

SHORT COMMUNICATIONS

GRAMMIAN ASSIGNMENT FOR STOCHASTIC PARAMETER SYSTEMS AND THEIR STABILIZATION UNDER RANDOMLY VARYING DELAYS

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SUMMARY

The motivation for the work reported in this paper accrues from the necessity of finding stabilizing control laws for systems with randomly varying distributed delays. It reports the development of assignability conditions for mean square controllability and observability Grammians in discrete-time stochastic-parameter systems and shows how to parametrize the static output feedback gains to achieve a certain Grammian assignment. All assignable Grammians are expressed either as the simultaneous solutions of a Riccati-like matrix inequality and a linear equation or as a non-linear matrix equation with a free parameter.

KEY WORDS stochastic-parameter systems; stabilizing control; Grammian assignment; random delays

1. INTRODUCTION

The motivation for the work reported in this paper accrues from the necessity of finding stabilizing control laws for systems with randomly varying distributed delays. These delays could be induced by an asynchronous time-division-multiplexed network¹ which serves as a data communications link between the spatially dispersed components of the integrated decision and control system such as the vehicle management system of future generation aircraft.² In this context the key issue is that filters and controllers designed for non-networked systems may not satisfy the performance and stability requirements in the delayed environment of network-based systems. Therefore a control synthesis methodology is needed for compensation of the randomly varying delays.

The control system under consideration consists of a continuous-time plant (where some of the states may not be directly measurable) and a discrete-time controller which share a data communications network with other subscribers.¹ Furthermore, the plant is subjected to random disturbances and the sensor data are contaminated with noise. The sensor and controller data are also subjected to randomly varying delays induced by the network before they arrive at their

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respective destinations. A finite-dimensional discrete-time model of the delayed control system has been reported in our previous publications,^{3,4} where the effects of random delays are realized in the form of stochastic parameters. For compensation of such delayed systems, Ray² has proposed the concept of delay-compensated linear quadratic Gaussian (DCLQG) as a combination of the optimal full state feedback stochastic regulator^{5,6} and the minimum variance state estimator⁷ which respectively compensate for the randomly varying delays distributed between the controller and actuator and the sensor and controller. However, the certainty equivalence property⁸ of LQG does not hold in general for DCLQG because of the multiplicative uncertainties in the system matrices.⁹

This paper attempts to provide the framework for a possible solution to the above problem via Grammian assignment to stochastic parameter models of delayed control systems to guarantee stability in the mean square (m.s.) sense.¹⁰ Assigning controllability and observability Grammians to deterministic linear continuous-time systems was considered by Yasuda and Skelton¹¹ as an extension of the covariance control concept.¹² Grammian assignment not only guarantees closed loop system stability but also augments the robustness and disturbance rejection properties. These ideas have been pursued by Yaz *et al.*¹³ for deterministic systems. The covariance assignment theory has been developed by Hsieh and Skelton¹⁴ and Skelton and Iwasaki¹⁵ for discrete-time systems and by Yaz and Skelton¹⁶ for discrete stochastic parameter systems.

2. PROBLEM FORMULATION

The continuous-time linear time-invariant plant under no disturbances and the resulting sensor data with no measurement noise are modelled in the discrete-time setting as

$$\xi_{k+1} = A\xi_k + Bu_k \quad (1)$$

$$\theta_k = C\xi_k \quad (2)$$

where the plant state vector $\xi_k \in \mathbb{R}^n$, the control vector $u_k \in \mathbb{R}^m$, the output vector $\theta_k \in \mathbb{R}^l$ and the matrices A , B and C are of compatible dimensions. We also assume that the plant is asymptotically stable, i.e. the spectral radius $\rho(A) < 1$. The observability and controllability Grammians, denoted as P_ξ and Q_ξ respectively, satisfy the algebraic identities

$$P_\xi = A^T P_\xi A + C^T C, \quad Q_\xi = A Q_\xi A^T + B B^T \quad (3)$$

The deterministic plant model in equation (1) is modified as a stochastic model to incorporate the effects of disturbances:

$$\tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{B}_k u_k + \tilde{D}_k^1 \tilde{v}_k \quad (4)$$

where the state vector $\tilde{x}_k \in \mathbb{R}^n$ is modified but the deterministic control vector $u_k \in \mathbb{R}^m$ is unchanged; the stochastic matrices \tilde{A}_k , \tilde{B}_k and \tilde{D}_k^1 have white, weakly stationary elements which are independent of the additive white noise vector \tilde{v}_k . The (noisy) output vector $\tilde{y}_k \in \mathbb{R}^l$ is accordingly modelled as

$$\tilde{y}_k = \tilde{C}_k \tilde{x}_k + \tilde{D}_k^2 \tilde{w}_k \quad (5)$$

where \tilde{C}_k and \tilde{D}_k^2 have white, weakly stationary elements independent of the standard additive white noise vectors \tilde{w}_k and \tilde{v}_k . Let \tilde{w}_k and \tilde{v}_k be mutually uncorrelated and, in addition, be uncorrelated to all stochastic parameters and let every stochastic element be independent of the plant initial state \tilde{x}_0 . Let the regulated output be

$$\tilde{z}_k = \tilde{G} \tilde{x}_k \quad (6)$$

The objective is to identify a static output feedback law

$$\mathbf{u}_k = \tilde{\mathbf{K}}\tilde{\mathbf{y}}_k \quad (7)$$

where $\tilde{\mathbf{K}}$ is the constant state feedback matrix. The m.s. observability and controllability Grammians $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ are defined respectively as

$$\tilde{\mathbf{P}} = E\{(\tilde{\mathbf{A}}_o + \tilde{\mathbf{B}}_o\tilde{\mathbf{K}}\tilde{\mathbf{C}})^T\tilde{\mathbf{P}}(\tilde{\mathbf{A}}_o + \tilde{\mathbf{B}}_o\tilde{\mathbf{K}}\tilde{\mathbf{C}}_o)\} + \tilde{\mathbf{G}}^T\tilde{\mathbf{G}} \quad (8)$$

$$\tilde{\mathbf{Q}} = E\{(\tilde{\mathbf{A}}_o + \tilde{\mathbf{B}}_o\tilde{\mathbf{K}}\tilde{\mathbf{C}}_o)\tilde{\mathbf{Q}}(\tilde{\mathbf{A}}_o + \tilde{\mathbf{B}}_o\tilde{\mathbf{K}}\tilde{\mathbf{C}}_o)^T\} + E\{\tilde{\mathbf{B}}_o\tilde{\mathbf{K}}\tilde{\mathbf{D}}_o^2\tilde{\mathbf{D}}_o^T\tilde{\mathbf{K}}^T\tilde{\mathbf{B}}^T\} + E\{\tilde{\mathbf{D}}_o\tilde{\mathbf{D}}_o^T\} \quad (9)$$

Both $E\{\cdot\}$ and $\bar{\cdot}$ are used to denote the expectation of \cdot . In the above equations and also in the sequel, weak stationarity of the random sequences is used to replace the subscript 'k' by 'o'. If the following additional conditions are imposed,

$$\tilde{\mathbf{G}}^T\tilde{\mathbf{G}} > \mathbf{0}, \quad E\{\tilde{\mathbf{D}}_o\tilde{\mathbf{D}}_o^T\} = \mathbf{W}_1 > \mathbf{0}, \quad E\{\tilde{\mathbf{D}}_o^2\tilde{\mathbf{D}}_o^T\} = \mathbf{W}_2 > \mathbf{0} \quad (10)$$

then, owing to Lemma 2.2 of DeKoning,¹⁷ existence of a $\tilde{\mathbf{P}} > \mathbf{0}$ or $\tilde{\mathbf{Q}} > \mathbf{0}$ that solves (8) or (9) respectively is sufficient to guarantee that the closed loop system is m.s. stable. For this class of systems the above statement implies that the closed loop system is also almost surely (a.s.) stable.¹⁸ The problem is to characterize all assignable mean square (m.s.) controllability/observability Grammians and then parametrize all feedback gains $\tilde{\mathbf{K}}$ that assign a particular Grammian matrix. We will show in Section 5 how these results can be used for the m.s. stabilization of linear time-invariant control systems with randomly varying delays between the controller and actuator or between the sensor and controller.

3. MEAN SQUARE OBSERVABILITY GRAMMIAN ASSIGNMENT

Let the sensor model be time-invariant, which implies that the measurement matrix $\tilde{\mathbf{C}}_k = \tilde{\mathbf{C}}$ is constant for all $k \geq 0$. The input matrix $\tilde{\mathbf{B}}_k$ may have stochastically varying elements, but $\tilde{\mathbf{B}}_k$, $k \geq 0$, is assumed to be of full column rank without loss of generality. The following results characterize all solutions of the m.s. observability Grammian $\tilde{\mathbf{P}} > \mathbf{0}$ in equation (8) assignable by a choice of the controller gain $\tilde{\mathbf{K}}$.

Theorem 1

The set $\mathbf{S}^{\text{ob}}(\tilde{\mathbf{P}})$ of all observability Grammians assignable by a choice of $\tilde{\mathbf{K}}$ is given by

$$\mathbf{S}^{\text{ob}}(\tilde{\mathbf{P}}) \equiv \{\tilde{\mathbf{P}} > \mathbf{0}: \Psi_{\text{ob}}(\tilde{\mathbf{P}}) = \tilde{\mathbf{P}} - \overline{\tilde{\mathbf{A}}_o^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}_o} + \overline{\tilde{\mathbf{A}}_o^T\tilde{\mathbf{P}}\tilde{\mathbf{B}}_o}(\overline{\tilde{\mathbf{B}}_o^T\tilde{\mathbf{P}}\tilde{\mathbf{B}}_o})^{-1}\overline{\tilde{\mathbf{B}}_o^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}_o} - \tilde{\mathbf{G}}^T\tilde{\mathbf{G}} \geq \mathbf{0}; \\ \text{rank}\Psi_{\text{ob}}(\tilde{\mathbf{P}}) \leq m; (\mathbf{I} - \tilde{\mathbf{C}}^T\tilde{\mathbf{C}})(\tilde{\mathbf{P}} - \overline{\tilde{\mathbf{A}}_o^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}_o} - \tilde{\mathbf{G}}^T\tilde{\mathbf{G}})(\mathbf{I} - \tilde{\mathbf{C}}^T\tilde{\mathbf{C}}) = \mathbf{0}\} \quad (11)$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudoinverse of (\cdot) .

Proof. The proof is similar to that of Theorem 1 of Yaz and Skelton¹⁶ and is therefore omitted. \square

Corollary to Theorem 1

The set $\mathbf{S}^{\text{ob}}(\tilde{\mathbf{P}})$ of all observability Grammians can be equivalently expressed as

$$\mathbf{S}^{\text{ob}}(\tilde{\mathbf{P}}) = \{\tilde{\mathbf{P}} > \mathbf{0}: \Psi_{\text{ob}}(\tilde{\mathbf{P}}) = [(\mathbf{I} - \tilde{\mathbf{C}}^T\tilde{\mathbf{C}})\overline{\tilde{\mathbf{A}}_o^T\tilde{\mathbf{P}}\tilde{\mathbf{B}}_o} + \tilde{\mathbf{C}}^T\tilde{\mathbf{C}}\tilde{\mathbf{Z}}_1](\overline{\tilde{\mathbf{B}}_o^T\tilde{\mathbf{P}}\tilde{\mathbf{B}}_o})^{-1} \\ \times [(\mathbf{I} - \tilde{\mathbf{C}}^T\tilde{\mathbf{C}})\overline{\tilde{\mathbf{A}}_o^T\tilde{\mathbf{P}}\tilde{\mathbf{B}}_o} + \tilde{\mathbf{C}}^T\tilde{\mathbf{C}}\tilde{\mathbf{Z}}_1]^T; \tilde{\mathbf{Z}}_1 \text{ arbitrary}\} \quad (12)$$

where $\Psi_{\text{ob}}(\tilde{\mathbf{P}})$ is as defined in equation (11).

Proof. First we show that equation (12) implies equation (11). The matrix $\Psi_{ob}(\tilde{P})$ in equation (12) is non-negative definite with rank $\leq m$. It also follows from (12) that

$$(\mathbf{I} - \tilde{C}^+ \tilde{C})(\tilde{P} - \overline{\tilde{A}_0^T \tilde{P} \tilde{A}_0} - \tilde{G}^T \tilde{G})(\mathbf{I} - \tilde{C}^+ \tilde{C}) = \mathbf{0} \quad (13)$$

because $\tilde{C}^+ \tilde{C}$ has the properties of a projection matrix.

Next we show that equation (11) implies equation (12). Combining the first and second assignability conditions in equation (11), we obtain $\Psi_{ob}(\tilde{P}) = \mathbf{L}_1^T \mathbf{L}_1$, with $\mathbf{L}_1 \in \mathbb{R}^{m \times n}$. Substituting this result into the third assignability condition of equation (11) yields

$$(\mathbf{I} - \tilde{C}^+ \tilde{C}) \mathbf{L}_1^T \mathbf{L}_1 (\mathbf{I} - \tilde{C}^+ \tilde{C}) = (\mathbf{I} - \tilde{C}^+ \tilde{C}) \overline{\tilde{A}_0^T \tilde{P} \tilde{B}_0} (\overline{\tilde{B}_0^T \tilde{P} \tilde{B}_0})^{-1} \overline{\tilde{B}_0^T \tilde{P} \tilde{A}_0} (\mathbf{I} - \tilde{C}^+ \tilde{C}) \quad (14)$$

Using the matrix factorization result given by Yaz and Skelton,¹⁶ we obtain the following relationship for an orthogonal matrix \mathbf{U} :

$$\mathbf{L}_1 (\mathbf{I} - \tilde{C}^+ \tilde{C}) = \mathbf{U}^T \mathbf{T}_1^{-T} \overline{\tilde{B}_0^T \tilde{P} \tilde{A}_0} (\mathbf{I} - \tilde{C}^+ \tilde{C}) \quad (15)$$

Equation (15) always has a solution \mathbf{L}_1 , since the existence condition is satisfied because $\tilde{C}^+ \tilde{C}$ has the properties of a projection matrix. The solution \mathbf{L}_1 is given in terms of an arbitrary matrix \mathbf{Z} as

$$\mathbf{L}_1 = \mathbf{U}^T \mathbf{T}_1^{-T} [\overline{\tilde{B}_0^T \tilde{P} \tilde{A}_0} (\mathbf{I} - \tilde{C}^+ \tilde{C}) + \mathbf{Z} \tilde{C}^+ \tilde{C}] \quad (16)$$

Therefore $\mathbf{L}_1^T \mathbf{L}_1$ satisfies equation (12) based on the previously derived equation $\Psi_{ob}(\tilde{P}) = \mathbf{L}_1^T \mathbf{L}_1$. \square

The next task is to identify a static controller \tilde{K} . For an assignable observability Grammian \tilde{P} the set of all gains \tilde{K} that assigns this \tilde{P} is presented below.

Theorem 2

Let the observability Grammian \tilde{P} satisfy the assignability conditions of Theorem 1. Then the parametrization of all gains is given by

$$\tilde{K} = (\mathbf{T}_1^{-1} \mathbf{U}_1 \mathbf{L}_1 - \tilde{K}^{ob}) \tilde{C}^+ + \mathbf{Z}_1 (\mathbf{I} - \tilde{C}^+ \tilde{C}) \quad (17)$$

where \tilde{K}^{ob} , \mathbf{T}_1 and \mathbf{L}_1 are given as

$$\tilde{K}^{ob} = \overline{(\tilde{B}_0^T \tilde{P} \tilde{B}_0)^{-1} \tilde{B}_0^T \tilde{P} \tilde{A}_0} \quad (18)$$

$$\mathbf{T}_1^T \mathbf{T}_1 = \overline{\tilde{B}_0^T \tilde{P} \tilde{B}_0}, \quad \det(\mathbf{T}_1) \neq 0, \quad \mathbf{L}_1^T \mathbf{L}_1 = \Psi_{ob}(\tilde{P}) \quad (19)$$

and \mathbf{U}_1 is found as

$$\mathbf{U}_1 = \mathbf{M}_3 \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{U}}_1 \end{bmatrix} \mathbf{M}_1^T \quad (20)$$

from the singular value decompositions

$$\mathbf{L}_1 (\mathbf{I} - \tilde{C}^+ \tilde{C}) = \mathbf{M}_1 \begin{bmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{M}_2^T \quad (21)$$

$$\mathbf{T}_1 \tilde{K}^{ob} (\mathbf{I} - \tilde{C}^+ \tilde{C}) = \mathbf{M}_3 \begin{bmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{M}_2^T \quad (22)$$

$\bar{\mathbf{U}}_1$ is an arbitrary orthogonal matrix of compatible dimension; \mathbf{Z}_1 is an arbitrary matrix of compatible dimension; r is the number of positive singular values in Λ_1 ; and \mathbf{M}_1 , \mathbf{M}_2 and \mathbf{M}_3 are

appropriate orthogonal matrices in the singular value decompositions of equations (21) and (22).

Proof. The proof is similar to that of Theorem 1 of Yaz and Skelton¹⁶ and is therefore omitted. \square

4. MEAN SQUARE CONTROLLABILITY GRAMMIAN ASSIGNMENT

Following the duality between controllability and observability,¹⁹ we now assume that the actuator model is time-invariant, which implies that $\tilde{\mathbf{B}}_k = \tilde{\mathbf{B}}$ is a constant matrix for all $k \geq 0$ instead of $\tilde{\mathbf{C}}_k = \tilde{\mathbf{C}}$. The measurement matrix $\tilde{\mathbf{C}}_k$ may have stochastically varying elements, but $\tilde{\mathbf{C}}_k$, $k \geq 0$, is assumed to be of full row rank without loss of generality. Then the m.s. controllability Grammian in equation (6) yields

$$\tilde{\mathbf{Q}} = E\{(\tilde{\mathbf{A}}_0 + \tilde{\mathbf{B}}\tilde{\mathbf{K}}\tilde{\mathbf{C}}_0)\mathbf{Q}(\tilde{\mathbf{A}}_0 + \tilde{\mathbf{B}}\tilde{\mathbf{K}}\tilde{\mathbf{C}}_0)^T\} + \tilde{\mathbf{B}}\tilde{\mathbf{K}}\mathbf{W}_2\tilde{\mathbf{K}}^T\tilde{\mathbf{B}}^T + \mathbf{W}_1 \quad (23)$$

The following results, which are the counterparts of Theorems 1 and 2 in Section 3, characterize all solutions of the m.s. controllability Grammian $\tilde{\mathbf{Q}} > \mathbf{0}$ in equation (9) assignable by a choice of the controller gain $\tilde{\mathbf{K}}$.

Theorem 3

The set $\mathbf{S}^c(\tilde{\mathbf{Q}})$ of all controllability Grammians assignable by a choice of $\tilde{\mathbf{K}}$ is given by

$$\begin{aligned} \mathbf{S}^c(\tilde{\mathbf{Q}}) = \{ \tilde{\mathbf{Q}} > \mathbf{0}; \Psi_c(\tilde{\mathbf{Q}}) = \tilde{\mathbf{Q}} - \tilde{\mathbf{A}}_0\tilde{\mathbf{Q}}\tilde{\mathbf{A}}_0^T + \tilde{\mathbf{A}}_0\tilde{\mathbf{Q}}\tilde{\mathbf{C}}_0^T(\tilde{\mathbf{C}}_0\tilde{\mathbf{Q}}\tilde{\mathbf{C}}_0^T + \mathbf{W}_2)^{-1}\tilde{\mathbf{C}}_0\tilde{\mathbf{Q}}\tilde{\mathbf{A}}_0^T - \mathbf{W}_1 \geq \mathbf{0}; \\ \text{rank } \Psi_c(\tilde{\mathbf{Q}}) \geq l; (\mathbf{I} - \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T)(\tilde{\mathbf{Q}} - \tilde{\mathbf{A}}_0\tilde{\mathbf{Q}}\tilde{\mathbf{A}}_0^T - \mathbf{W}_1)(\mathbf{I} - \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T) = \mathbf{0} \} \end{aligned} \quad (24)$$

where $\mathbf{W}_1 = E\{\tilde{\mathbf{D}}_0\tilde{\mathbf{D}}_0^T\} > \mathbf{0}$ is as defined in equation (10) in Section 2.

Proof. The proof is similar to that of Theorem 1 of Yaz and Skelton¹⁶ and is therefore omitted. \square

Corollary to Theorem 3

The set $\mathbf{S}^c(\tilde{\mathbf{Q}})$ of all controllability Grammians defined in equation (24) can be equivalently expressed as

$$\begin{aligned} \mathbf{S}^c(\tilde{\mathbf{Q}}) = \{ \tilde{\mathbf{Q}} > \mathbf{0}; \Psi_c(\tilde{\mathbf{Q}}) = [(\mathbf{I} - \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T)\tilde{\mathbf{A}}_0\tilde{\mathbf{Q}}\tilde{\mathbf{C}}_0^T + \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T\tilde{\mathbf{Z}}_2](\tilde{\mathbf{C}}_0\tilde{\mathbf{Q}}\tilde{\mathbf{C}}_0^T + \mathbf{W}_2)^{-1} \\ \times [(\mathbf{I} - \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T)\tilde{\mathbf{A}}_0\tilde{\mathbf{Q}}\tilde{\mathbf{C}}_0^T + \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T\tilde{\mathbf{Z}}_2]^T; \tilde{\mathbf{Z}}_2 \text{ arbitrary} \} \end{aligned} \quad (25)$$

where $\Psi_c(\tilde{\mathbf{Q}})$ is as defined in equation (24).

Proof. The proof is similar to that of Corollary to Theorem 1 and is therefore omitted. \square

Theorem 4

Let the controllability Grammian $\tilde{\mathbf{Q}}$ satisfy the assignability conditions of Theorem 3. Then the parametrization of all gains is given by

$$\tilde{\mathbf{K}} = \tilde{\mathbf{B}}^T(\mathbf{L}_2\mathbf{U}_2\mathbf{T}_2^{-1} - \tilde{\mathbf{K}}^c) + (\mathbf{I} - \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T)\mathbf{Z}_2 \quad (26)$$

where $\tilde{\mathbf{K}}^c$, \mathbf{T}_2 and \mathbf{L}_2 are given as

$$\tilde{\mathbf{K}}^c = \overline{\mathbf{A}_0 \tilde{\mathbf{Q}} \tilde{\mathbf{C}}_0^T} (\overline{\mathbf{C}_0 \tilde{\mathbf{Q}} \tilde{\mathbf{C}}_0^T} + \mathbf{W}_2)^{-1} \quad (27)$$

$$\mathbf{T}_2 \mathbf{T}_2^T = \overline{\mathbf{C}_0 \tilde{\mathbf{Q}} \tilde{\mathbf{C}}_0^T} + \mathbf{W}_2 > \mathbf{0}, \quad \det(\mathbf{T}_2) \neq 0, \quad \mathbf{L}_2 \mathbf{L}_2^T = \Psi_c(\tilde{\mathbf{Q}}) \quad (28)$$

and \mathbf{U}_2 is found as

$$\mathbf{U}_2 = \mathbf{N}_1 \begin{bmatrix} \mathbf{I}_\rho & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}_2 \end{bmatrix} \mathbf{N}_3^T \quad (29)$$

from the singular value decompositions

$$(\mathbf{I} - \tilde{\mathbf{B}} \tilde{\mathbf{B}}^\dagger) \mathbf{L}_2 = \mathbf{N}_2 \begin{bmatrix} \mathbf{L}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{N}_2^T \quad (30)$$

$$(\mathbf{I} - \tilde{\mathbf{B}} \tilde{\mathbf{B}}^\dagger) \tilde{\mathbf{K}}^c \mathbf{T}_2 = \mathbf{N}_2 \begin{bmatrix} \mathbf{L}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{N}_3^T \quad (31)$$

$\tilde{\mathbf{U}}_2$ is an arbitrary orthogonal matrix of compatible dimension; \mathbf{Z}_2 is an arbitrary matrix of compatible dimension; ρ is the number of positive singular values in Λ_2 ; and \mathbf{N}_1 , \mathbf{N}_2 and \mathbf{N}_3 are appropriate orthogonal matrices in the singular value decompositions of equations (30) and (31).

Proof. A detailed proof can be constructed by following Yaz and Skelton.¹⁶ \square

5. APPLICATIONS TO DELAY-COMPENSATED STABILIZATION

Now we consider the application of the m.s. observability Grammian assignment to the problem of guaranteed stabilization in the presence of a randomly varying delay between the controller and actuator. Because of this random but a.s. bounded delay and zero-order hold (see assumptions in Reference 2), the input to the plant is piecewise constant during a sampling interval $[kT, (k+1)T)$ in the controller time frame where the changes in the input occur at random instants $kT + t^k$, where $\{t^k\}$ is a random sequence as explained in Reference 3. The plant model in equation (1) is discretized in the controller frame as

$$\tilde{\mathbf{x}}_{k+1} = \Phi((k+1)T, kT) \tilde{\mathbf{x}}_k + \sum_{i=0}^1 \mathbf{b}_k^i \mathbf{u}_{k-i} + \tilde{\mathbf{D}}_k^1 \tilde{\mathbf{v}}_k \quad (32)$$

where $\Phi(\cdot, \cdot)$ is the state transition matrix and

$$\mathbf{b}_k^0 = \int_{kT+t^k}^{(k+1)T} \Phi((k+1)T, \tau) \mathbf{b}(\tau) d\tau, \quad \mathbf{b}_k^1 = \int_{kT}^{(k+1)T} \Phi((k+1)T, \tau) \mathbf{b}(\tau) d\tau - \mathbf{b}_k^0 \quad (33)$$

This gives rise to the augmented model of the delayed system as

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k + \mathbf{D}_k^1 \tilde{\mathbf{v}}_k \quad (34)$$

with

$$\mathbf{x}_k = \begin{bmatrix} \tilde{\mathbf{x}}_k \\ \mathbf{u}_{k-1} \end{bmatrix}, \quad \mathbf{A}_k = \begin{bmatrix} \mathbf{F}((k+1)T, kT) & \mathbf{b}_k^1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_k = \begin{bmatrix} \mathbf{b}_k^0 \\ \mathbf{I}_m \end{bmatrix}, \quad \mathbf{D}_k^1 = \begin{bmatrix} \tilde{\mathbf{D}}_k^1 \\ \mathbf{0} \end{bmatrix} \quad (35)$$

where the white noise $\mathbf{D}_k^1 \tilde{\mathbf{v}}_k$, which is independent of both \mathbf{A}_k and \mathbf{B}_k , represents the additive uncertainty. Similarly, the sensor output in equation (2), contaminated by the measurement noise $\tilde{\mathbf{D}}_k^2 \tilde{\mathbf{w}}_k$, is expressed in compact form as

$$\begin{bmatrix} \tilde{\mathbf{y}}_k \\ \mathbf{u}_{k-1} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_k \\ \mathbf{u}_{k-1} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{D}}_k^2 \\ \mathbf{0} \end{bmatrix} \tilde{\mathbf{w}}_k = \mathbf{C} \mathbf{x}_k + \mathbf{w}_k \quad (36)$$

In order to make the control system analysis easily tractable, we assume the plant dynamics to be time-invariant. That is, the state transition matrix $\Phi((k+1)T, kT) = \mathbf{F}$ is constant in addition to the earlier assumption that the measurement matrix $\tilde{\mathbf{C}}_k = \tilde{\mathbf{C}}$ is constant for all k . Furthermore, for control synthesis the regulated output $\mathbf{z}_k = \mathbf{G} \mathbf{x}_k$ is defined as a linear function of the augmented state \mathbf{x}_k :

$$\mathbf{z}_k = \mathbf{G}_1 \tilde{\mathbf{x}}_k + \mathbf{G}_2 \mathbf{u}_{k-1} \quad (37)$$

where the weighting matrix $\mathbf{G} = [\mathbf{G}_1 \ \mathbf{G}_2]$ and $\mathbf{G}^T \mathbf{G} = \text{diag}[\mathbf{V}_1 \ \mathbf{V}_2] > \mathbf{0}$. In view of time-invariance of the plant dynamics, we modify the assignability conditions for a suitably partitioned observability Grammian \mathbf{P} as

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^T & \mathbf{P}_{22} \end{bmatrix} > \mathbf{0} \quad (38)$$

to yield the inequality

$$\begin{bmatrix} \mathbf{F}^T \mathbf{P}_{11} \bar{\mathbf{b}}_0 + \mathbf{F}^T \mathbf{P}_{12} \\ \bar{\mathbf{b}}_0^T \mathbf{P}_{11} \bar{\mathbf{b}}_0 + \bar{\mathbf{b}}_0^T \mathbf{P}_{12} \end{bmatrix} (\bar{\mathbf{b}}_0^{0T} \mathbf{P}_{11} \bar{\mathbf{b}}_0 + \bar{\mathbf{b}}_0^T \mathbf{P}_{12} \bar{\mathbf{b}}_0 + \bar{\mathbf{b}}_0^{0T} \mathbf{P}_{12} + \mathbf{P}_{22})^{-1} [\bar{\mathbf{b}}_0^{0T} \mathbf{P}_{11} \mathbf{F} + \bar{\mathbf{b}}_0^T \mathbf{P}_{12} \mathbf{F} \ \bar{\mathbf{b}}_0^{0T} \mathbf{P}_{11} \bar{\mathbf{b}}_0^1 + \bar{\mathbf{b}}_0^T \mathbf{P}_{12} \bar{\mathbf{b}}_0^1] + \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^T & \mathbf{P}_{22} \end{bmatrix} - \begin{bmatrix} \mathbf{F}^T \mathbf{P}_{11} \mathbf{F} & \mathbf{F}^T \mathbf{P}_{11} \bar{\mathbf{b}}_0^1 \\ \bar{\mathbf{b}}_0^{1T} \mathbf{P}_{11} \mathbf{F} & \bar{\mathbf{b}}_0^{1T} \mathbf{P}_{11} \bar{\mathbf{b}}_0^1 \end{bmatrix} \geq \mathbf{0} \quad (39)$$

with the rank of the left-hand side not exceeding m and

$$(\mathbf{I} - \tilde{\mathbf{C}}^+ \tilde{\mathbf{C}})(\mathbf{P}_{11} - \mathbf{F}^T \mathbf{P}_{11} \mathbf{F} - \mathbf{V}_1)(\mathbf{I} - \tilde{\mathbf{C}}^+ \tilde{\mathbf{C}}) = \mathbf{0} \quad (40)$$

If for a given model one can find an observability Grammian \mathbf{P} that satisfies the above conditions, then the control gain $\tilde{\mathbf{K}}$ given in Theorem 2 can achieve this assignment. In this formulation each of the following three requirements needs to be satisfied: (i) the plant dynamics and sensor configuration are time-invariant, i.e. both the state transition and measurement matrices are constant; (ii) the input matrix \mathbf{B}_k is of full column rank, but from (48) this is always satisfied by construction; and (iii) the square of the regulated output weighting matrix \mathbf{G} is positive definite, i.e. $\mathbf{G}^T \mathbf{G} > \mathbf{0}$, which can be chosen as such.

Remark. In view of the characterizations of $\mathbf{S}^{\text{ob}}(\tilde{\mathbf{P}})$ and $\mathbf{S}^c(\tilde{\mathbf{Q}})$ given in Theorems 1 and 3, the results in equations (39) and (40) with the rank condition can be equivalently expressed by a single non-linear matrix equation, which is omitted here for reasons of brevity.

The application of m.s. controllability Grammian assignment to the output feedback stabilization of systems will be shown next where no delay exists between the sensor and controller. Effects of the sensor-to-controller random delays on the closed loop control system are different from those of the controller-to-actuator delays as pointed out in

References 2, 3 and 7. If the plant is linear time-invariant with weak sense stationary plant and measurement noise and if there is no controller-to-actuator delay, then the plant and the sensor data without the sensor-to-controller delay are modelled in the sensor time frame as

$$\tilde{\mathbf{x}}_{k+1}^s = \mathbf{F}\tilde{\mathbf{x}}_k^s + \mathbf{\Gamma}\mathbf{u}_k + \tilde{\mathbf{D}}^1\tilde{\mathbf{v}}_k \quad (41)$$

$$\tilde{\mathbf{y}}_k^s = \tilde{\mathbf{C}}\tilde{\mathbf{x}}_k^s + \tilde{\mathbf{D}}_2\tilde{\mathbf{w}}_k^s \quad (42)$$

where $\tilde{\mathbf{v}}_k$ and $\tilde{\mathbf{w}}_k^s$ are sequences of standard white noise vectors, \mathbf{F} , $\mathbf{\Gamma}$, $\tilde{\mathbf{D}}^1$ and $\tilde{\mathbf{C}}$ are constant matrices and $\tilde{\mathbf{D}}_2$, defined in equation (10), has full row rank. The sensor-to-controller delays are responsible for randomly delayed arrival of sensor data at the controller. This phenomenon caused by bounded sensor-to-controller delays is modelled as

$$\mathbf{y}_k^s = (1 - \zeta_k)\tilde{\mathbf{y}}_k^s + \zeta_k\tilde{\mathbf{y}}_{k-1}^s \quad (43)$$

where $\zeta_k \in \{0, 1\}$ is a white binary sequence with $E\{\zeta_k\} = 1 - \alpha$, i.e. α is the probability of late arrival of the sensor data at the controller. By combining equations (41)–(43), we obtain the plant and sensor models in the sensor time frame² as

$$\mathbf{x}_{k+1}^s = \mathbf{A}\mathbf{x}_{k+1}^s + \mathbf{B}\mathbf{u}_k + \mathbf{D}^1\tilde{\mathbf{v}}_k, \quad \mathbf{y}_k^s = \mathbf{C}_k\mathbf{x}_k^s + \mathbf{D}_k^2\tilde{\mathbf{w}}_k \quad (44)$$

where

$$\begin{aligned} \mathbf{x}_k^s &= \begin{bmatrix} \tilde{\mathbf{x}}_k^s \\ \tilde{\mathbf{x}}_{k-1}^s \end{bmatrix}, & \mathbf{A} &= \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{I}_n & \mathbf{0} \end{bmatrix}, & \mathbf{B} &= \begin{bmatrix} \mathbf{\Gamma} \\ \mathbf{0} \end{bmatrix}, & \mathbf{C}_k &= [(1 - \zeta_k)\tilde{\mathbf{C}} \quad \zeta_k\tilde{\mathbf{C}}] \\ \mathbf{D}^1 &= \begin{bmatrix} \tilde{\mathbf{D}}^1 \\ \mathbf{0} \end{bmatrix}, & \mathbf{D}_k^2 &= [(1 - \zeta_k)\tilde{\mathbf{D}}^2 \quad \zeta_k\tilde{\mathbf{D}}^2], & \mathbf{w}_k^s &= \begin{bmatrix} \tilde{\mathbf{w}}_k^s \\ \tilde{\mathbf{w}}_k^s \end{bmatrix} \end{aligned} \quad (45)$$

The measurement noise $\mathbf{D}_k^2\tilde{\mathbf{w}}_k^s$ in the delayed sensor model of equation (44) is non-white even though $\{\tilde{\mathbf{w}}_k^s\}$ is assumed to be a white noise sequence. The non-whiteness results from the usage of the same sensor data at two consecutive samples,⁷ i.e. if $\zeta_{k-1} = 0$ and $\zeta_k = 1$. The probability of such occurrences diminishes to zero if α approaches either zero or unity. If α is small, $\mathbf{D}_k^2\tilde{\mathbf{w}}_k^s$ can be approximated as white and then the controllability Grammian assignment conditions in Theorem 3 could be considered valid. Under these conditions the m.s. controllability Grammian for the augmented system in equation (44) is

$$\mathbf{Q} = \mathbf{A}\mathbf{Q}\mathbf{A}^T + \mathbf{A}\mathbf{Q}\overline{\mathbf{C}}_0^T\mathbf{K}^T\mathbf{B}^T + \mathbf{B}\mathbf{K}\overline{\mathbf{C}}_0\mathbf{Q}\mathbf{A}^T + \mathbf{B}\mathbf{K}(\overline{\mathbf{C}}_0\mathbf{Q}\overline{\mathbf{C}}_0^T + \mathbf{W}_2)\mathbf{K}^T\mathbf{B}^T + \mathbf{W}_1 \quad (46)$$

with

$$\begin{aligned} \overline{\mathbf{C}}_0 &= [\alpha\tilde{\mathbf{C}} \quad (1 - \alpha)\tilde{\mathbf{C}}], & \overline{\mathbf{C}}_0\mathbf{Q}\overline{\mathbf{C}}_0^T &= \tilde{\mathbf{C}}\mathbf{Q}_{11}\tilde{\mathbf{C}}^T + (1 - \alpha)\tilde{\mathbf{C}}\mathbf{Q}_{22}\tilde{\mathbf{C}}^T, \\ \mathbf{Q} &= \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^T & \mathbf{Q}_{22} \end{bmatrix}, & \tilde{\mathbf{D}}_2\tilde{\mathbf{D}}_2^T &> \mathbf{0} \end{aligned} \quad (47)$$

It follows from equations (44) and (45) that \mathbf{W}_1 defined in equation (10) is now constrained as $\mathbf{W}_1 = \text{diag}[\tilde{\mathbf{D}}^1\tilde{\mathbf{D}}^{1T}, \mathbf{0}]$. However, equation (10) requires that \mathbf{W}_1 must be positive definite. This problem can be circumvented by adding $\mathbf{W}_3 > \mathbf{0}$ to the right side of equation (46), which increases the value of \mathbf{Q} . However, if $\mathbf{W}_3 > \mathbf{0}$ is chosen such that the maximum singular value $\bar{\sigma}(\mathbf{W}_3) \ll 1$, then the assigned \mathbf{Q} will be close to the \mathbf{Q} in equation (46). In this way the

assignability conditions become

$$\begin{aligned} \Psi_c(Q) = & \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} - \begin{bmatrix} FQ_{11}F^T & FQ_{11} \\ Q_{11}F^T & Q_{11} \end{bmatrix} - \begin{bmatrix} \tilde{D}_1\tilde{D}_1^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - W_3 \\ & + \begin{bmatrix} F[\alpha Q_{11} + (1-\alpha)Q_{12}]\tilde{C} \\ [\alpha Q_{11} + (1-\alpha)Q_{12}]\tilde{C} \end{bmatrix} \{\tilde{C}[Q_{11} + (1-\alpha)Q_{22}]\tilde{C}^T + W_2\}^{-1} \\ & \times [\tilde{C}^T[\alpha Q_{11} + (1-\alpha)Q_{12}]\tilde{C}^T] \end{aligned} \quad (48)$$

with $\text{rank}\Psi_c(Q) \leq l$ and

$$\begin{bmatrix} \mathbf{I} - \Gamma\Gamma^+ & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} - \begin{bmatrix} FQ_{11}F^T & FQ_{11} \\ Q_{11}F^T & Q_{11} \end{bmatrix} - \begin{bmatrix} \tilde{D}_1\tilde{D}_1^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - W_3 \right) \begin{bmatrix} \mathbf{I} - \Gamma\Gamma^+ & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \mathbf{0} \quad (49)$$

If these assignability conditions are satisfied for a particular $Q > \mathbf{0}$, then Theorem 4 will provide the necessary gains to assign this Q .

Remark. In view of equation (48), the rank condition of $\text{rank}\Psi_c(Q) \leq l$ and equation (49) can be equivalently expressed as

$$\begin{aligned} \Psi_c(Q) = & \left(\begin{bmatrix} (\mathbf{I} - \Gamma\Gamma^+)F[\alpha Q_{11} + (1-\alpha)Q_{12}]\tilde{C} \\ [\alpha Q_{11} + (1-\alpha)Q_{12}]\tilde{C} \end{bmatrix} + \begin{bmatrix} \Gamma\Gamma^+ & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{Z}_2 \right) \\ & \times \{\tilde{C}[Q_{11} + (1-\alpha)Q_{22}]\tilde{C}^T + W_2\}^{-1} \\ & \times \left(\begin{bmatrix} (\mathbf{I} - \Gamma\Gamma^+)F[\alpha Q_{11} + (1-\alpha)Q_{12}]\tilde{C} \\ [\alpha Q_{11} + (1-\alpha)Q_{12}]\tilde{C} \end{bmatrix} + \begin{bmatrix} \Gamma\Gamma^+ & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{Z}_2 \right)^T \end{aligned}$$

6. CONCLUSIONS

This paper characterizes the sets of m.s. controllability and observability Grammians that are assignable to discrete time stochastic parameter systems by static output feedback and parametrizes the feedback gains to achieve a certain Grammian assignment. All assignable Grammians are given as the simultaneous solutions of a Riccati-like matrix inequality and a linear equation. It is shown that an alternative characterization can be obtained in terms of a non-linear algebraic matrix equation with a free matrix parameter. Apparently, non-linear programming is a viable approach for finding an assignable Grammian. Since Grammians are constrained to be positive definite to guarantee stability, this optimization problem is non-differentiable. Upon finding a suitable Grammian via numerical means, it can easily be assigned to the system by feedback gain matrices specified in Theorems 2 and 4. This guarantees m.s. stability of the closed loop system. It may be possible to quantify the performance bounds of the control system by properly defining appropriate cost criteria. This issue and the selection of a numerical method for solving the Grammian assignment equations are subjects of future research.

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