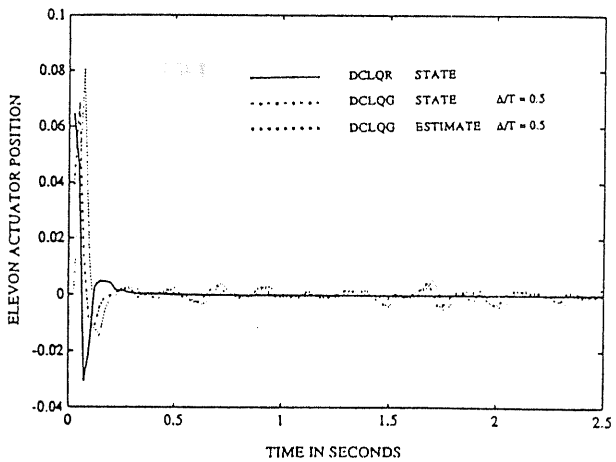

Fig. 4 Transients of elevon actuator position.

sated. On the other hand, the overall performance of DCLQG is degraded with an increase in Δ. From Proposition 2 and the fact that the probability α of timely arrival of the sensor data at the controller is a monotonically increasing function of Δ, it follows that as Δ/T approaches 1, the state estimator would perform like an optimum filter. However, this advantage of a larger Δ is overcome by the increased error in extrapolation of the estimated state, $\hat{\xi}_{\bar{k}|k}$, at the sensor frame to $\hat{\xi}_{k|k}$ at the controller frame. Since the performance of DCLQG relative to Δ depends

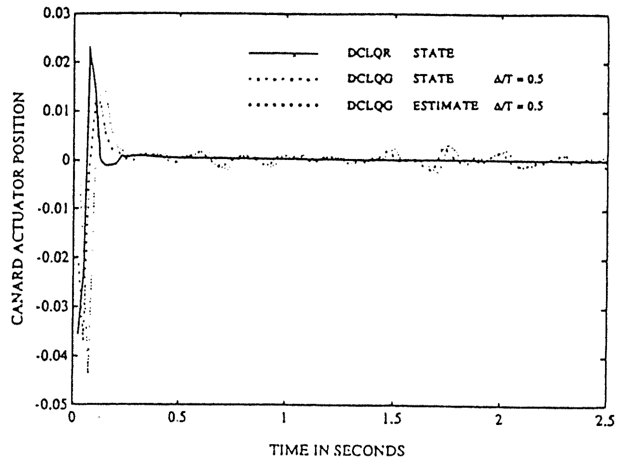

Fig. 5 Transients of canard actuator position.

on other factors, such as statistics of the network-induced delays and the plant dynamics, the observation of monotonic degradation of DCLQG performance with increase in Δ may not be valid in general. Figures 4 and 5 show that both elevon and canard actuators respond fast under DCLQR to bring the closed-loop system to a steady state. In contrast, these actuators exhibit small oscillations to compensate for plant and sensor noise. As expected, the state estimates lag behind the respective states because of the dynamic effects of filtering.

Extensive simulation experiments were conducted with a wide range of parameters representing the statistics of network-induced delays, plant noise, sensor noise, and the skew Δ as well as with different initial conditions for the plant model and the state estimation filter. In general, the performance of DCLQG degraded with larger noise covariance, but there was no evidence of instability of the closed-loop control system. In particular, the filter performance was very good in terms of the covariance of the state estimation error. Simulation experiments were also conducted after injecting structured (e.g., errors in plant model parameters) and unstructured (e.g., additional dynamics in the plant model representing the flexible modes of the airframe) uncertainties in the plant model. DCLQG was found to be sensitive relative to these uncertainties in terms of both stability and performance robustness. This is expected because LQG has poor robustness property (unless appropriately tuned), and the injected delays in the control loop further deteriorate robustness of DCLQG.

Conclusions

An output feedback control law for compensation of randomly varying distributed delays has been formulated in the setting of linear quadratic Gaussian, hereafter called the delay compensated linear quadratic Gaussian. The minimum-variance state estimator and the optimal state feedback controller in this delay compensated control law are synthesized on the basis of a stochastic model of the sensor and plant dynamics and also the delay statistics of the sensor and control data arrival at the controller and actuator terminals, respectively. These delays are assumed to be bounded (by one sampling period) in the analysis reported in this paper. Nevertheless, the filter and control algorithms can be extended for delays of more than one sampling period, but such conditions are not of much practical significance because the network design should not allow such overload even under the most severe conditions. Another major assumption in the construction of the delay compensator is that the statistics of the induced delays are white and independent of the plant and measurement noise statistics. In the context of a network, whiteness of the delay sequence can be viewed as the result of having a large number of subscribers with random traffic, which is an approximation for an onboard network for an aircraft.

In this approach, the state estimation and state feedback control laws are separately synthesized based on the principles of optimality, and then they are integrated to obtain the delay compensated output feedback controller. However, this application of the separation principle does not retain optimality of the integrated controller because of multiplicative uncertainties in the plant model resulting from the delays in the loop. Therefore, the proposed output feedback controller is suboptimal even though the state estimator and state feedback controller are individually optimal.

The proposed output feedback control law for delay compensation has been tested by simulation of the flight control system of an advanced aircraft which is unstable in the open loop. The simulation results show that the state estimator and the state feedback regulator are together capable of compensating for randomly varying distributed delays. The proposed technique of delay compensated linear quadratic Gaussian has the potential of emerging as a practical tool for analysis and synthesis of on-line control systems that are subjected to randomly varying distributed delays, in general, and particularly for future generation aircraft that are equipped with computer networks to serve the vehicle management system. However, robustness of this control concept is yet to be established under external disturbances and uncertainties in modeling of both plant dynamics and network-induced delays. Further research is needed in this area.

Appendix: Proofs of Propositions 1, 2, and 3

Proof of Proposition 1

Starting at the $(N-1)$th stage, the cost is

$$\begin{aligned} J_{N-1}(x_{N-1}, u_{N-1}) &= \tfrac{1}{2}(x_{N-1}^T Q_{N-1} x_{N-1} + u_{N-1}^T R_{N-1} u_{N-1}) \\ &+ E\{J_N^*(x_N) | x_{N-1}\} \end{aligned} \tag{A1}$$

Using Eqs. (3) and (7) in Eq. (A1) yields

$$\begin{aligned} J_{N-1}(x_{N-1}, u_{N-1}) &= \tfrac{1}{2}(x_{N-1} Q_{N-1} x_{N-1} + u_{N-1} R_{N-1} u_{N-1}) \\ &+ \tfrac{1}{2} E\{(A_{N,N-1} x_{N-1} + B_{N-1} u_{N-1})^T \\ &\times P_N (A_{N,N-1} x_{N-1} + B_{N-1} u_{N-1}) | x_{N-1}\} \\ &= \tfrac{1}{2}(x_{N-1} Q_{N-1} x_{N-1} + u_{N-1} R_{N-1} u_{N-1}) \\ &+ \tfrac{1}{2} x_{N-1}^T E\{A_{N,N-1}^T P_N A_{N,N-1}\} x_{N-1} \\ &+ u_{N-1}^T E\{B_{N-1}^T P_N A_{N,N-1}\} x_{N-1} \\ &+ \tfrac{1}{2} u_{N-1}^T E\{B_{N-1}^T P_N B_{N-1}\} u_{N-1} \end{aligned} \tag{A2}$$

The optimal control at the $(N-1)$th stage is obtained by minimizing the quadratic cost functional in Eq. (A2) with respect to u_{N-1}. Setting

$$\begin{aligned} \frac{\partial J_{N-1}(x_{N-1}, u_{N-1})}{\partial u_{N-1}} &= (R_{N-1} + E\{B_{N-1}^T P_N B_{N-1}\}) u_{N-1} \\ &+ E\{B_{N-1}^T P_N A_{N,N-1}\} x_{N-1} \end{aligned}$$

to zero, we have

$$u_{N-1}^* = -F_{N-1} x_{N-1} \tag{A3}$$

where

$$\begin{aligned} F_{N-1} &= (R_{N-1} + E\{B_{N-1}^T P_N B_{N-1}\})^{-1} \\ &\times E\{B_{N-1}^T P_N A_{N,N-1}\} \end{aligned} \tag{A4}$$

The following facts have been used in the derivation: $A_{N,N-1}$ and B_{N-1} are independent of x_{N-1} on the basis of the condition laid out in Proposition 1 and u_{N-1} is a deterministically structured function.

The optimal performance is then obtained by substituting Eqs. (A3) and (A2) into Eq. (A1)

$$\begin{aligned} J_{N-1}^*(x_{N-1}) &= J_{N-1}(x_{N-1}, u_{N-1}^*) \\ &= \tfrac{1}{2} x_{N-1}^T P_{N-1} x_{N-1} \end{aligned} \tag{A5}$$

where

$$\begin{aligned} P_{N-1} &= Q_{N-1} + F_{N-1}^T R_{N-1} F_{N-1} \\ &+ E\{(A_{N,N-1} - B_{N-1}F_{N-1})^T P_N (A_{N,N-1} - B_{N-1}F_{N-1})\} \\ &= Q_{N-1} + E\{A_{N,N-1}^T P_N (A_{N,N-1} - B_{N-1}F_{N-1})\} \\ &+ F_{N-1}^T (R_{N-1} + E\{B_{N-1}^T P_N B_{N-1}\}) F_{N-1} - F_{N-1}^T \\ &\times E\{B_{N-1}^T P_N A_{N,N-1}\} \end{aligned}$$

or

$$P_{N-1} = Q_{N-1} + E\{A_{N,N-1}^T P_N (A_{N,N-1} - B_{N-1}F_{N-1})\} \tag{A6}$$

because

$$\begin{aligned} F_{N-1}^T &(R_{N-1} + E\{B_{N-1}^T P_N B_{N-1}\}) F_{N-1} \\ &= F_{N-1}^T (R_{N-1} + E\{B_{N-1}^T P_N B_{N-1}\}) \\ &\times (R_{N-1} + E\{B_{N-1}^T P_N B_{N-1}\})^{-1} \times E\{B_{N-1}^T P_N A_{N,N-1}\} \\ &= F_{N-1}^T E\{B_{N-1}^T P_N A_{N,N-1}\} \end{aligned}$$

Now we step back to the $(N-2)$th stage,

$$\begin{aligned} J_{N-2}(x_{N-2}, u_{N-2}) &= \tfrac{1}{2}(x_{N-2}^T Q_{N-2} x_{N-2} + u_{N-2}^T R_{N-2} u_{N-2}^T) \\ &+ E\{J_{N-1}^*(x_{N-1}) | x_{N-2}\} \end{aligned} \tag{A7}$$

Using Eqs. (A5) and (3) in Eq. (A7) yields

$$\begin{aligned} J_{N-2}(x_{N-2}, u_{N-2}) &= \tfrac{1}{2}(x^T{}_{N-2} Q_{N-2} x_{N-2} + u^T{}_{N-2} R_{N-2} u^T{}_{N-2}) \\ &+ \tfrac{1}{2} E\{(A_{N-1,N-2} x_{N-2} + B_{N-2} u_{N-2})^T \\ &P_{N-1}(A_{N-1,N-2} + B_{N-2} u_{N-2}) | x_{N-1}\} \\ &= \tfrac{1}{2}(x^T{}_{N-2} Q_{N-2} x_{N-2} + u^T{}_{N-2} R_{N-2} u^T{}_{N-2}) \\ &+ \tfrac{1}{2} x^T{}_{N-2} E\{A^T{}_{N-1,N-2} P_{N-1} A_{N-1,N-2}\} x_{N-2} \\ &+ u^T{}_{N-2} E\{B^T{}_{N-2} P_{N-1} A_{N-1,N-2}\} x_{N-2} \\ &+ \tfrac{1}{2} u^T{}_{N-2} E\{B^T{}_{N-2} P_{N-1} B_{N-2}\} u_{N-2} \end{aligned}$$

Setting $\partial J(x_{N-2}, u_{N-2})/\partial u_{N-2}$ to zero yields the optimal control law

$$u_{N-2}^* = -F_{N-2} x_{N-2} \tag{A9}$$

where

$$\begin{aligned} F_{N-2} &= (R_{N-2} + E\{B_{N-2}^T P_{N-1} B_{N-2}\}) \\ &\times E\{B^T{}_{N-2} P_N A_{N-1,N-2}\} \end{aligned} \tag{A10}$$

and the optimal performance is

$$\begin{aligned} J_{N-2}^*(x_{N-2}) &= J_{N-2}(x_{N-2}, u^T{}_{N-2}) \\ &= \tfrac{1}{2} x_{N-2}^T P_{N-2} x_{N-2} \end{aligned} \tag{A11}$$

where

$$P_{N-2} = Q_{N-2} + E\{A^T_{N-1,N-2}P_{N-1}(A_{N-1,N-2} - B_{N-2}F_{N-2})\} \quad \text{(A12)}$$

The proof can now be completed by induction.

Proof of Proposition 2

It suffices to show that the limiting performance cost is bounded. That is,

$$J_0^*(x_0) = \tfrac{1}{2} x_0^T P_0 x_0 < \infty \quad \text{(A13)}$$

From Eqs. (6) and (7) it follows that as $N \to \infty$,

$$J_0^*(x_0) = \tfrac{1}{2} E\left(\sum_{k=0}^{\infty} [x_k^T Q x_k + u_k^T R u_k]\right) \quad \text{(A14)}$$

Mean square stabilizability of the pair $(A_{k+1,k}, B_k)$ implies that there exists a gain matrix F such that the closed-loop system $x_{k+1} = (A1_{k+1,} - B_k F)x_k$ is asymptotically stable in the mean square sense. It follows from Yaz[10] that the limiting gain matrix F in Proposition 1 satisfies this condition.

We define an operator Ξ: $\Theta_{n+m} \to \Theta_{n+m}$ where Θ_{n+m} is the linear space of $(n+m) \times (n+m)$ real symmetric positive semidefinite matrices, and

$$\Xi M = E\{(A_{k+1,k} - B_k F)^T M (A_{k+1,k} - B_k F)\} \quad \forall M \in \Theta_{n+m} \quad \text{(A15)}$$

Since the sequences of stochastic matrices $A_{k+1,k}$ and B_k are identically distributed for all k, the operator Ξ is invariant. Further, because of asymptotic stability of the closed-loop system, the spectral radius $\rho(\Xi) < 1$. Therefore, the first part of the infinite sum on the right-hand side of Eq. (A14) can be expressed as

$$E\left(\sum_{k=0}^{\infty} x_k^T Q x_k\right) = x_0^T\left(\sum_{k=0}^{\infty} \Xi^k\right) Q x_0 < \infty \qquad \forall x_0 \quad \text{(A16)}$$

Since $u_k = -Fx_k$, the second part of the infinite sum in Eq. (A14) involving u_k also converges. Hence the optimal cost $J_0^*(x_0)$ is bounded. □

Proof of Proposition 3

Using the concept of a standard (i.e., without measurement delays) state estimator, we propose a linear estimator for randomly delayed measurements, which will minimize the cost functional J_k in Eq. (28), to have the following recursive structure:

$$\hat{\xi}^s_{k|k} = L_k \hat{\xi}^s_{k|k-1} + K_k z_k \quad \text{(A17)}$$

$$\hat{\xi}^s_{k|k-1} = \Phi_{k,k-1} \hat{\xi}^s_{k-1|k-1} + \sum_{l=0}^{2} \beta_k^l u_{k-l} \quad \text{(A18)}$$

where the matrices β_k^l are deterministic for on-line state estimation because the arrival instants $\{t^k\}$ of past control signals at the actuator terminal are recorded at the controller terminal (see Remark 8 in the second section). The following relationship must be satisfied for the linear stochastic filter to be unbiased, i.e., $E\{e_{k|k}\} = E\{\hat{\xi}^s_{k|k} - \xi^s_k\} = 0\ \forall k$, we must have

$$L_k = I_n - K_k[\alpha_k H_k + (1 - \alpha_k) H_{k-1} \Phi^{-1}_{k,k-1}] \quad \text{(A19)}$$

where I_n is the $(n \times n)$ identity matrix. We first present three lemmas that are needed to prove the proposition.

Lemma 1 to Proposition 3

For the linear stochastic filter to be unbiased, i.e., $E\{e_{k|k-1}\} = E_x\{\hat{\xi}^s_{k|k-1} - \xi^s_{k|k}\} = 0$, $\forall k$, the following relationship must be satisfied:

$$\hat{\xi}^s_{k|k} = \hat{\xi}^s_{k|k-1} + K_k\{z_k - [\alpha_k H_k + (1-\alpha_k) H_{k-1} \Phi^{-1}_{k,k-1}] \hat{\xi}^s_{k|k-1}\} \quad \text{(A20)}$$

Proof of Lemma 1: We need to establish two additional lemmas for proving Lemma 1.

Lemma 2 for Proposition 3

If the filter is unbiased, i.e., $E\{e_{k|k-1}\} = 0$, $\forall k$, then the predictor is also unbiased, i.e.,

$$E\{e_{k|k-1}\} = 0\ \forall k \quad \text{(A21)}$$

where $e_{k|k-1} \equiv (\hat{\xi}^s_{k|k-1} - \xi^s_k)$.

Proof of Lemma 2: Using Eqs. (20) and (A18) the prediction error $e_{k|k-1}$ can be expressed as

$$\begin{aligned} e_{k|k-1} &= \Phi_{k,k-1} \hat{\xi}^s_{k|k-1} - (\Phi_{k,k-1} \xi^s_{k-1} + w_{k-1}) \\ &= \Phi_{k,k-1}(\hat{\xi}^s_{k|k-1} - \xi^s_{k-1}) - w_{k-1} \\ &= \Phi_{k,k-1} e_{k-1|k-1} - w_{k-1} \end{aligned}$$

Since the filter is given to the unbiased, i.e., $E\{e_{k-1|k-1}\} = 0$ and $\{w_k\}$ is a zero mean sequence, the expectation of the right-hand side in the preceding equation is zero. □

Lemma 3 for Proposition 3

For an unbiased filter, i.e., $E\{e_{k|k}\} = 0$, $\forall k$, the gain matrix L_k in Eq. (A17) can be expressed in terms of K_k as

$$L_k = I_n - K_k[\alpha_k H_k + (1-\alpha_k) H_{k-1} \Phi^{-1}_{k,k-1}] \quad \text{(A22)}$$

where I_n is the $(n \times n)$ identity matrix.

Proof of Lemma 3: Using Lemma 1, $E\{e_{k|k}\} = 0$ implies $E\{e_{k|k-1}\} = 0$. Substituting Eq. (A17) in the expression for $e_{k|k}$ yields

$$\begin{aligned} e_{k|k} &= \hat{\xi}^s_{k|k} - \xi^s_k \\ &= L_k \hat{\xi}^s_{k|k-1} + K_k z_k - \xi^s_k \\ &= L_k(\xi^s_k + e_{k|k-1}) + K_k[(1-\zeta_k) y_k + \zeta_k y_{k-1}] - \xi^s_k \end{aligned}$$

Substituting the relationships for ξ_k and y_k from Eqs. (20–22) into the preceding equation, we obtain

$$\begin{aligned} e_{k|k} = {} & [(1-\zeta_k) K_k H_k \Phi_{k,k-1} + L_k \Phi_{k,k-1} - \Phi_{k,k-1} \\ & + \zeta_k K_k H_{k-1}] \xi^s_{k-1} \\ & + [(1-\zeta_k) K_k H_k + L_k - I_n] G_{k-1} w_{k-1} \\ & + L_k e_{k|k-1} + (1-\zeta_k) K_k v_k + \zeta_k K_k v_{k-1} \end{aligned}$$

Since $\{w_k\}$ and $\{v_k\}$ are zero-mean sequences and are independent of $\{\zeta_k\}$, taking expectation $E\{\cdot\}$ on both sides yields

$$E\{[(1-\zeta_k) K_k H_k \Phi_{k,k-1} + L_k \Phi_{k,k-1} - \Phi_{k,k-1} + \zeta_k K_k H_{k-1}]\} E\{\xi^s_{k-1}\} = 0$$

to guarantee the zero mean of $e_{k|k}$. Since $E\{\xi^s_{k-1}\} \neq 0$, in general, its coefficient matrix must be zero. Noting that the plant state transition matrix $\Phi_{k,k-1}$ is invertible for $\forall k$ and substituting the first moment of ζ_k in the preceding equation, the result follows after some algebraic manipulations. □

The proof of Lemma 1 is now completed by using Lemma 3 for L_k into Eq. (A17). □

Proof of Proposition 3: The cost function J_k is to be minimized at each instant k based on the measurement history Z_k; the problem is recast as

$$\min_{K_k} J_k = \min_{K_k} \text{tr}[E\{e_{k|k}e_{k|k}^T | Z_k\}] \tag{A23}$$

such that the gain matrix K_k achieves the minimum for the estimator structure laid out in Proposition 3.

We express the filter error $e_{k|k}$ in terms of the prediction error $e_{k|k-1}$ and the measurement z_k by first using Eq. (A17) and then using Eqs. (20–22) for the delayed measurements

$$\begin{aligned} e_{k|k} &= \hat{\xi}^s_{k|k} - \xi^s_k \\ &= L_k\hat{\xi}^s_{k|k-1} + K_k z_k - \xi^s_k \\ &= L_k\hat{\xi}^s_{k|k-1} + K_k[(1-\zeta_k)(H_k\xi^s_k + v_k) \\ &\quad + \zeta_k(H_{k-1}\xi^s_{k-1} + v_{k-1})] - \xi^s_k \end{aligned} \tag{A24}$$

Substituting the inverse relationship of the plant model (20) and rearranging Eq. (A24) yields

$$\begin{aligned} e_{k|k} &= L_k(\hat{\xi}^s_{k|k-1} - \xi^s_k) \\ &\quad + L_k\xi^s_k - [I_n - (1-\zeta_k)K_kH_k - \zeta_kK_kH_{k-1}\Phi^{-1}_{k,k-1}] \\ &\quad \times \xi^s_k - \zeta_kK_kH_{k-1}\Phi^{-1}_{k,k-1}w_{k-1} \\ &\quad + (1-\zeta_k)K_kv_k + \zeta_kK_kv_{k-1} \\ &= L_ke_{k|k-1} + (L_k - \ell_k)\xi^s_k - \zeta_kK_kH_{k-1}\Phi^{-1}_{k,k-1}w_{k-1} \\ &\quad + (1-\zeta_k)K_kv_k + \zeta_kK_kv_{k-1} \end{aligned} \tag{A25}$$

where $\ell_k \equiv I_n - (1-\zeta_k)K_kH_k - \zeta_kK_kH_{k-1}\Phi^{-1}_{k,k-1}$. The following relationships are used for subsequent derivations: Moments of measurement delay statistics given as

$$\Pr[\zeta_k = 0] = \alpha_k; \qquad \text{and} \qquad \Pr[\zeta_k = 1] = 1 - \alpha_k$$

and

$$E\{w_kw_k^T|Z_k\} = E\{w_kw_k^T\} = Q_k$$

$$E\{v_kv_k^T|Z_k\} = E\{v_kv_k^T\} = R_k$$

All other cross terms involving $\{w_k\}$ and $\{v_k\}$ are zero because of their mutual independence and

$$E\{\ell_k|Z_k\} = L_k$$

using the result of assumption 9,

$$\begin{aligned} &E\{e_{k|k-1}w^T_{k-1}|Z_k\} \\ &= E\{(\Phi_{k,k-1}\hat{\xi}_{k-1|k-1} - \Phi_{k,k-1}\xi_{k-1} - w_{k-1})w^T_{k-1}|Z_k\} \\ &= -Q_{k-1} \end{aligned}$$

and

$$\begin{aligned} &E\{(1-\xi_k)(L_k - \ell_k)|Z_k\} \\ &= (1-\alpha_k)L_k - E\{(1-\zeta_k)\ell_k\} \\ &= \alpha_k(1-\alpha_k)K_k(H_{k-1}\Phi^{-1}_{k,k-1} - H_k) \end{aligned}$$

Using Eq. (A25) and the relationships just given, we have after some algebraic manipulations

$$\begin{aligned} \Sigma_{k|k} &= E\{e_{k|k}e^T_{k|k}|Z_k\} = L_kE\{e_{k|k-1}e^T_{k|k-1}|Z_k\}L_k^T \\ &\quad + \alpha_k(1-\alpha_k)K_k[H_k - H_{k-1}\Phi^{-1}_{k,k-1}] \\ &\quad \times E\{\xi^s_k\xi^{sT}_k|Z_k\}[H_k - H_{k-1}\Phi^{-1}_{k,k-1}]^TK_k^T \\ &\quad - (1-\alpha_k)K_kH_{k-1}\Phi^{-1}_{k,k-1} \\ &\quad \times Q_{k-1}\Phi^{-T}_{k,k-1}H^T_{k-1}K^T_k \\ &\quad + (1-\alpha_k)(Q_{k-1}\Phi^{-T}_{k,k-1}H^T_{k-1}K^T_k \\ &\quad + K_kH_{k-1}\Phi^{-1}_{k,k-1}Q_{k-1}) \\ &\quad + \alpha_kK_kR_kK_k^T + (1-\alpha_k)K_kR_{k-1}K_k^T \end{aligned} \tag{A26}$$

and the error covariance for prediction error is obtained as

$$\begin{aligned} \Sigma_{k|k-1} &= E\{e_{k|k-1}e^T{}_{k|k-1}|Z_k\} \\ &= E\{(\xi^s_k - \hat{\xi}^s_{k|k-1})(\xi^s_k - \hat{\xi}^s_{k|k-1})^T|Z_k\} \\ &= E\{(\Phi_{k,k-1}\xi^s_{k-1} + w_{k-1} - \Phi_{k,k-1}\hat{\xi}^s_{k-1|k-1}) \\ &\quad \times (\Phi_{k,k-1}\xi^s_{k-1} + w_{k-1} - \Phi_{k,k-1}\hat{\xi}^s_{k-1|k-1})^T|Z_k\} \\ &= E\{(\Phi_{k,k-1}e_{k-1|k-1} + w_{k-1}) \\ &\quad \times (\Phi_{k,k-1}e_{k-1|k-1} + w_{k-1})^T|Z_k\} \\ &= \Phi_{k,k-1}\Sigma_{k-1|k-1}\Phi^T_{k,k-1} + Q_{k-1} \end{aligned} \tag{A27}$$

Now we can substitute the right-hand side of Eq. (A26) in the cost functional J_k in Eq. (A27) to find the optimal K_k. To minimize J_k we set the partial derivative of J_k w.r.t. K_k to zero

$$\frac{\partial J_k}{\partial K_k} = \frac{\partial\, \text{tr}(E\{e_{k|k}e^T_{k|k}|Z_k\})}{\partial K_k} = 0 \tag{A28}$$

By using the following facts about matrix manipulations:

$$\frac{\partial\, \text{tr}[A(B+B^T)A^T]}{\partial A} = 2A(B+B^T); \quad \text{and} \quad \frac{\partial\, \text{tr}(AB)}{\partial A} = B^T$$

the substituted terms of Eq. (A26) in Eq. (A28) can be expanded for evaluation of the partial derivatives. Collecting the terms containing K_k yields

$$\begin{aligned} &K_k\{[\alpha_kH_k \\ &\quad + (1-\alpha_k)H_{k-1}\Phi_{k,k-1}]\Sigma_{k|k-1}[\alpha_kH_k \\ &\quad + (1-\alpha_k)H_{k-1}\Phi_{k,k-1}]^T + \alpha_k(1-\alpha_k)[H_k \\ &\quad - H_{k-1}\Phi^{-1}_{k,k-1}]E\{\xi^s{}_k\xi^{sT}_k|Z_k\}[H_k \\ &\quad - H_{k-1}\Phi^{-1}_{k,k-1}]^T \\ &\quad - (1-\alpha_k)H_{k-1}\Phi^{-1}_{k,k-1}Q_{k-1}\Phi^{-T}_{k,k-1}H^T_{k-1} \\ &\quad + (1-\alpha_k)R_{k-1} + \alpha_kR_k\} - \alpha_k\Sigma_{k|k-1}H^T_k \\ &\quad - (1-\alpha_k)(\Sigma_{k|k-1} - Q_{k-1})\Phi^{-T}_{k,k-1}H^T_{k-1} = 0 \end{aligned} \tag{A29}$$

The filter gain K_k is obtained by solving Eq. (A29) and using Eq. (A27) to substitute for $\Sigma_{k|k-1}$ in the last term on the left-hand side of Eq. (A29). □

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