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Stochastic optimal control under randomly varying distributed delays

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An output feedback control law is presented, hereafter called the linear quadratic coupled delay compensator (LQCDC), for application to processes that are subjected to randomly varying distributed delays. An example is future generation aircraft, which are equipped with computer networks to serve as the communications link for the vehicle management system. The LQCDC is synthesized via dynamic programming in the stochastic setting. A pair of discrete-time modified matrix Riccati equations and a pair of modified matrix Lyapunov equations are constructed by using lagrangian multipliers and the matrix minimum principle. The performance cost is formulated as the conditional expectation of a quadratic functional adjoined with an equality constraint involving the dynamics of the conditional covariance of the closed-loop system state. Results of simulation experiments are presented to demonstrate the efficacy of the LQCDC for control of longitudinal motion of an advanced aircraft.

1. Introduction

Computer networks are often employed in complex dynamic systems, such as advanced aircraft, autonomous manufacturing and chemical plants, to interconnect the spatially distributed subsystems or components (Ray 1987). Time-division-multiplexed networks such as fibre distributed data interface protocols (Dykeman and Bux 1988) are usually adequate for meeting the requirements of data rate, data latency and reliability for information exchange, and have distinct advantages over conventional point-to-point connections in terms of reduced wiring, flexibility of operations and evolutionary design. However, for real-time control applications, the randomly varying distributed delays induced by the network could degrade the system performance because the timely transfer of sensor and control signals from one device to another is not guaranteed, and therefore these delays are sources of potential instability (Halevi and Ray 1988, Ray and Halevi 1988). An application example is future generation aircraft which are equipped with computer networks to serve as the communications link for the vehicle management system. The communication and control system specifications are expected to have very stringent standards on safety and performance.

Liou and Ray (1991) proposed the synthesis of a stochastic regulator based on the principles of dynamic programming and optimality. This control law follows the structure of the standard linear quadratic regulator (LQR) and is formulated in the presence of randomly varying delays from the controller to actuator under full state feedback with no plant noise and disturbances; we will refer to this control structure as the delay compensated linear quadratic regulator (DCLQR). Ray *et al.* (1993)

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formulated a minimum-variance state estimation filter to account for random delays in the measurements based on the stipulation that the state vector can be replaced by its estimate in this delay-compensated linear quadratic regulator. A control structure, called the delay compensated linear quadratic regulator. A control structure, called the delay compensated linear quadratic gaussian (DCLQG), has been proposed by Ray (1994) for the compensation of randomly varying distributed delays in the stochastic setting. However, the certainty equivalence property (Bar-Shalom and Tse 1974), which is valid for LQG, does not hold for DCLQG in general. Even worse, if the plant state vector is replaced by its minimum variance estimate in the optimal state feedback control law, then the resulting closed-loop control system, namely DCLQG, is not guaranteed to retain its nominal stability.

Hyland and Bernstein (1984) proposed optimal projection equations for a fixed structure compensator in the continuous-time setting in the presence of additive noise. Bernstein *et al.* (1986 a) proposed a fixed-structure sampled-data dynamic compensator for systems that are subjected to computation delays in control signal processing without and randomly varying delays such as those due to computer communication networks. Bernstein and Haddad (1987) presented the concept of optimal projection for discrete-time reduced-order dynamic compensation in the presence of multiplicative white noise in the plant model and sensor data. The controller and state estimator were coupled in the formulation of an optimal output feedback law. The optimal projection technique was used to arrive at the coupled structure of the reduced-order classical LQG problem. This approach, in fact, implies that the separation between state estimation and state feedback control is no longer valid, i.e. the certainty equivalence (Bar-Shalom and Tse 1974) breaks down.

The main objective of this paper is to formulate a methodology for synthesis of an optimal output feedback control law to compensate for the detrimental effects of randomly varying distributed delays. The resulting controller is referred to as linear quadratic coupled delay compensator (LQCDC) in which the state estimator and the state feedback controller cannot be separated. The concept of simultaneous design of the stochastic controller and state estimator is brought in to circumvent for the problem of violation of the certainty equivalence principle. Furthermore, the effects of the time skew due to mis-synchronization of the sampling instants of the (possibly) non-collocated digital controller and sensor are compensated for as well. Mean square stability of the stochastic control system has been established based on the nominal models of plant dynamics and delay statistics. Simulation results for longitudinal motion dynamics of an advanced aircraft are presented to compare the proposed LQCDC with DCLQR (Liou and Ray 1991) and DCLQG (Ray 1994). Although the modelling and analysis of uncertainties and the attendant issues of robustness are not addressed in this paper, simulation results are presented to illustrate the sensitivity of the LQCDC relative to probability distribution of the random delays.

Apparently, no analytical methods are available to practising engineers for the synthesis of control systems under randomly varying distributed delays. The work reported in this paper is intended to serve as a step toward formulation of a control synthesis methodology for dynamic processes that are subjected to randomly varying distributed delays. The paper is organized into seven sections including the introduction: § 2 presents the pertinent assumptions. The discretized plant and measurement models are established in § 3. The LQCDC law is proposed and proved in § 4. The design steps and their implications are briefly described in § 5. Results of simulation experiments under randomly varying distributed delays are presented in § 6 to demonstrate the super performance of the LQCDC law relative to other control laws. Finally, the paper is summarized and concluded in § 7, along with recommendations for future research.

2. Pertinent assumptions

The pertinent assumptions needed to formulate the algorithm of the linear quadratic coupled delay compensator (LQCDC) are delineated below. The underlying justifications are laid out in the paragraphs following each of these assumptions.

Assumption 1: *The sensor and controller have the same sampling period T with a skew Δ_S between the sensor and controller sampling instants; Δ_S is a very slowly varying parameter to be periodically reset, and it is treated as a constant parameter.*

As the clock rates for the sensor and controller computers are almost identical with zero probability of being exactly identical, the skew Δ_S varies quasi-statically.

Assumption 2: *The delay Δ_p in processing of the control signal is a constant. Therefore, the gross time skew between the instants of sensor and control signal generation, which is equal to the sum $\Delta = \Delta_S + \Delta_p$ is also a constant. The processing delay Δ_p is set to zero without loss of generality.*

The actual delay in the control signal processing is of the order of microseconds in a high-speed control computer, whereas the sampling period T is at least of the order of tens of milliseconds.

Assumption 3: *The random delays from sensor to controller θ^k_{sc} and that from controller to actuator t^k are mutually independently and identically distributed (i.i.d.), and each of them is white. Each of the induced delays has* a priori *known statistics, and is bounded between* 0 *and T with probability* 1. *Therefore, the number of sensor signals arriving at the controller terminal is* 0, 1 *or* 2 *during a controller sampling period $[kT, (k+1)T)$. On the other hand, exactly one control signal arrives at the actuator during a sampling period relative to the controller time frame.*

This assumption follows the standard practice of network design (Ray 1987, 1994) in which the maximum data latency is constrained not to exceed the sampling interval.

Assumption 4: *In the sensor time frame, the plant disturbance, sensor noise and binary measurement delay sequences $\{\omega^s_k\}$, $\{\upsilon^s_k\}$, and $\{\zeta_k\}$, respectively, are mutually independent.*

These assumptions are in line with the standard LQG problem and they represent an approximation of the real situation for the mathematical tractability of control systems analysis.

Assumption 5: *The sampler is ideal, and the digital-to-analogue conversion is implemented via a zero-order-hold (ZOH). The actuator operates as a continuous device, i.e. the control input acts on the plant immediately after its arrival at the actuator terminal.*

This assumption is justified in view of the fact that the actuator is equipped with a dedicated microprocessor which is sampled much more quickly (ten times or more, for example) than the controller computer.

Assumption 6: *The probability of data loss, due to noise in the communication medium and protocol malfunctions, is zero.*

The data communication network in future generation aircraft is expected to be based on fibre-optics in which the bit error rate is extremely small (10^{-12} or less, for example).

3. Discretized system modelling

The discrete-time frame k is based on the controller sampling instant instead of the sensor sampling instant, except for certain specific cases. Consider a continuous-time, linear, finite-dimensional plant model and the associated measurement model:

$$\dot{\xi}(t) = a(t)\xi(t) + b(t)u(t) + g(t)\omega(t) \tag{1}$$

$$y(t) = c(t)\xi(t) = \upsilon(t) \tag{2}$$

where the plant state $\xi(t) \in \mathbb{R}^n$, control $u(t) \in \mathbb{R}^n$ and measurement $y(t) \in \mathbb{R}^r$; matrices $a(t)$, $b(t)$, $c(t)$ and $g(t)$ are real deterministic with appropriate dimensions; $\omega(t)$ and $\upsilon(t)$ are independent, zero-mean, white noise for the plant and sensor models, respectively.

To distinguish from controller time frame, the superscript s is used to particularly indicate the time instant based on the sensor time frame, i.e.

$$(\cdot)_k^{\mathrm{s}} \equiv (\cdot)_{k-\delta} \tag{3}$$

where $(\cdot)$ is any arbitrary vector or matrix which may be hereafter referred to. The normalized time skew $\delta := \Delta/T$ where Δ represents the time skew due to mis-synchronization of sensor and controller sampling instants, and T is the sampling period.

Two sets of input matrices $\{\varphi_0^k, \varphi_1^k\}$ and $\{\psi_0^{k-1}, \psi_1^{k-1}\}$ during the kth sensor's sampling period $[kT - \Delta, (k+1)T - \Delta)$ are defined as follows:

$$\varphi_0^k = \begin{cases} \displaystyle\int_{kT+t^k}^{(k+1-\delta)T} \Phi((k+1-\delta)T, \tau) b(\tau)\, \mathrm{d}\tau, & \text{if } t^k < (1-\delta)T \\ 0, & \text{if } t^k \geq (1-\delta)T \end{cases} \tag{4a}$$

$$\varphi_1^k = \int_{kT}^{(k+1-\delta)T} \Phi((k+1-\delta)T, \tau) b(\tau)\, \mathrm{d}\tau - \varphi_0^k \tag{4b}$$

$$\psi_0^k = \begin{cases} \displaystyle\int_{kT+t^k}^{(k+1)T} \Phi((k+1)T, \tau) b(\tau)\, d\tau, & \text{if } t^k \geq (1-\delta)T \\ \displaystyle\int_{(k+1-\delta)T}^{(k+1)T} \Phi((k+1)T, \tau) b(\tau)\, d\tau, & \text{if } t^k < (1-\delta)T \end{cases} \tag{4c}$$

$$\psi_1^k = \int_{(k+1-\delta)T}^{(k+1)T} \Phi((k+1)T, \tau) b(\tau)\, d\tau - \psi_0^k \tag{4d}$$

$\Phi(\cdot, \cdot)$ is the state transition matrix. The random time epoch t^k is the arrival instant of control command u_k at the actuator where the instant of the kth controller sampling is set to be the time origin. That is, $t^k = 0$ if the control command u_k arrives exactly at the kth controller sampling instant. Based on Assumption 2, the plant model and its measurement based on sensor time frame now can be described as follows.

3.1. *Plant model*

$$\xi_{k+1}^s = \Phi_k^s \xi_k^s + \sum_{i=0}^{2} \beta_i^k u_{k-i} + \omega_k^s \tag{5a}$$

where

$$\Phi_k^s = \Phi((k+1-\delta)T, (k-\delta)T) \tag{5b}$$

$$\beta_0^k = \varphi_0^k \tag{5c}$$

$$\beta_1^k = \varphi_1^k + \psi_0^{k-1} \tag{5d}$$

$$\beta_2^k = \psi_1^{k-1} \tag{5e}$$

$$\omega_k^s = \int_{(k-\delta)T}^{(k+1-\delta)T} \Phi((k+1-\delta)T, \tau) g(\tau) \omega(\tau)\, d\tau \tag{5f}$$

Φ_k^s is defined as the state transition matrix from the kth to the $(k+1)$th sampling instant in the sensor time frame; u_k, u_{k-1} and u_{k-2} are the three consecutive control commands applied during the time interval $[(k-\delta)T, (k+1-\delta)T)$; ω_k^s is the zero-mean white discretized plant noise.

3.2. *Measurement model*

We have considered continuous-time processes with sampled data sensing (Franklin *et al.* 1990) similar to what is practised in aircraft inertial navigational system (INS) sensors and power and chemical plant instrumentation. In reality, the standard deviation of the measurement noise is often reduced by sampled data averaging but it can never be eliminated. The sampled measurements are modelled in the following form regardless of whether the A/D device in the sensing system uses sample averaging (Bernstein *et al.* 1986a).

$$y_k^s = c_k^s \xi_k^s + \upsilon_k^s \tag{6}$$

In (5) and (6) the noise vectors ω_k^s and υ_k^s are assumed to be zero-mean, mutually independent, white gaussian and strictly stationary with symmetric covariances V_1

and V_2, respectively. As usual, V_1 is assumed to be non-negative definite and V_2 positive definite, respectively. In addition, the random sequence $\{\omega_k^s\}$ or $\{\upsilon_k^s\}$ is individually assumed to be independently identically distributed.

The sensor data to be used for generating the $(k+1)$th control command, denoted as z_k, is subjected to binary random delays such that $z_k = y_k^s$ or $z_k = y_{k-1}^s$, depending on whether the fresh sensor data or the previous data is to be used for generation of u_{k+1}. That is

$$z_k = (1 - \zeta_k) y_k^s + \zeta_k y_{k-1}^s \tag{7}$$

where $\{\zeta_k\}$ is the random delay sequence from sensor to controller with binary distribution with the expected value

$$E[1 - \zeta_k] = \alpha_k \tag{8}$$

It is noted that α_k, in fact, represents the probability of timely arrival of fresh sensor data, y_k^s. With z_k as the dynamic output feedback instead of y_k^s, which is normally used in the standard stochastic output feedback regulation problem, a linear delay compensated state estimator is proposed:

$$\hat{\xi}_{k+1}^s + \hat{L}_k \hat{\xi}_k^s + \hat{K}_k z_k + \sum_{i=0}^{2} \beta_i^k u_{k-i} \tag{9}$$

Remark 1: The plant state estimator in (9) is of full order. The rationale for a full-order plant state estimator is that the sensor data are subjected to communication network delays and noise in addition to the sensor noise. A reduced-order estimator is not suitable because of direct feedback of the delayed sensor data into the control signal. This is avoided by constructing a full-order estimator. □

To compensate the random delays of control commands, the control law is constructed to include the feedback of three past consecutive control signals.

$$u_k = \hat{f}_1^k u_{k-1} + \hat{f}_2^k u_{k-2} + \hat{f}_3^k u_{k-3} + \hat{F}_k \hat{\xi}_k^s = \hat{F}_{\mathrm{au}}^k \hat{x}_{\mathrm{au}}^k \tag{10a}$$

where

$$\hat{F}_{\mathrm{au}}^k \equiv \left[\hat{f}_1^k \ \hat{f}_2^k \ \hat{f}_3^k \ \hat{F}_k\right] \text{ is the augmented control gain matrix} \tag{10b}$$

and

$$\hat{x}_{\mathrm{au}}^k \equiv \left[u_{k-1}^{\mathrm{T}} \ u_{k-2}^{\mathrm{T}} \ u_{k-3}^{\mathrm{T}} \ (\hat{\xi}_k^s)^{\mathrm{T}}\right]^{\mathrm{T}} \text{ is the augmented state estimate.} \tag{10c}$$

With the augmentation of the plant state to include state estimate and the three past consecutive control commands, the closed-loop system under randomly varying distributed delays is therefore established as follows.

3.3. *Closed-loop model*

$$\tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{G}_k \tilde{\Omega}_k \tag{11}$$

where

$$\tilde{x}_k = \left[(\xi_k^s)^{\mathrm{T}} \quad u_{k-1}^{\mathrm{T}} \quad u_{k-2}^{\mathrm{T}} \quad u_{k-3}^{\mathrm{T}} \quad (\hat{\xi}_k^s)^{\mathrm{T}} \right. \tag{12}$$

$$\tilde{A}_k = \begin{bmatrix} A^k_{\mathrm{au}} + \Delta^k_A & B^k_{\mathrm{au}}\hat{F}_k \\ \hat{K}_k C^k_{\mathrm{au}} + \Delta^k_C & \hat{L}_{\mathscr{C}\mathscr{L}} \end{bmatrix} \tag{13a}$$

$$A^k_{\mathrm{au}} = \begin{bmatrix} \Phi^{\mathrm{s}}_k & \beta^k_1 & \beta^k_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_m & 0 \end{bmatrix} \tag{13b}$$

$$B^k_{\mathrm{au}} = \begin{bmatrix} (\beta^k_0)^{\mathrm{T}} & I_m & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{14}$$

$$C^k_{\mathrm{au}} \equiv \begin{bmatrix} c^k_0 & c^k_{q1} & c^k_{q2} & c^k_{q3} \end{bmatrix} \tag{15a}$$

$$c^k_0 \equiv (1 - \zeta_k) c^{\mathrm{s}}_k + \zeta_k c^{\mathrm{s}}_{k-1} (\Phi^{\mathrm{s}}_{k-1})^{-1} \tag{15b}$$

$$c^k_{q1} \equiv - \zeta_k c^{\mathrm{s}}_{k-1} (\Phi^{\mathrm{s}}_{k-1})^{-1} \beta^{k-1}_0 \tag{15c}$$

$$c^k_{q2} \equiv - \zeta_k c^{\mathrm{s}}_{k-1} (\Phi^{\mathrm{s}}_{k-1})^{-1} \beta^{k-1}_1 \tag{15d}$$

$$c^k_{q3} \equiv - \zeta_k c^{\mathrm{s}}_{k-1} (\Phi^{\mathrm{s}}_{k-1})^{-1} \beta^{k-1}_2 \tag{15e}$$

$$\hat{L}_{\mathscr{C}\mathscr{L}} = \hat{L}_k + \Delta^k_0 \tag{16a}$$

$$\Delta^k_0 = \beta^k_0 \hat{F}_k \tag{16b}$$

$$\Delta^k_A = B^k_{\mathrm{au}} \hat{f}^k_{\mathrm{au}} \Pi_{01} \tag{17a}$$

$$\Pi_{01} = \begin{bmatrix} 0_{3m \times n} & I_{3m} \end{bmatrix} \tag{17b}$$

$$\hat{f}^k_{\mathrm{au}} = \begin{bmatrix} \hat{f}^k_1 & \hat{f}^k_2 & \hat{f}^k_3 \end{bmatrix} \tag{17c}$$

$$\Delta^k_{\mathrm{C}} = \begin{bmatrix} 0_n & \Delta^k_{q1} & \Delta^k_{q2} & \Delta^k_{q3} \end{bmatrix} \tag{18a}$$

$$\Delta^k_{q1} = \beta^k_1 + \beta^k_0 \hat{f}^k_1 \tag{18b}$$

$$\Delta^k_{q2} = \beta^k_2 + \beta^k_0 \hat{f}^k_2 \tag{18c}$$

$$\Delta^k_{q3} = \beta^k_0 \hat{f}^k_3 \tag{18d}$$

$$\tilde{G}_k = \begin{bmatrix} I_{n \times n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g^k_{q1} & g^k_{q2} & g^k_{q3} \end{bmatrix} \tag{19a}$$

$$g^k_{q1} = - \zeta_k \hat{K}_k c^{\mathrm{s}}_{k-1} (\Phi^{\mathrm{s}}_{k-1})^{-1} \tag{19b}$$

$$g^k_{q2} = (1 - \zeta_k) \hat{K}_k \tag{19c}$$

$$g_{q3}^k = \zeta_k \hat{K}_k \tag{19d}$$

$$\tilde{\Omega}_k = \left[(\omega_k^s)^{\mathrm{T}} \quad (\omega_{k-1}^s)^{\mathrm{T}} \quad (\upsilon_k^s)^{\mathrm{T}} \quad (\upsilon_{k-1}^s)^{\mathrm{T}} \right]^{\mathrm{T}} \tag{20}$$

It is noted that the triplet $(A_{\mathrm{au}}^k, B_{\mathrm{au}}^k, C_{\mathrm{au}}^k)$ is nothing but the state-space representation of the delayed system based on the sensor time frame. Thre noise sequence $\{\tilde{\Omega}_k\}$ is neither white nor Markov. If $\Delta_A^k = 0$ and $\Delta_C^k = 0$, the closed-loop system matrix $\tilde{A}_k$ reduces to a linear quadratic guassian (LQG) model.

In this paper the linear quadratic random delay compensation law characterized by $\{\hat{L}_k^*, \hat{K}_k^*, (\hat{F}_{\mathrm{au}}^k)^*\}$ is to be constructed to minimize the performance cost functional

$$\tilde{J}_k(\hat{L}_k, \hat{K}_k, \hat{F}_{\mathrm{au}}^k, \tilde{Z}_{k-1}) = \tfrac{1}{2} E\left[\tilde{x}_k^{\mathrm{T}} \tilde{R}_k \tilde{x}_k \middle| \tilde{Z}_{k-1}\right] + E\left[\tilde{J}_{k+1}^*(\tilde{Z}_k) \middle| \tilde{Z}_{k-1}\right] \tag{21a}$$

with the terminal condition

$$\tilde{J}_N^*(\tilde{Z}_{N-1}) = \tfrac{1}{2} E\left[\tilde{x}_N^{\mathrm{T}} \tilde{R}_N \tilde{x}_N \middle| \tilde{Z}_{N-1}\right] \tag{21b}$$

where

$$\tilde{J}_k^*(\tilde{Z}_{k-1}) \equiv \tilde{J}_k(\hat{L}_k^*, \hat{K}_k^*, (\hat{F}_{\mathrm{au}}^k)^*, \tilde{Z}_{k-1}) \tag{22}$$

$$\tilde{R}_k = \begin{bmatrix} R_1 & 0_{n\times(n+3m)} \\ 0_{(n+3m)\times n} & (\hat{F}_{\mathrm{au}}^k)^2 \hat{F}_{\mathrm{au}}^k \end{bmatrix} \tag{23}$$

$$R_N = \begin{bmatrix} \Re & 0_{n\times(n+3m)} \\ 0_{(n+3m)\times n} & 0_{n+3m} \end{bmatrix} \tag{24}$$

The superscript * is used to denote optimality; $\tilde{Z}_k$ is the random delay history: $\tilde{Z}_k \equiv \{\zeta_k, \zeta_{k-1}, \ldots, \zeta_1; t^k, t^{k-1}, \ldots, t^1\}$; N is the time horizon over which the performance is evaluated; $R_1 \geq 0$ and $R_2 > 0$ are the state deviation and control penalty matrices, respectively, as $k < N$; $\Re \geq 0$ is the final-stage state deviation matrix at $k = N$. For the synthesis of the compensator triplet $\{\hat{L}_k, \hat{K}_k, \hat{F}_{\mathrm{au}}^k\}$, the estimation gain matrix $\hat{K}_k$ and the control gain matrix $\hat{F}_{\mathrm{au}}^k$ are identified off-line; and the open-loop state transition matrix $\hat{L}_k$ of the stochastic state etimator is calculated on-line to take advantage of all available recorded data.

Remark 2: As explained in Remark 1, there is no direct feedback of the measurements in the feedback control law proposed in (10a), i.e. the direct coupling matrix is zero in the control law. Therefore, the performance functional in (21a) does not include the cross-weighting of the state and control vectors (see equation (3.3) and Remark 4.3 of Hyland and Kapila (1995)). □

4. Formulation of the LQCDC law

If the augmented state transition matrix of the closed-loop system $\tilde{A}_k$ is stable in the mean square sense, then the compensator $\{\hat{L}_k, \hat{K}_k, \hat{F}_{\mathrm{au}}^k\}$ is mean-square stabilizing. Therefore, attention is focused on the set of stabilizing and minimum compensator:

$$\hat{S}_k = \left\{(\hat{L}_k, \hat{K}_k, \hat{F}_{\mathrm{au}}^k) \middle| \tilde{A}_k \text{ is stable in mean square sense and } \{\hat{L}_k, \hat{K}_k, \hat{F}_{\mathrm{au}}^k\} \text{ is minimal}\right\}$$

The following lemma for projective factorization (Bernstein *et al.* 1986b) is summarized and required to prove Proposition 1.

Lemma 1: *Let* $\tau \in \mathbb{R}^{n_{\mathrm{au}} \times n_{\mathrm{au}}}$, *then*

$$\tau^2 = \tau \tag{25}$$

$$\operatorname{rank}(\tau) = n_c \tag{26}$$

if and only if there exist G *and* $\Gamma \in \mathbb{R}^{n_c \times n_{\mathrm{au}}}$ *such that*

$$G^{\mathrm{T}} \Gamma = \tau \tag{27}$$

$$\Gamma G^{\mathrm{T}} = I_{n_c} \tag{28}$$

Proof: For the proof, see Bernstein *et al.* (1986a,b).

Let G and Γ satisfying (27) and (28) be named as a projective factorization of τ. Furthermore, define the set of contragradiently diagonalizing transformation (Rao and Mitra 1971) as

$$\tilde{\mathcal{D}}(Q,P) \equiv \left\{\Psi \in \mathbb{R}^{n \times n} : \Psi^{-1} Q \Psi^{-\mathrm{T}} \text{and } \Psi^{\mathrm{T}} P \Psi \text{ are diagonal}\right\}$$

It directly follows from Theorem 6.2.5 of Rao and Mitra (1971) that $\tilde{\mathcal{D}}(Q,P)$ is always non-empty if Q and P are both symmetric and non-negative definite. □

Define the conditional covariance matrix for zero-input regulation problems as

$$Q_k = E\left[\tilde{x}_k \quad \tilde{x}_k^{\mathrm{T}} \middle| \tilde{Z}_{k-1}\right] \tag{29}$$

which has the dynamics presented in Lemma 2 below.

Lemma 2:

$$Q_{k+1} = E\left[\tilde{A}_k Q_k \tilde{A}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] + \tilde{V}_{\mathrm{eq}}^k, \quad k = 1,2,\ldots,N-1 \tag{30}$$

where

$$\tilde{V}_{\mathrm{eq}}^k = \begin{bmatrix} V_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{V}_2^k \end{bmatrix} \tag{31a}$$

$$\tilde{V}_2^k = \hat{K}_k \left\{ V_2 + E\left[\zeta_k^2 \middle| \tilde{Z}_k\right] c_{k-1}^{\mathrm{s}} (\Phi_{k-1}^{\mathrm{s}})^{-1} V_1 \left[c_{k-1}^{\mathrm{s}} (\Phi_{k-1}^{\mathrm{s}})^{-1}\right]^{\mathrm{T}} \right\} \hat{K}_k^{\mathrm{T}} \tag{31b}$$

Proof: It follows from (11) that

$$Q_{k+1} = E\left[\tilde{A}_k \tilde{x}_k \tilde{x}_k^{\mathrm{T}} \tilde{A}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] + E\left[\tilde{A}_k \tilde{x}_k \tilde{\Omega}_k^{\mathrm{T}} \tilde{G}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] + E\left[\tilde{G}_k \tilde{\Omega}_k \tilde{x}_k^{\mathrm{T}} \tilde{A}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] + E\left[\tilde{G}_k \tilde{\Omega}_k \tilde{\Omega}_k^{\mathrm{T}} \tilde{G}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] \tag{32}$$

As $\tilde{A}_k = \tilde{A}_k(\tilde{Z}_k)$

$$E\left[\tilde{A}_k \tilde{x}_k \tilde{x}_k^{\mathrm{T}} \tilde{A}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] = E\left[\tilde{A}_k E\left[\tilde{x}_k \tilde{x}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] \tilde{A}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] \tag{33}$$

From the closed-loop system model in (11), the delay data $\{\zeta_k, t^k\}$ provided by $\tilde{Z}_k$ is superfluous in the conditional expectation $E\left[\tilde{x}_k \tilde{x}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right]$. Therefore

$$E\left[\tilde{A}_k \tilde{x}_k \tilde{x}_k^{\mathrm{T}} \tilde{A}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] = E\left[\tilde{A}_k Q_k \tilde{A}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] \tag{34}$$

For estimation process, randomly delayed measurement z_k is available when $\tilde{Z}_k$ has occurred. Hence the plant state ξ_{k+1}^{s} and its estimation $\hat{\xi}_{k+1}^{\mathrm{s}}$ are always uncorrelated for given $\tilde{Z}_k$, i.e.

$$E\left[\xi_{k+1}^{\mathrm{s}} (\hat{\xi}_{k+1}^{\mathrm{s}})^{\mathrm{T}} \middle| \tilde{Z}_k\right] = E\left[\xi_{k+1}^{\mathrm{s}} \middle| \tilde{Z}_k\right] E\left[(\hat{\xi}_{k+1}^{\mathrm{s}})^{\mathrm{T}} \middle| \tilde{Z}_k\right] \tag{35}$$

Moreover, because the plant disturbance and sensor noise are both assumed to be zero-mean, the noise cross-covariance between the plant state ξ_{k+1}^{s} and its estimation $\hat{\xi}_{k+1}^{\mathrm{s}}$ is zero. Similarly

$$E\left[u_m (\hat{\xi}_{k+1}^{\mathrm{s}})^{\mathrm{T}} \middle| \tilde{Z}_k\right] = E\left[u_m \middle| \tilde{Z}_k\right] E\left[(\hat{\xi}_{k+1}^{\mathrm{s}})^{\mathrm{T}} \middle| \tilde{Z}_k\right] \tag{36a}$$

$$E\left[u_m (\xi_{k+1}^{\mathrm{s}})^{\mathrm{T}} \middle| \tilde{Z}_k\right] = E\left[u_m \middle| \tilde{Z}_k\right] E\left[(\xi_{k+1}^{\mathrm{s}})^{\mathrm{T}} \middle| \tilde{Z}_k\right] \tag{36b}$$

where $m = k$, $k = 1, k - 2$. Equations (35) and (36) imply that the noise cross-covariances in (32) are all zeros.

$$E\left[\tilde{A}_k \tilde{x}_k \tilde{\Omega}_k^{\mathrm{T}} \tilde{G}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right] = 0 \tag{37}$$

Lastly, calculating $E\left[\tilde{G}_k \tilde{\Omega}_k \tilde{\Omega}_k^{\mathrm{T}} \tilde{G}_k^{\mathrm{T}} \middle| \tilde{Z}_k\right]$ yields $\tilde{V}_{\mathrm{eq}}^k$. Equation (32) is reduced to (30). □

The main results are now presented as Proposition 1 to provide an optimal solution to the control problem under randomly varying delays.

Proposition 1: *Let the random delay history, $\{t^j, \zeta_j; j = 0, \ldots, k-1\}$, be available for synthesizing the control command u_k and the optimal linear quadratic coupled delay compensator (LQCDC) triplet $\{\hat{L}_k^*, \hat{K}_k^*, (\hat{F}_{\mathrm{au}}^k)^*\} \in \hat{S}_k$ at the kth sensor sampling interval. Then the optimal LQCDC, consisting of a linear modified dynamic output feedback controller in which the state estimator is embedded to circumvent the randomly varying distributed delays, can be presented as follows.*

For $k = 0, 1, \ldots, N-1$, there exist non-negative definite matrices, $\bar{\bar{Q}}_x^k$, $\hat{\hat{Q}}_x^k$, $\bar{\bar{P}}_x^k$ and $\hat{\hat{P}}_x^k$ all with dimensions $(n+3m) \times (n+3m)$ such that the following state estimation law and control law are formed.

State estimation law:

$$\hat{\xi}_{k+1}^{\mathrm{s}} = \hat{L}_k^* \hat{\xi}_k^{\mathrm{s}} + \hat{K}_k^* z_k + \sum_{i=0}^{2} \beta_i^k u_{k-i} \tag{38}$$

Optimal control law:

$$u_k = (\hat{F}_{\mathrm{au}}^k)^* \hat{x}_{\mathrm{au}}^k \tag{39}$$

where

$$\hat{L}_k^* = \Gamma_k((A_{\mathrm{au}}^k + \Delta_A^k) - B_{\mathrm{au}}^k (\hat{R}_{2\mathrm{s}}^k)^{-1} \hat{P}_{x0}^k - \hat{Q}_{\mathrm{s}}^k (\hat{V}_{2\mathrm{s}}^k)^{-1} C_{\mathrm{au}}^k] G_k^{\mathrm{T}}$$
$$- \beta_0^k \hat{F}_k^* - (\Delta_{\mathrm{C}}^k)^* G_k^{\mathrm{T}} \tag{40a}$$

$$(\Delta_C^k)^* = \left[0_n \quad \beta_1^k + \beta_0^k (\hat{f}_1^k)^* \quad \beta_2^k + \beta_0^k (\hat{f}_2^k)^* \quad \beta_0^k (\hat{f}_3^k)^*\right] \tag{40b}$$

$$\hat{K}_k^* = \Gamma_k \hat{Q}_s^k (\hat{V}_{2s}^k)^{-1} \tag{41}$$

$$(\hat{F}_{au}^k)^* = \left[\hat{F}_k^* \quad (\hat{f}_1^k)^* \quad (\hat{f}_2^k)^* \quad (\hat{f}_3^k)^*\right] \tag{42a}$$

$$\hat{F}_k^* = -(\hat{R}_{2s}^k)^{-1} \hat{P}_{x0}^k G_k^{\mathrm{T}} \tag{42b}$$

$$(\hat{f}_1^k)^* = -(\hat{R}_{2s}^k)^{-1} \hat{P}_{q1} \tag{42c}$$

$$(\hat{f}_2^k)^* = -(\hat{R}_{2s}^k)^{-1} \hat{P}_{q2} \tag{42d}$$

$$(\hat{f}_3^k)^* = 0_{m\times m} \tag{42e}$$

with the following matrix definitions:

$$\hat{V}_{2s}^k = \bar{\bar{V}}_2^k + E\left[C_{au}^k \bar{\bar{Q}}_x^k (C_{au}^k)^{\mathrm{T}}\right] \tag{43a}$$

$$\bar{\bar{V}}_2^k = V_2 + (1-\alpha_k) c_{k-1}^s (\Phi_{k-1}^s)^{-1} V_1 \left[c_{k-1}^s (\Phi_{k-1}^s)^{-1}\right]^{\mathrm{T}} \tag{43b}$$

$$\hat{Q}_s^k = E\left\{\left[A_{au}^k + \Delta_A^k - G_{k+1}^{\mathrm{T}} (\Delta_C^k)^*\right] \bar{\bar{Q}}_x^k (C_{au}^k)^{\mathrm{T}}\right\} \tag{44}$$

$$\hat{R}_{2s}^k = R_2 + E\left[(B_{au}^k)^{\mathrm{T}} \bar{\bar{Q}}_x^k B_{au}^k\right] \tag{45}$$

$$\hat{P}_{xs}^k = E\left[(B_{au}^k)^{\mathrm{T}} \bar{\bar{P}}_x^k A_{au}^k\right] \tag{46a}$$

$\hat{P}_{xs}^k$ can be partitioned into four submatrices of dimension $m\times n$, $m\times m$, $m\times m$ and $m\times m$ as follows:

$$\bar{P}_{xs}^k = \left[\hat{P}_0^k \quad \hat{P}_{q1}^k \quad \hat{P}_{q2}^k \quad 0_{m\times m}\right] \tag{46b}$$

based on which $\hat{P}_{x0}^k$ in (42b) is defined as

$$\hat{P}_{x0}^k = \left[\hat{P}_0^k \quad 0_{m\times 3m}\right] \tag{47}$$

where $\bar{\bar{Q}}_x^k$, $\hat{\hat{Q}}_x^k$, $\bar{\bar{P}}_x^k$ and $\hat{\hat{P}}_x^k$ satisfy the following pairs of modified matrix Riccati and Lyapunov equations.

$$\begin{aligned}
\bar{\bar{Q}}_x^{k+1} = {} & E(A_{au}^k + \Delta_A^k) \bar{\bar{Q}}_x^k (A_{au}^k + \Delta_A^k)^{\mathrm{T}} \left| \tilde{Z}_k \right] + \bar{\bar{V}}_x^k + \tau_\perp^k \hat{\hat{Q}}_x^{k+1} (\tau_\perp^k)^{\mathrm{T}} \\
& - \hat{Q}_s^k (\hat{V}_{2s}^k)^{-1} (\hat{Q}_s^k)^{\mathrm{T}} - G_{k+1}^{\mathrm{T}} E\left[\Delta_C^k \bar{\bar{Q}}_x^k (\Delta_C^k)^{\mathrm{T}} \left| \tilde{Z}_k \right.\right] G_{k+1} \\
& - \hat{Q}_s^k (\hat{V}_{2s}^k)^{-1} E\left[C_{au}^k \bar{\bar{Q}}_x^k (\Delta_C^k)^{\mathrm{T}} \left| \tilde{Z}_k \right.\right] G_{k+1} \\
& - \left\{\hat{Q}_s^k (\hat{V}_{2s}^k)^{-1} E\left[C_{au}^k \bar{\bar{Q}}_x^k (\Delta_C^k)^{\mathrm{T}} \left| \tilde{Z}_k \right.\right] G_{k+1}\right\}^{\mathrm{T}}
\end{aligned} \tag{48}$$

$$\hat{\hat{Q}}_x^{k+1} = E\Big\{\big[A_{\mathrm{au}}^k + \Delta_A^k - B_{\mathrm{au}}^k(\hat{R}_{2\mathrm{s}}^k)^{-1}\hat{P}_{x0}^k\big]\tau_{k-1}\hat{\hat{Q}}_x^k\tau_{k-1}^{\mathrm{T}}\big[A_{\mathrm{au}}^k + \Delta_A^k - B_{\mathrm{au}}^k(\hat{R}_{2\mathrm{s}}^k)^{-1}\hat{P}_{x0}^k\big]^{\mathrm{T}}\Big|\tilde{Z}_k\Big\}$$
$$+ \hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}(\hat{Q}_{\mathrm{s}}^k)^{\mathrm{T}} + G_{k+1}^{\mathrm{T}}E\big[\Delta_{\mathrm{C}}^k\bar{\bar{Q}}_x^k(\Delta_{\mathrm{C}}^k)^{\mathrm{T}}\big|\tilde{Z}_k\big]G_{k+1}$$
$$+ \hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}E\big[C_{\mathrm{au}}^k\bar{\bar{Q}}_x^k(\Delta_{\mathrm{C}}^k)^{\mathrm{T}}\big|\tilde{Z}_k\big]G_{k+1}$$
$$+ \big\{\hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}E\big[C_{\mathrm{au}}^k\bar{\bar{Q}}_x^k(\Delta_{\mathrm{C}}^k)^{\mathrm{T}}\big|\tilde{Z}_k\big]G_{k+1}\big\}^{\mathrm{T}} \quad (49)$$

$$\bar{\bar{P}}_x^{k+1} = E\big[(A_{\mathrm{au}}^k)^{\mathrm{T}}\bar{\bar{P}}_x^kA_{\mathrm{au}}^k\big|\tilde{Z}_k\big] + \bar{\bar{R}}_x^k + (\tau_{\perp}^{k-1})^{\mathrm{T}}\hat{\hat{P}}_x^{k-1}\tau_{\perp}^{k-1}$$
$$- (\hat{P}_{x\mathrm{s}}^k)^{\mathrm{T}}(\hat{R}_{x\mathrm{s}}^k)^{-1}\hat{P}_{x\mathrm{s}}^k \quad (50)$$

$$\hat{\hat{P}}_x^{k-1} = E\Big\{\big[A_{\mathrm{au}}^k + \Delta_A^k - \hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}C_{\mathrm{au}}^k - G_{k+1}^{\mathrm{T}}\Delta_C^k\big]^{\mathrm{T}}\tau_k^{\mathrm{T}}\hat{\hat{P}}_x^k\tau_k$$
$$\times\big[A_{\mathrm{au}}^k + \Delta_A^k - \hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}C_{\mathrm{au}}^k - G_{k+1}^{\mathrm{T}}\Delta_C^k\big]\Big|\tilde{Z}_k\Big\} + (\hat{P}_{x0}^k)^{\mathrm{T}}(\hat{R}_{2s}^k)^{-1}\hat{P}_{x0}^k \quad (51)$$

where

$$\bar{\bar{R}}_x^k = \begin{bmatrix} R_1 & 0_{n\times 3m} \\ 0_{3m\times n} & 0_{3m} \end{bmatrix} \quad (52)$$

$$\bar{\bar{V}}_x^k = \begin{bmatrix} V_1 & 0_{n\times 3m} \\ 0_{3m\times n} & 0_{3m} \end{bmatrix} \quad (53)$$

$$\tau_{\perp}^k = I_{n\times 3m} - \tau_k \quad (54)$$

and τ_k is the optimal projection matrix, defined as

$$\tau_k = G_{k+1}^{\mathrm{T}}\Gamma_k \quad (55)$$

with the factorization

$$\tau_k = \Psi_k I_x \Psi_k^{-1} \quad (56a)$$

$$I_x = \begin{bmatrix} I_n & 0_{n\times 3m} \\ 0_{3m\times n} & 0_{3m\times 3m} \end{bmatrix} \quad (56b)$$

for some $\Psi_k \in \tilde{\mathcal{D}}(\hat{\hat{Q}}_x^{k+1}, \hat{\hat{P}}_x^k)$ such that $\Psi_k^{-1}(\hat{\hat{Q}}_x^{k+1}\hat{\hat{P}}_x^k)\Psi_k$ is diagonal.

Proof:

(1) *Create the langrangian to be minimized.* By applying the conventional minimization method using N lagrangian multipliers P_k $0 \le k \le N-1$. we define lagrangian $\mathscr{L}_k$, $0 \le k \le N-1$, by adjoining the following equality constraint obtained from Lemma 2 to the performance cost functional, as follows:

$$\mathscr{L}_k(Q_k, Q_{k+1}, \ldots, Q_N; P_k, P_{k+1}, \ldots, P_{N-1}; \hat{L}_k, \hat{K}_k, \hat{F}_{\mathrm{au}}^k, \tilde{Z}_k)$$
$$= \tfrac{1}{2}\mathrm{tr}\,\big\{Q_k\tilde{R}_k + (Q_{k+1} - E\big[\tilde{A}_kQ_k\tilde{A}_k^{\mathrm{T}}\big|\tilde{Z}_k\big] + \tilde{V}_{\mathrm{eq}}^k)P_k\big\}$$
$$+ E\big[\mathscr{L}_{k+1}(Q_{k+1}, Q_{k+2}, \ldots, Q_N; P_{k+1}, P_{k+2}, \ldots, P_{N-1}; \tilde{Z}_{k+1})\big|\tilde{Z}_k\big],$$
$$0 \le k \le N-1 \quad (57a)$$

with the terminal condition

$$\mathscr{J}_N^*(Q_N) = Q_N\tilde{\mathfrak{R}} \tag{57b}$$

where

$$\begin{aligned}\mathscr{J}_k(Q_k, Q_{k+1}, \dots, Q_N; P_k, P_{k+1}, \dots, P_{N-1}; \tilde{Z}_k) \\ = \mathscr{J}_k^*(Q_k, Q_{k+1}, \dots, Q_N; P_k, P_{k+1}, \dots, P_{N-1}; \hat{L}_k^*, \hat{K}_k^*, (\hat{F}_{\mathrm{au}}^k)^*; \tilde{Z}_k), \\ 0 \le k \le N-1\end{aligned} \tag{58}$$

Gor simplicity and consistency, Q_k, P_k and $\tilde{V}_{\mathrm{eq}}^k$ are partitioned into $(n+3m)\times(n+3m)$, $(n+3m)\times n$ and $n \times n$ submatrices, respectively, as follows:

$$Q_k = \begin{bmatrix} Q_x^k & Q_{xq}^k \\ (Q_{xq}^k)^{\mathrm{T}} & Q_q^k \end{bmatrix} \tag{59}$$

$$P_k = \begin{bmatrix} P_x^k & P_{xq}^k \\ (P_{xq}^k)^{\mathrm{T}} & P_q^k \end{bmatrix} \tag{60}$$

$$\tilde{V}_{\mathrm{eq}}^k = \begin{bmatrix} \bar{\bar{V}}_x^k & 0_{(n\times 3m)\times n} \\ 0_{(n\times 3m)\times n} & \tilde{V}_2^k \end{bmatrix} \tag{61}$$

(2) *Create the projection equations.* As the state estimator is designed to minimize the performance index $\mathscr{J}_k$, the derivative of $\mathscr{J}_k$ with respect to the estimator matrix $\hat{\varepsilon}_k$ must be equal to zero as a necessary condition. For a linear state estimator, the estimator matrix $\hat{\varepsilon}_k$ is restricted and formulated in the following form based on the plant dynamic model in (11):

$$\hat{\varepsilon}_k = \left[\hat{K}_k C_{\mathrm{au}}^k + \Delta_C^k \quad \hat{L}_{\mathscr{C}\mathscr{L}}\right] \tag{62}$$

To synthesize the optimal linear state estimator, the derivatives of $\mathscr{J}_k$ with respect to the closed-loop transition matrix $\hat{L}_{\mathscr{C}\mathscr{L}}$ and the estimation gain matrix $\hat{K}_k$ (which has been assumed to be synthesized off-line) in expectation are both forced to be equal to zero. That is $\partial\mathscr{J}_k/\partial\hat{\varepsilon}_k = 0$, $\partial\mathscr{J}_k/\partial\hat{L}_{\mathscr{C}\mathscr{L}} = 0$ and $E\left[\partial\mathscr{J}_k/\partial\hat{K}_k\right] = 0$. In the matrix expansion

$$\begin{aligned}P_q^k(\hat{L}_{\mathscr{C}\mathscr{L}}^k)^*(Q_{xq}^k)^{\mathrm{T}} + (P_{xq}^k)^{\mathrm{T}}(A_{\mathrm{au}}^k + \Delta_A^k)Q_x^k + (P_{xq}^k)^{\mathrm{T}}B_{\mathrm{au}}^k\hat{F}_k(Q_{xq}^k)^{\mathrm{T}} \\ + P_q^k(\hat{K}_k^* C_{\mathrm{au}}^k + \Delta_C^k)Q_x^k = 0\end{aligned} \tag{63}$$

$$(P_{xq}^k)^{\mathrm{T}}(A_{\mathrm{au}}^k + \Delta_A^k)Q_{xq}^k + (P_{xq}^k)^{\mathrm{T}}B_{\mathrm{au}}^k\hat{F}_k Q_q^k + P_q^k(\hat{L}_{\mathscr{C}\mathscr{L}}^k)^* Q_q^k + P_q^k(\hat{K}_k^* C_{\mathrm{au}}^k + \Delta_C^k)Q_x^k = 0 \tag{64}$$

$$\begin{aligned}E\{(P_{xq}^k)^{\mathrm{T}}(A_{\mathrm{au}}^k + \Delta_A^k)Q_x^k(C_{\mathrm{au}}^k)^{\mathrm{T}} + (P_{xq}^k)^{\mathrm{T}}B_{\mathrm{au}}^k F_k(Q_{xq}^k)^{\mathrm{T}}(C_{\mathrm{au}}^k)^{\mathrm{T}} \\ + P_q^k\hat{L}_{\mathscr{C}\mathscr{L}}^k(Q_{xq}^k)^{\mathrm{T}}(C_{\mathrm{au}}^k)^{\mathrm{T}} + P_q^k\hat{K}_k^*\left[\bar{\bar{V}}_2^k + C_{\mathrm{au}}^k Q_x^k(C_{\mathrm{au}}^k)^{\mathrm{T}}\right] + P_q^k\Delta_C^k Q_x^k(C_{\mathrm{au}}^k)^{\mathrm{T}}\} = 0\end{aligned} \tag{65}$$

Similarly, on the control side, after taking matrix derivatives on $\mathscr{J}_k$ with respect to $\hat{F}_k^*$ and $(\hat{f}_{\mathrm{au}}^k)^*$, $0 \le k \le N-1$, by the matrix minimum principle and then setting their expected values to zero, respectively, $E[\partial \ell_k / \partial \hat{F}_k] = 0$ and $E[\partial \ell_k / \partial \hat{f}^k_{\mathrm{au}}] = 0$ can be described together in a single matrix equation as follows:

$$E\left\{ \left[(B^k_{\mathrm{au}})^{\mathrm{T}} P^k_x B^k_{\mathrm{au}} + R_2\right]\left[\hat{F}^*_k \quad (\hat{f}^k_{\mathrm{au}})^*\right]\begin{bmatrix} Q^k_q & (Q^k_{xq})^{\mathrm{T}} \Pi^{\mathrm{T}}_{01} \\ \Pi_{01} Q^k_{xq} & \Pi_{01} Q^k_x \Pi^{\mathrm{T}}_{01} \end{bmatrix} \right.$$

$$+ (B^k_{\mathrm{au}})^{\mathrm{T}} P^k_x A^k_{\mathrm{au}}\left[Q^k_{xq} \quad Q^k_x \Pi^{\mathrm{T}}_{01}\right] + (B^k_{\mathrm{au}})^{\mathrm{T}} P^k_{xq}\left[\hat{L}^k_{\mathscr{A}} \quad \hat{K}_k C^k_{\mathrm{au}} + \Delta^k_C\right]$$

$$\left. \times \begin{bmatrix} Q^k_q & (Q^k_{xq})^{\mathrm{T}} \Pi^{\mathrm{T}}_{01} \\ Q^k_{xq} & Q^k_x \Pi^{\mathrm{T}}_{01} \end{bmatrix} \right\} = 0 \quad (66)$$

(3) *Establish optimal gain relations of* $\hat{L}^*_k$, $\hat{K}^*_k$, $\hat{F}^*_k$ *and* $(\hat{f}^k_{\mathrm{au}})^*$: It follows from (64) that

$$(\hat{L}^k_{\mathscr{A}})^* = \Gamma_k (A^k_{\mathrm{au}} + \Delta^k_A) G^{\mathrm{T}}_k + \Gamma_k B^k_{\mathrm{au}} \hat{F}_k - (\hat{K}_k C^k_{\mathrm{au}} + \Delta^k_C) G^{\mathrm{T}}_k \quad (67)$$

where G_k and Γ_k are a pair of projective factors (Bernstein and Haddad 1987) defined as follows:

$$G_k = (Q^k_{\mathrm{q}})^{-1} (Q^k_{xq})^{\mathrm{T}} \quad (68a)$$

$$\Gamma_k = -(P^k_{\mathrm{q}})^{-1} (P^k_{xq})^{\mathrm{T}} \quad (68b)$$

In addition, define

$$\bar{\bar{Q}}^k_x = Q^k_x - \hat{Q}^k_x \quad (69a)$$

$$\hat{Q}^k_x = Q^k_{xq} (Q^k_q)^{-1} (Q^k_{xq})^{\mathrm{T}} \quad (69b)$$

Equation (41) is obtained by substituting (67) into (65). Applying (63) and (64) to (66) yields

$$E\left[(B^k_{\mathrm{au}})^{\mathrm{T}} \bar{\bar{P}}^k_x B^k_{\mathrm{au}} + R_2\right]\left[\hat{F}^*_k \quad (\hat{f}^k_{\mathrm{au}})^*\right]\begin{bmatrix} Q^k_q & (Q^k_{xq})^{\mathrm{T}} \Pi^{\mathrm{T}}_{01} \\ \Pi_{01} Q^k_{xq} & \Pi_{01} Q^k_x \Pi^{\mathrm{T}}_{01} \end{bmatrix}$$

$$= -E\left[(B^k_{\mathrm{au}})^{\mathrm{T}} \bar{\bar{P}}^k_x A^k_{\mathrm{au}}\left[Q^k_{xq} \quad Q^k_x \Pi^{\mathrm{T}}_{01}\right]\right] \quad (70)$$

where

$$\bar{\bar{P}}^k_x = P^k_x - \hat{\hat{P}}^k_x \quad (71a)$$

$$\hat{P}^k_x = P^k_{xq} (P^k_q)^{-1} (P^k_{xq})^{\mathrm{T}} \quad (71b)$$

As the last m columns of A^k_{au} are zero, from (70) we must have $(\hat{f}^k_3)^* = 0_m$. Let

$$(\hat{f}^k_{au})^* = -(\hat{R}^k_{2\mathrm{s}})^{-1}\left[\hat{P}^k_{q1} \quad \hat{P}^k_{q2} \quad 0_m\right] \quad (72)$$

then from (70)

$$\hat{F}^*_k = -(\hat{R}^k_{2\mathrm{s}})^{-1} \hat{P}^k_{x0} G^{\mathrm{T}}_k \quad (73)$$

where

$$\hat{P}_{x0}^{k} = E\Big[(B_{\rm au}^{k})^{\rm T} P_{x}^{k}(A_{\rm au}^{k} + \Delta_{A}^{k})\Big] + R_{2}(\hat{f}_{\rm au}^{k})^{*}\Pi_{01} \tag{74}$$

Substituting (72) into (74) yields (47). The gain relations in (40) and (42) are constructed.

(4) *Create two pairs of modified matrix Riccati and Lyapunov equations.* After taking derivatives of $\mathscr{L}_k$ with respect to P_k and Q_k and by setting them to be zero

$$\frac{\partial \mathscr{L}_k}{\partial P_k} = 0: \quad Q_{k+1} = E\Big[\tilde{A}_k Q_k \tilde{A}_k^{\rm T} \Big| \tilde{Z}_k\Big] + \tilde{V}_{\rm eq}^{k} \tag{75}$$

$$\frac{\partial \mathscr{L}_k}{\partial Q_k} = 0: \quad P_{k+1} = E\Big[\tilde{A}_k^{\rm T} P_k \tilde{A}_k \Big| \tilde{Z}_k\Big] + \tilde{R}_k \tag{76}$$

Expansion of (75) and (76) yields

$$E\Big\{\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - (\hat{R}_{2s}^{k})^{-1}\hat{P}_{x0}^{k}\Big]\hat{Q}_{x}^{k}\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - (\hat{R}_{2s}^{k})^{-1}\hat{P}_{x0}^{k}\Big]^{\rm T}\Big|\tilde{Z}_k\Big\}$$
$$+ E\Big[(A_{\rm au}^{k} + \Delta_{A}^{k})\bar{\bar{Q}}_{x}^{k}(A_{\rm au}^{k} + \Delta_{A}^{k})^{\rm T}\Big|\tilde{Z}_k\Big] + \bar{\bar{V}}_{x}^{k} - \bar{\bar{Q}}_{x}^{k+1} - \hat{Q}_{x}^{k+1} = 0 \tag{77a}$$

$$\Big\{E\Big\{\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - (\hat{R}_{2s}^{k})^{-1}\hat{P}_{x0}^{k}\Big]\hat{Q}_{x}^{k}\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - (\hat{R}_{2s}^{k})^{-1}\hat{P}_{x0}^{k}\Big]^{\rm T}\Big|\tilde{Z}_k\Big\}$$
$$+ \Big\{\hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}E\Big[C_{\rm au}^{k}\bar{\bar{Q}}_{x}^{k}(\Delta_{\rm C}^{k})^{\rm T}\Big|\tilde{Z}_k\Big](\Gamma_k^{-R})^{\rm T}\Big\}^{\rm T}$$
$$+ E\Big[(A_{\rm au}^{k} + \Delta_{A}^{k})\bar{\bar{Q}}_{x}^{k}(\Delta_{\rm C}^{k})^{\rm T}\Big|\tilde{Z}_k\Big](\Gamma_k^{-R})^{\rm T} + \hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}(\hat{Q}_{\rm s}^{k})^{\rm T}\Big\}\Gamma_k^{\rm T} - Q_{xq}^{k+1} = 0 \tag{77b}$$

$$\Gamma_k\Big\{E\Big\{\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - (\hat{R}_{2s}^{k})^{-1}\hat{P}_{x0}^{k}\Big]\hat{Q}_{x}^{k}\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - (\hat{R}_{2s}^{k})^{-1}\hat{P}_{x0}^{k}\Big]^{\rm T}\Big|\tilde{Z}_k\Big\}$$
$$+ \Gamma_k^{-R}E\Big[\Delta_{C}^{k}\bar{\bar{Q}}_{x}^{k}(\Delta_{C}^{k})^{\rm T}\Big|\tilde{Z}_k\Big](\Gamma_k^{-R})^{\rm T} + \hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}E\Big[C_{\rm au}^{k}\bar{\bar{Q}}_{x}^{k}(\Delta_{C}^{k})^{\rm T}\Big|\tilde{Z}_k\Big](\Gamma_k^{-R})^{\rm T}$$
$$+ \Big\{\hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}E\Big[C_{\rm au}^{k}\bar{\bar{Q}}_{x}^{k}(\Delta_{\rm C}^{k})^{\rm T}\Big|\tilde{Z}_k\Big](\Gamma_k^{-R})^{\rm T}\Big\}^{\rm T} + \hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}(\hat{Q}_{\rm s}^{k})^{\rm T}\Big\}\Gamma_k^{\rm T} - Q_{xq}^{k+1} = 0 \tag{77c}$$

$$E\Big[(A_{\rm au}^{k} + \Delta_{A}^{k})^{\rm T}\bar{\bar{P}}_{x}^{k}(A_{\rm au}^{k} + \Delta_{A}^{k})\Big|\tilde{Z}_k\Big] + E\Big\{\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - \hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}C_{\rm au}^{k} - \Gamma_k^{-R}\Delta_{\rm C}^{k}\Big]^{\rm T}\hat{P}_{x}^{k}$$
$$\times\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - \hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}C_{\rm au}^{k} - \Gamma_k^{-R}\Delta_{C}^{k}\Big]\Big|\tilde{Z}_k\Big\} + \hat{R}_{x}^{k} - \bar{\bar{P}}_{x}^{k-1} - \hat{P}_{x}^{k-1} = 0 \tag{78a}$$

$$\Big\{E\Big\{\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - \hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}C_{\rm au}^{k} - \Gamma_k^{-R}\Delta_{C}^{k}\Big]^{\rm T}\hat{P}_{x}^{k}$$
$$\times\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - \hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}C_{\rm au}^{k} - \Gamma_k^{-R}\Delta_{C}^{k}\Big]\Big|\tilde{Z}_k\Big\}$$
$$+ (\hat{P}_{x0}^{k})^{\rm T}(\hat{R}_{2s}^{k})^{-1}\hat{P}_{x0}^{k}\Big\}G_k^{\rm T} + P_{xq}^{k-1} = 0 \tag{78b}$$

$$G_k\Big\{E\Big\{\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - \hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}C_{\rm au}^{k} - \Gamma_k^{-R}\Delta_{C}^{k}\Big]^{\rm T}\hat{P}_{x}^{k}$$
$$\times\Big[A_{\rm au}^{k} + \Delta_{A}^{k} - \hat{Q}_{\rm s}^{k}(\hat{V}_{2s}^{k})^{-1}C_{\rm au}^{k} - \Gamma_k^{-R}\Delta_{C}^{k}\Big]\Big|\tilde{Z}_k\Big\}$$
$$+ (\hat{P}_{x0}^{k})^{\rm T}(\hat{R}_{2s}^{k})^{-1}\hat{P}_{x0}^{k}\Big\}G_k^{\rm T} - P_{xq}^{k-1} = 0 \tag{78c}$$

with the matrix definitions

$$\Gamma_k \Gamma_k^{-R} = I_n \tag{79}$$

$$\overset{\leftrightarrow}{R}{}_x^k = \begin{bmatrix} R_1 & 0_{n\times 3m} \\ 0_{3m\times n} & (\hat{f}_{\mathrm{au}}^k)^2 \hat{f}_{\mathrm{au}}^k \end{bmatrix} \tag{80}$$

Let

$$\begin{aligned} \hat{\hat{Q}}{}_x^{k+1} = {} & E\left\{\left[A_{\mathrm{au}}^k + \Delta_A^k - B_{\mathrm{au}}^k(\hat{R}_{2\mathrm{s}}^k)^{-1}\hat{P}_{x0}^k\right]\hat{\hat{Q}}{}_x^k \times \left[A_{\mathrm{au}}^k + \Delta_A^k - B_{\mathrm{au}}^k(\hat{R}_{2\mathrm{s}}^k)^{-1}\hat{P}_{x0}^k\right]^{\mathrm{T}}\middle|\tilde{Z}_k\right\} \\ & + \Gamma_k^{-R}E\left[\Delta_C^k\bar{\bar{Q}}{}_x^k(\Delta_C^k)^{-\mathrm{T}}\middle|\tilde{Z}_k\right](\Gamma_k^{-R})^{\mathrm{T}} + \hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}E\left[C_{\mathrm{au}}^k\bar{\bar{Q}}{}_x^k(\Delta_C^k)^{-\mathrm{T}}\middle|\tilde{Z}_k\right](\Gamma_k^{-R})^{\mathrm{T}} \\ & + \left\{\hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}E\left[C_{\mathrm{au}}^k\bar{\bar{Q}}{}_x^k(\Delta_C^k)^{-\mathrm{T}}\middle|\tilde{Z}_k\right](\Gamma_k^{-R})^{\mathrm{T}}\right\}^{\mathrm{T}} + \hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}(\hat{Q}_{\mathrm{s}}^k)^{-\mathrm{T}} \end{aligned} \tag{81}$$

and

$$\begin{aligned} \hat{\hat{P}}{}_x^{k-1} = {} & E\left\{\left[A_{\mathrm{au}}^k + \Delta_A^k - \hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}C_{\mathrm{au}}^k - \Gamma_k^{-R}\Delta_C^k\right]^{-\mathrm{T}}\hat{P}_x^k\right. \\ & \left.\times\left[A_{\mathrm{au}}^k + \Delta_A^k - \hat{Q}_{\mathrm{s}}^k(\hat{V}_{2\mathrm{s}}^k)^{-1}C_{\mathrm{au}}^k - \Gamma_k^{-R}\Delta_C^k\right]\middle|\tilde{Z}_k\right\} + (\hat{P}_{x0}^k)^{\mathrm{T}}(\hat{R}_{2\mathrm{s}}^k)^{-1}\hat{P}_{x0}^k \end{aligned} \tag{82}$$

By substituting (67) into (65), a relation between (77 *b*) and (77 *c*) is established as follows:

$$\Gamma_k Q_{xq}^{k+1} = Q_q^{k+1} \tag{83}$$

Hence, $\Gamma_k G_{k+1}^{-\mathrm{T}} = I_n$. From Lemma 1, $\tau_k^2 = \tau_k$ by defining the projection matrix $\tau_k = G_{k+1}^{\mathrm{T}}\Gamma_k$. Moreover, the following equalities always hold.

$$\tau_k\hat{Q}_x^{k+1}\tau_k^{\mathrm{T}} = \tau_k\hat{Q}_x^{k+1} = \hat{Q}_x^{k+1}\tau_k^{\mathrm{T}} = \hat{Q}_x^{k+1} \tag{84a}$$

$$\tau_k^{\mathrm{T}}\hat{P}_x^k\tau_k = \tau_k^{\mathrm{T}}\hat{P}_x^k = \hat{P}_x^k\tau_k = \hat{P}_x^k \tag{84b}$$

It follows from (77 *b*) that by applying (81) and (83)

$$\tau_k(\hat{\hat{Q}}{}_x^{k+1} - \hat{Q}_x^{k+1})\Gamma_k^{\mathrm{T}}G_{k+1} = 0 \tag{85}$$

Equation (85), via application of (84 *a*), becomes

$$\hat{Q}_x^{k+1} = \tau_k\bar{\bar{Q}}{}_x^{k+1}\tau_k^{\mathrm{T}} \tag{86}$$

Similarly, derived from (78 *b*), a dual result ends up with

$$\hat{P}_x^k = \tau_k^{\mathrm{T}}\hat{\hat{P}}{}_x^k\tau_k \tag{87}$$

Substituting (86) and (87) into (81) and (82), respectively, yields (49) and (51). Finally, successive application of (86) and the calculation of

$$(77a) + G_{k+1}^{\mathrm{T}}\Gamma_k(77b)G_{k+1} - (77b)G_{k+1} - \left\{(77b)G_{k+1}\right\}^{\mathrm{T}}$$

yields (48). Similarly, successive application of (72) and (87) and calculation of

$$(78a) + \Gamma_k^{\mathrm{T}} G_{k+1}(78b)\Gamma_k - (78b)\Gamma_k - \{(78b)\Gamma_k\}^{\mathrm{T}}$$

yields (50). Thus the two pairs of modified matrix Riccati and Lyapunov recursions in (48)–(51) are all established.

(5) *Establish the eigenprojection.* It follows from Lemma 1 that $\mathrm{rank}(\tau_k) = n$. As τ_k is idempotent and has rank n, there exists some modal matrix S_k such that

$$\tau_k = S_k I_x S_k^{-1} \tag{88}$$

It follows from (84a) or (84b) that

$$\tau_k = \hat{Q}_x^{k+1}\hat{P}_x^k(\hat{Q}_x^{k+1}\hat{P}_x^k)^{\#} \tag{89}$$

where $^{\#}$ indicates the general group matrix inverse (Rao and Mitra 1971). For $\hat{\hat{Q}}_x^{k+1}$ and $\hat{\hat{P}}_x^k$ are both symmetric and non-negative definite, there exists some $\Psi_k \in \tilde{\mathcal{D}}(\hat{\hat{Q}}_x^{k+1}, \hat{\hat{P}}_x^k)$ such that $D_{\mathrm{Q}}^k = \Psi_k^{-1}\hat{\hat{Q}}_x^{k+1}\Psi_k^{-\mathrm{T}}$ and $D_P^k = \Psi_k^{\mathrm{T}}\hat{\hat{P}}_x^k\Psi_k$ are diagonal. Therefore

$$\hat{\hat{Q}}_x^{k+1}\hat{\hat{P}}_x^k = \Psi_k D_Q^k D_P^k \Psi_k^{-1} \tag{90}$$

Substituting (86), (87), (88) and (90) into (89) yields

$$\tau_k = H_k H_k^{\#} \tag{91a}$$

where

$$H_k = S_k I_x (S_k^{-1}\Psi_k) D_Q^k (S_k^{-1}\Psi_k)^{-\mathrm{T}} I_x (S_k^{-1}\Psi_k)^{-\mathrm{T}} D_P^k (S_k^{-1}\Psi_k)^{-1} I_x S_k^{-1} \tag{91b}$$

Let $S_k = \Psi_k$, then

$$\tau_k = H_k H_k^{\#} = S_k I_x S_k^{-1} = \Psi_k I_x \Psi_k^{-1} \tag{92}$$

The factorization of the projection matrix τ_k is accomplished. The proposition is proved. □

Remark 3: The optimal projection equations (48)–(51) become similar in structure to equations (4.9)–(4.12) of Haddad and Kapila (1995) in the absence of random delays, i.e. if the conditional expectation operator $E[\cdot|\cdot]$ is taken out. Instead of the pair $(\bar{\bar{Q}}_x^k, \hat{\hat{Q}}_x^k)$ as used in our paper, Haddad and Kapila (1995) have used an equivalent form of the pair $(\bar{\bar{Q}}_x^k, \hat{Q}_x^k)$. The relationship, via a transformation, between $\hat{Q}_x^k$ and $\hat{\hat{Q}}_x^k$ is stated in (86) where the transformation matrix τ_k has rank equal to n. Similarly, the transformation between $\hat{P}_x^k$ and $\hat{\hat{P}}_x^k$ is given in (87) by duality. The tank conditions of the two Lyapunov matrices are

$$\mathrm{rank}(\hat{\hat{Q}}_x^k) = n, \quad \mathrm{rank}(\hat{\hat{P}}_x^k) = n$$

5. Controller design procedure and implications

So that the structure of each design iteration is satisfied and remains tractable, we present a systematic procedure for sequentially refining estimates of the optimal projection and synthesizing the steady-state LQCDC gain matrix for linear time-invariant systems that are excited by wide sense stationary noise. The design steps are as follows.

Step 1. Let the projection matrix $\tau_0 = I_{n+3m}$ and its factorization factor $G_0 = \begin{bmatrix} I_n & 0_{n\times 3m} \end{bmatrix}$ to obtain a set of starting values of control and estimation gains $(\hat{F}^0_{au}, \hat{K}_0)$ after a preset convergence tolerance is achieved such that the initial design of LQCDC is set as close as to the full-order dynamic output feedback compensator. Meanwhile, the pair of non-negative definite matrices $(\bar{\bar{Q}}^0_x, \bar{\bar{P}}^0_x)$ is simultaneously found by iteration of the pair of modified matrix Riccati equations.

Step 2. Using $(\hat{F}^0_{au}, \hat{K}_0)$ from step 1, the starting values of $(\hat{\hat{Q}}^0_x, \hat{\hat{P}}^0_x)$ are thus set by iteration of the pair of modified matrix Lyapunov equations.

Step 3. Establish steady-state projection matrix τ_s. Applying the procedure of constructing the projection matrix τ_k described in the last section, and the pair of modified matrix Lyapunov recursions of $(\hat{\hat{Q}}^k_x, \hat{\hat{P}}^k_x)$, the steady-state projection matrix τ_s is uniquely determined. So is the steady-state pair $(\hat{\hat{Q}}^s_x, \hat{\hat{P}}^s_x)$.

Step 4. The steady-state values of LQCDC gains $(\hat{F}^s_{au}, \hat{K}_s)$ and estimator transition matrix $\hat{L}_s$ are finally synthesized by convergent recursion of $(\bar{\bar{Q}}^k_x, \bar{\bar{P}}^k_x)$ once the steady state projection matrix, τ_s, is employed. The steady state pair $(\bar{\bar{Q}}^s_x, \bar{\bar{P}}^s_x)$ is obtained at the same time.

Details of the control synthesis procedure and the design implications are given in § 4.3 of Tsai (1995). The motivation and implied meaning of each step are stated below.

As the optimal projection matrix of LQCDC becomes oblique for the state estimation problem, the initial projection matrix $\tau_0 = I_{n+3m}$ is selected so that the induced steady-state projection matrix is modified to be close to the full-order case through the convergence of the pair of modified matrix Lyapunov recursion of $(\hat{\hat{Q}}^k_x, \hat{\hat{P}}^k_x)$. The full-order projection matrix thus becomes a special case in which no modification of the initial projection matrix is needed. The only purpose of step 2 is to prepare the initial values of $(\hat{\hat{Q}}^k_x.\hat{\hat{P}}^k_x)$ such that its recursion and the steady-state projection matrix in step 3 converges quickly without having to start from zeros. For numerical sensitivity of finding eigenvalues and eigenvectors of $\hat{\hat{Q}}^{k+1}_x \hat{\hat{P}}^k_x, (\hat{\hat{Q}}^k_x, \hat{\hat{P}}^k_x)$ is made to converge as smoothly and quickly as possible. During the procedure of establishing projection matrix in step 3, it is also noted that the suggested approximation of the projection matrix described in (42) is only used for faster and smoother convergence of the recursion of the pair $(\hat{\hat{Q}}^k_x, \hat{\hat{P}}^k_x)$. When the matrix $\gamma_k \leq I_{3m}$ becomes sufficiently small, it is dropped off finally. In other words, (34) is the final solution of the steady-state projection matrix. This implies that the eigenspaces of $\hat{\hat{Q}}^s_x, \hat{\hat{P}}^s_x$ associated with the $3m$ smallest eigenvalues are deleted. By inspecting the sensor output with binary random delay z_k through an estimation gain, $\hat{K}^*_k = \Gamma_k \hat{Q}^k_s \hat{V}^k_{2s}$, as the state estimator input, i.e. the second term in the equation

$$\hat{\xi}^s_{k+1} = \hat{L}_k \hat{\xi}^s_k + \hat{K}_k z_k + \sum_{i=0}^{2} \beta^k_i u_{k-i} \tag{93}$$

it is noted that the state estimator input $\hat{K}_k z_k$ is annihilated unless it belongs to the subspace $N^{\perp}(\Gamma_k)$ where N and $\perp$ denote null space and orthogonal compliment space, respectively (Kailath 1980). As the equality $N^{\perp}(\Gamma_k) = R(\Gamma_k^{\mathrm{T}})$ always holds, where R denotes the range space and $\tau_k^{\mathrm{T}} = \Gamma_k^{\mathrm{T}} G_{k+1}$, $R(\tau_k^{\mathrm{T}})$ represents the estimation subspace of LQCDC. On the control side, the first term of the plant input

$$u_k = (\hat{R}_{2s}^k)^{-1} P_{x0}^k G_k^{\mathrm{T}} \hat{\xi}_k^{\mathrm{s}} + \hat{f}_1^k u_{k-1} + \hat{f}_2^k u_{k-2} \tag{94}$$

must belong to the range space of G_k^{T}, or $R(\tau_k)$, which represents the control subspace of LQCDC instead. Equivalently, the column and row spaces of G_k^{T} and Γ_k, factored from the projection matrix, respectively, constitute the control and estimation subspaces and jointly comprises the projection transformation by $\tau_k = G_{k+1}^{\mathrm{T}} \Gamma_k$. This also illustrates the significance of the eigenvalues of $\hat{\hat{Q}}_x^{\mathrm{s}} \hat{\hat{P}}_x^{\mathrm{s}}$ as the natural measures of the relative importance of the various eigenspaces in the optimal projection. Furthermore, because $\hat{\hat{Q}}_x^{k+1}$ and $\hat{\hat{P}}_x^k$ are balanced by means of the transformation $\Psi_k \in \widetilde{T}(\hat{\hat{Q}}_x^{k+1}, \hat{\hat{P}}_x^k)$, it follows that $\Psi_k^L \hat{\hat{Q}}_x^{k+1} \hat{\hat{P}}_x^k \Psi_k^R$ is diagonal. Hence, $\hat{\hat{Q}}_x^{k+1} \hat{\hat{P}}_x^k$ is semisimple, i.e. the geometric multiplicity of each eigenvalue of $\hat{\hat{Q}}_x^{k+1} \hat{\hat{P}}_x^k$ is always equal to unity. The eigenvalues of $\hat{\hat{Q}}_x^{k+1} \hat{\hat{P}}_x^k$ are always real and non-negative because $D_{QP}^k = D_Q^k D_P^k$ holds, where $D_Q^k \equiv \Psi_k^{\mathrm{L}} \hat{\hat{Q}}_x^{k=1} (\Psi_k^L)^{\mathrm{T}}$ and $D_P^k \equiv (\Psi_k^R)^{\mathrm{T}} \hat{\hat{P}} \Psi_k^R$. The final step, step 4, is made to find the steady-state LQCDC by using the same procedure as the synthesizing LQG as usual with a fixed projection matrix which is not the identity matrix, in general.

6. Simulation of a flight control system

This section presents simulation experiments of an application example that utilizes the LQCDC technique for the control of the longitudinal motion of an advanced aircraft where the sensor and actuator signals are subjected to randomly varying delays. The preformance of the LQCDC control law is compared with those of DCLQR (Liou and Ray 1991 a, b) and DCLQG (Ray 1994) for the compensation of randomly varying distributed delays. The control laws were synthesized in the MATLAB environment on a Pentium. The following notations are used in the simulation example after discretization of the continuous-time linear time-invariant plant model.

c denotes the output matrix; V_1 and V_2 denote the covariances of zero-mean plant disturbance and zero-mean sensor noise, respectively; R_1 and R_2 are the state deviation and cost penalty matrices orderly; $\Re$ dictates the state deviation matrix at final stage, i.e. when $k = N$. The plant state vector and its estimate are, respectively, denoted as ξ_k^{s} and $\hat{\xi}_k^{\mathrm{s}}$ at the kth sensor sampling period.

The longitudinal motion of the aircraft under consideration is characterized by the pitch, pitch rate, and the velocity components along the X-axis and the Y-axis of the aircraft. The X-axis is chosen to coincide with the longitudinal axis when the aircraft performs a stationary horizontal flight. The control variables and system inputs for this motion are the engine thrust and the elevator deflection, whereas the controlled variables, system outputs, are the speed along the X-axis and the pitch. From the inertial and aerodynamic laws governing the motion of the aircraft (Nelson 1989), a linearized longitudinal equations of motion around the nominal point which consists of horizontal flight with constant speed can be established, with contamination of the white zero-mean plant disturbance and sensor noise.

Let

$$\xi \equiv \begin{bmatrix} U_X^{\mathrm{T}} & U_Y^{\mathrm{T}} & \zeta^{\mathrm{T}} & \mu^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \quad \text{and} \quad u = \begin{bmatrix} \Theta^{\mathrm{T}} & \sigma^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

as the state and input vectors, respectively, where U_X is the incremental velocity in m/s along the X-axis of aircraft, U_Y is the velocity in m/s along the Y-axis, of the aircraft, ζ is the pitch in rad and μ is the pitch rate in rad/s; Θ is the incremental engine thrust in N and σ is the elevator deflection in rad. The numerical continuous-time system and input matrices are given as follows:

$$a = \begin{bmatrix} -1{\cdot}580\times 10^{-2} & 2{\cdot}633\times 10^{-2} & -9{\cdot}810 & 0 \\ -1{\cdot}571\times 10^{-1} & -1{\cdot}030 & 0 & 1{\cdot}205\times 10^{2} \\ 0 & 0 & 0 & 1 \\ 5{\cdot}274\times 10^{-4} & -1{\cdot}652\times 10^{-2} & 0 & -1{\cdot}466 \end{bmatrix},$$

$$b = \begin{bmatrix} 6{\cdot}056\times^{-4} & 0 \\ 0 & -9{\cdot}496 \\ 0 & 0 \\ 0 & -5{\cdot}556 \end{bmatrix},$$

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The sampling time in this example is chosen as $T = 0{\cdot}025$ s. Assuming that a deviation of 10 m/s in the slight speed along the X-axis of aircraft is considered to be unacceptable; a deviation of 0·2 rad in the pitch and a deviation of 500 N in the engine thrust are about as acceptable as a deviation of 0·2 rad in the elevator deflection. This leads to the construction of the following weighting matrices $R1$ and $R2$ for state deviation and control input vectors, respectively:

$$R_1 = \begin{bmatrix} 5\times 10^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1{\cdot}25\times 10^{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 4\times 10^{-9} & 0 \\ 0 & 2{\cdot}5\times 10^{-2} \end{bmatrix}$$

The other matrices needed for the synthesis of the LQCDC law are

$$V_1 = \mathrm{diag}(6{\cdot}25\times 10^{-2}, 6{\cdot}25\times 10^{-2}, 3{\cdot}5156\times 10^{-6}, 6{\cdot}25\times 10^{-14})$$

$$V_2 = \mathrm{diag}(6{\cdot}25\times 10^{-4}, 6{\cdot}25\times 10^{-8}),$$

$$\Re = 0_{4\times 4}$$

The following initial conditions for the perturbed plant state and the unperturbed plant state estimate are prescribed in the sensor time frame to obtain the transient responses:

$$\xi_0^s = \begin{bmatrix} 50{\cdot}0 & 0 & 0{\cdot}5 & 0 \end{bmatrix}^{\mathrm{T}} \quad \text{and} \quad \hat{\xi}_0^s = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

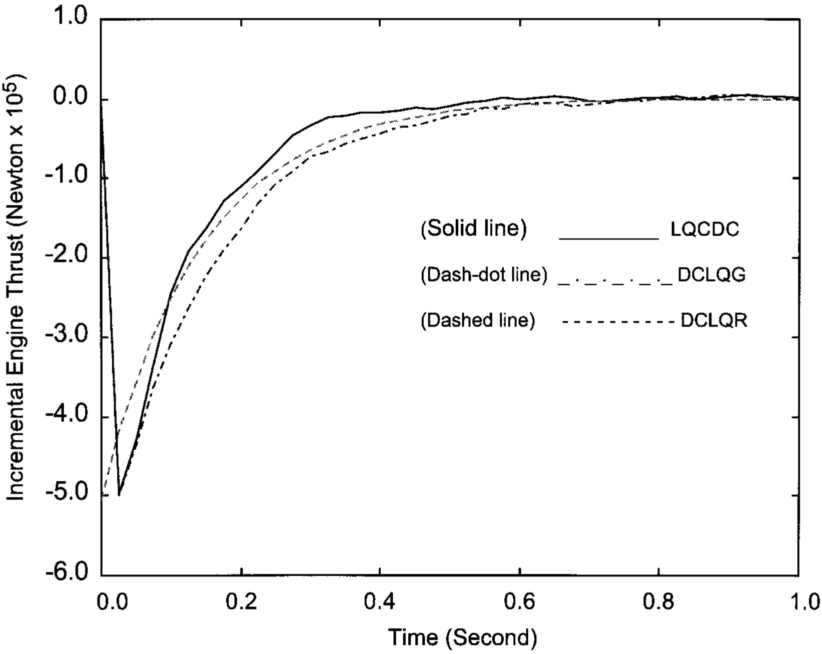

Figure 1. Transient response of incremental engine thrust.

Note that the plant initial condition has been perturbed from the equilibrium point (i.e. the zero state) by 50 m/s in the indremental velocity U_X along the X -axis and by 0·5 rad in the pitch angle ζ. These perturbations are equivalent to having impulse response in the respective variables at time zero. Extensive simulation experiments were conducted with a wide range of parameters respresenting the statistics of network-induced delays, plant noise, sensor noise and the skew Δ as well as with different initial conditions for the plant model and the state estimation filter. Some of the results are summarized below and plotted in Figs 1 and 2.

Figure 1 shows the transient response of the incremental engine thrust under the three control laws, LQCDC, DCLQG amd DCLQR. In general, for output feedback in the presence of noise, the performance of LQCDC is superior (in terms of faster response) to that of DCLQG because the composite design of DCLQR preserves the optimality of the closed loop system via coupled estimation and feedback control. However, the performance of LQCDC with full state feedback is not superior to that of DCLQR because DCLQR generates the best performance provided that there is no plant disturbance and the full plant state is available. The coupled design of the controller and state estimator yields the stochastic optimal performance. For the time skew $\Delta = T$ and time horizon $N = 200$, the performance cost of LQCDC was 25·6281 ($\times$200) and that of DCLQG goes up to 30·9176 ($\times$200).

In the standard LQR, the control penalty matrix R_2 plays an esential role for placing the poles of the closed-loop system; it also indicates how fast the plant states are regulated to zeros. In other words, with large R_2 a conservative action is expected to be taken so that only unstable poles are moved with slow regulation and relatively poor performance since the control energy is heavily penalized. Figure 2 shows a comparison to demonstrate that similar results are obtained by LQCDC. With total time skew fixed at $0{\cdot}25T$, where T is the sampling period, the cost penalty matrix R_2

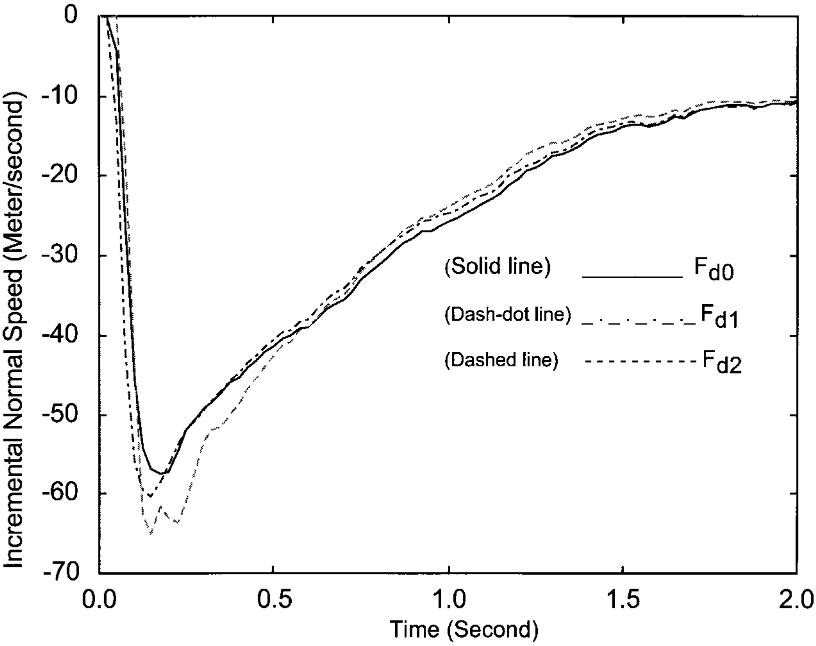

Figure 2. Transient response of incremental normal speed.

is replaced by $\gamma_c R_2$ to inspect the effect of the weighting factor γ_c. As expected, it is found that the performance is indeed degraded as γ_c is increased. Although the poles of the closed-loop system show no obvious evidence that LQCDC has the same behaviour as LQR for pole allocation.

For different delay statistics, LQCDC appears to be robust in stability and performance. Results for the following three different probability distribution functions (PDFs) are demonstrated:

$$\left.\begin{aligned} F_{d0}(q) &= \left[1 - \exp(8{\cdot}0q^2/T^2)\right]/\left[1 - \exp(-8{\cdot}0)\right] \\ F_{d1}(q) &= \left[1 - \exp(-80q^2/T^2)\right] \\ F_{d2}(q) &= \exp(-16(1 - q/T^2)\right] \end{aligned}\right\}, \quad \text{for } 0 < q < T \tag{48}$$

The PDF F_{d0} represents the most likely case in network-induced delay traffic with approximate $0{\cdot}25T$ as the delay that corresponds to the peak value of probability density function (p.d.f.). The PDF F_{d1} represents a delay traffic with smaller variance and lower expected value than F_{d0}. It implies that the problem of random delays in the traffic is greatly alleviated. F_{d2} represents a more severe case of random delay whose lowest limit is $0{\cdot}4T$ and the expected value is approximately located at $0{\cdot}8T$. Fig. 2 shows the transients of normal speed (along the Y-axis) to demonstrate the impact of the delay statistic on robust stability and performance of LQCDC. As described above, LQCDC has the best performance for delay statistics F_{d1} which has a relatively less serious random delay problem. Nevertheless, even for the worst delay statistics F_{d2} LQCDC has an excellent performance with each plant state continuously exhibiting small oscillations and retaijing fast zero regulation. It has also demonstrated the deterioration of the performance when the delay problem is worse. Apparently, LQCDC remains stable no matter how severely the delay statistics are perturbed. This is a subject of further analytical research.

7. Summary and conclusions

The linear quadratic coupled delay compensator (LQCDC) presented in this paper is potentially applicable to network-based distributed control systems that are subjected to random delays between the sensor and controller and between the controller and actuator. The LQCDC algorithm is shown to be stochasitically optimal. Although a linear stochastic control problem can be solved as two separate optimal control and optimal estimation problems in the deterministic setting, this is not true in general for stochastic systems especially under the influence of random delays. In that case, the controller and estimator must be synthesized simultaneously as a single composite compensator. In contrast to the conventional LQG problem, the two pairs of modified matrix Riccati and matrix Lyapunov equations for LQCDC are coupled by a projection matrix whose column and row spaces are, respectively, the control and estimation subspaces. When the global optimality is achieved, the controller and the state estimator are simultaneously determined by its eigenprojection. Results of simulation experiments are presented to demonstrate efficacy of the LQCDC for control of longitudinal motion of an advanced aircraft in contrast to other techniques. It is also shown that LQCDC is not very sensitive to the small changes in the delay statistics.

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References

Bar-Shalom, Y., and Tse, E., 1974, Dual effect, certainty equivalence, and separation in stochastic control. *IEEE Transactions on Automatic Control*, **19**, 494–500.

Bernstein, D. S., Davis, L. D., and Greeley, S. W., 1986 a, The optimal projection equations for fixed-order sampled-data dynamic compensation with computation delay. *IEEE Transactions on Automatic Control*, **31**, 859–862.

Bernstein, D. S., Davis, L. D., and Hyland, D. C., 1986 b, Thee optimal projection equations for reduced-order discrete-time modeling, estimation and control. *AIAA, Journal of Guidance*, **9**, 288–292.

Bernstein, D. S., and Haddad, W. M., 1987, Optimal projection equations for discrete-time fixed-order dynamics compensation of linear systems with multiplicative white noise. *International Journal of Control*, **46**, 65–73.

Dykeman, D., and Bux, W., 1988, Analysis and tuning of the FDDI media access control protocol. *IEEE Journal on Selected Areas in Communications*, **6**, 997–1010.

Franklin, G. F., Powell, J. D., and Workman, M. L., 1990, *Digital Control of Dynamic Systems* (Reading, Massachusetts, U.S.A.: Addison-Wesley).

Halevi, Y., and Ray, A., 1988, Integrated communication and control systems: Part I—analysis. *ASME Journal of Dynamic Systems, Measurement and Control*, **110**, 367–373.

Haddad, W. M., and Kapila, V., 1995, A periodic fixed structure approach to multirate control. *IEEE Transactions on Automatic Control*, **40**, 301–307.

Hyland, D. C., and Bernstein, D. S., 1984, The optimal projection equations for fixed order dynamic compensator. *IEEE Transactions on Automatic Control*, **29**, 1034–1037.

Kailath, T., 1980, *Linear Systems* (Englewood Cliffs, New Jersey, U.S.A.: Prentice Hall).

Liou, L.-W., and Ray, A., 1991, A stochastic regulator for integrated communication and control systems: Parts I and II. *ASME Journal of Dynamic Systems, Measurement and Control*, **113**, 604–619.

Nelson, R. C., 1989, *Flight Stability and Automatic Control* (New York: McGraw-Hill).

Rao, C. R., and Mitra, S. K., 1971, *Generalized Inverse of Matrices and its Applications* (New York: Wiley).

Ray, A., 1987, Performance analysis of medium access control protocols for distributed digital avionics. *ASME Journal of Dynamic Systems, Measurement and Control*, **109**, 370–377; 1994, Output feedback control under randomly varying distributed delays. *Journal of Guidance, Control, and Dynamics*, **17**, 701–711.

Ray, A., and Halevi, Y., 1988, Integrated communication and control systems; Part II—design considerations. *ASME Journal of Dynamic Systems, Measurement and Control*, **110**, 374–381.

Ray, A., Liou, L.-W., and Shen, J. H., 1993, State estimation using randomly delayed measurements. *ASME Journal of Dynamic Systems, Measurement and Control*, **115**, 19–26.

Tsai, N.-C., 1995, *Stochastic optimal control under randomly varying distributed delays*. PhD dissertation in mechanical engineering, The Pennsylvania State University, University Park, Pennsylvania, U.S.A.