Pergamon

Appl. Math. Lett. Vol. 11, No. 4, pp. 27–32, 1998

Printed in Great Britain
0893-9659/98 $19.00 + 0.00

PII: S0893-9659(98)00051-2

Linear Unbiased State Estimation under Randomly Varying Bounded Sensor Delay

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(Received June 1997; revised and accepted October 1997)

Abstract—The motivation for the work reported in this paper accrues from the necessity of finding stabilizing control laws for systems with randomly varying bounded sensor delay. It reports the development of reduced-order linear unbiased estimators for discrete-time stochastic parameter systems and shows how to parametrize the estimator gains to achieve a certain estimation error covariance. Both finite-time and steady-state estimators are considered. The results are potentially applicable to state estimation for stabilizing output feedback control systems. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Control and estimation, Stochastic processes, Random delay.

INTRODUCTION

This paper presents a continuation of our previous work [1] which provided the framework for a partial solution to the problem of stabilizing randomly delayed control systems (in the mean square sense) via Grammian assignment by static output feedback to a stochastic-parameter model. The motivation for the work results from the necessity of finding stabilizing control laws for output feedback control systems with randomly varying distributed delays. In the present work, reduced-order estimators are presented for potential application to state-estimate feedback control schemes. Random delays could be induced in the sensor data by an asynchronous time-division-multiplexed network which serves as a data communications link between the spatially dispersed components of the integrated decision and control system such as the vehicle management system of future generation aircraft [2]. In this context, the key issue is that filters and controllers designed for non-networked systems may not satisfy the performance and stability requirements in the delayed environment of network-based systems. Therefore, a state-estimation methodology is needed for compensation of the randomly varying sensor delays.

Since, in this work, the state estimation problem for a model involving randomly varying bounded sensor delays is reformulated as a stochastic parameter estimation, the analysis is based on the previous work in the latter area. Special cases of such models have been considered for minimum variance estimation by Nahi [3] in an uncertain measurement context where the

The work reported in this paper was supported in part by the National Science Foundation under Grants ECS-9322798 and CMS-9531835.

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX measurement may or may not contain the signal with certain *a priori* probabilities and later by several investigators [4–7] for more general types of random parameters. Yaz [7–10] considered such models for stable full-order stochastic observer designs. Robustness issues and guaranteed convergence rates were discussed in [11–13]. Reduced order observer design was presented in [14], and [15] was on the application to a nonlinear version of the model in [3]. In the following development, the sensor delay model is presented first. Then, all linear unbiased finite-time fixed-order estimators are characterized. It is shown how linear unbiased minimum variance estimators can be obtained. Sufficient conditions for a convergent estimator are also established under steady-state conditions.

MODEL EQUATIONS

The control system under consideration consists of a continuous-time plant (where some of the states may not be directly measurable) and the data acquisition system of a discrete-time controller which share a communications network with other subscribers [16]. Furthermore, the plant is subjected to random disturbances and the sensor data is contaminated with noise. Typically, both the sensor and controller data are subjected to randomly varying bounded delays induced by the network before they arrive at their respective destinations. In the present context, only the sensor delay in the state estimation problem is considered. The plant dynamics and delayed sensor outputs are modeled as reported in [17]:

$$\text{Plant Model:} \qquad x_{k+1} = \Phi_k x_k + F_k^1 v_k, \tag{1}$$

$$\text{Delayed Sensor Model:} \qquad \tilde{y}_k = \tilde{C}_k x_k + F_k^2 \tilde{w}_k,$$

$$y_k = (1-\zeta_k)\tilde{y}_k + \zeta_k \tilde{y}_{k-1}, \tag{2}$$

where $x_k \in R^n$ is the plant state vector to be estimated; Φ_k and F_k^1 are varying deterministic matrices; v_k is a zero mean white noise sequence at the plant input having covariance V_k with arbitrary probability distribution such that: Trace $V_k < \infty$, $\forall\, k \geq 0$; the measurement $\tilde{y}_k \in R^p$ is contaminated by zero mean white noise $\tilde{w}_k$ with covariance W_k having arbitrary probability distribution such that: Trace $W_k < \infty$, $\forall\, k \geq 0$; ζ_k is a binary white noise sequence having the expected values $E\{\zeta_k\} = \alpha_k$ and $E\{\zeta_k^2\} = \alpha_k$, where Prob $\{\zeta_k = 1\} = \alpha_k$. This assumption of one-step sensor delay is based on the rationale that, following the standard practice of communication network design [16], the induced data latency from the sensor to the controller is restricted not to exceed the sampling period. The random processes x_o, v_k, $\tilde{w}_k$, and ζ_k are assumed to be mutually independent. It is also assumed that $F_k^2 W_k F_k^{2^\top} > 0$, $\forall\, k \geq 0$. Based on (1) and (2), a compact representation of the plant and measurement system is given as:

$$\tilde{x}_k := \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix}; \qquad A_k := \begin{bmatrix} \Phi_k & 0 \\ I_n & 0 \end{bmatrix}; \qquad C_k := \left[\, (1-\zeta_k)\tilde{C}_k \quad \zeta_k \tilde{C}_{k-1} \,\right];$$
$$D_k^2 := \left[\, (1-\zeta_k)F_k^2 \quad \zeta_k F_{k-1}^2 \,\right]; \qquad D_k^1 := \begin{bmatrix} F_k^1 \\ 0 \end{bmatrix}; \qquad w_k := \begin{bmatrix} \tilde{w}_k \\ \tilde{w}_{k-1} \end{bmatrix}; \tag{3}$$

and the augmented plant model becomes

$$\text{Plant Model:} \qquad \tilde{x}_{k+1} = A_k \tilde{x}_k + D_k^1 v_k, \tag{4}$$

$$\text{Sensor Model:} \qquad y_k = C_k \tilde{x}_k + D_k^2 w_k, \tag{5}$$

where the nonwhite noise w_k is of zero mean and it has the covariance:

$$E\{w_k w_k^\top\} = \begin{bmatrix} W_k & 0 \\ o & W_{k-1} \end{bmatrix} := \Sigma_k^w. \tag{6}$$

REDUCED-ORDER ESTIMATION

Consider the compact form of system and measurement equations (4) and (5) with the necessary definitions (3) and (6). We would like to estimate only a part of the composite state vector

$$x_k = [I \quad 0] \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} := T \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix}. \tag{7}$$

The objective is to estimate x_k by using a linear estimator of the following form:

$$\hat{x}_{k+1} = K_k^1 \hat{x}_k + K_k^2 y_{k+1}, \tag{8}$$

which is an n-dimensional (reduced order) estimator for a $2n$-dimensional model. Although the form of the observer may suggest that the measurement has to be processed immediately because of the use of y_{k+1} in generating the estimate $\hat{x}_{k+1}$, equation (8) can be rewritten by introducing the auxiliary variable

$$\hat{v}_k = \hat{x}_k - K_{k-1}^2 y_k \tag{9}$$

in the following way:

$$\hat{v}_{k+1} = K_k^1 \left(\hat{v}_k + K_{k-1}^2 y_k\right). \tag{10}$$

Equation (10) uses y_k to find the one-step-ahead prediction $\hat{v}_{k+1}$ so that it can be used in actual implementation, and the necessary estimate $\hat{x}_{k+1}$ can be calculated from (9). However, we will use (8) in our development due to better mathematical tractability. We use e_k to denote the resulting estimation error:

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\ &= T\left(A_k \tilde{x}_k + D_k^1 v_k\right) - K_k^1 \hat{x}_k - K_k^2 \left[C_{k+1}\left(A_k \tilde{x}_k + D_k^1 v_k\right) + D_{k+1}^2 w_{k+1}\right] \\ &= K_k^1 e_k + \left(TA_k - K_k^1 T - K_k^2 C_{k+1} A_k\right)\tilde{x}_k + \left(T - K_k^2 C_{k+1}\right) D_k^1 v_k - K_k^2 D_{k+1}^2 w_{k+1}. \end{aligned} \tag{11}$$

To have an unbiased estimator, we have to let

$$E\{x_o\} = \hat{x}_o \quad \text{and} \quad TA_k - K_k^1 T - K_k^2 C_{k+1} A_k = 0. \tag{12}$$

The above equation, upon substitution from (3) and (7), reduces to

$$\Phi_k - K_k^1 - K_k^2 \left(\alpha_{k+1}\tilde{C}_k + (1-\alpha_{k+1})\tilde{C}_{k+1}\Phi_k\right) = 0,$$

and letting

$$K_k^1 = \Phi_k - K_k^2 \left(\alpha_{k+1}\tilde{C}_k + (1-\alpha_{k+1})\tilde{C}_{k+1}\Phi_k\right), \tag{13}$$

equation (11) yields

$$\begin{aligned} e_{k+1} = &\left[\Phi_k - K_k^2 \begin{pmatrix} \alpha_{k+1}\tilde{C}_k \\ +(1-\alpha_{k+1})\tilde{C}_{k+1}\Phi_k \end{pmatrix}\right] e_k - K_k^2 \left(C_{k+1}^e A_k \tilde{x}_k + D_{k+1}^2 w_{k+1}\right) \\ &+ \left(T - K_k^2 C_{k+1}\right) D_k^1 v_k. \end{aligned} \tag{14}$$

Denoting the estimation error covariance matrix as $P_k := E\{e_k e_k^\top\}$, the recursive relation follows:

$$\begin{aligned} P_{k+1} = &\left[\Phi_k - K_k^2 \left(\begin{bmatrix} \alpha_{k+1}\tilde{C}_k \\ +\alpha_{k+1}(1-\alpha_{k+1})\tilde{C}_{k+1}\Phi_k \end{bmatrix}\right)\right] P_k \left[\Phi_k - K_k^2 \left(\begin{bmatrix} \alpha_{k+1}\tilde{C}_k \\ +(1-\alpha_{k+1})\tilde{C}_{k+1}\Phi_k \end{bmatrix}\right)\right]^\top \\ &+ E\left\{\begin{bmatrix} \left(T - K_k^2 C_{k+1}\right) D_k^1 V_k D_k^{1\top} \\ \times \left(T - K_k^2 C_{k+1}\right)^\top \end{bmatrix}\right\} + K_k^2 E\left\{\begin{bmatrix} C_{k+1}^e A_k \tilde{X}_k A_k^\top C_{k+1}^{e\top} \\ + D_{k+1}^2 \sum_{k+1}^w D_{k+1}^{2\top} \end{bmatrix}\right\} K_k^{2\top}, \end{aligned} \tag{15}$$

where we define $\tilde{X}_k \equiv E\{\tilde{x}_k\tilde{x}_k^\top\}$. Substituting (3) and completing the square, we obtain the following equation for the filter gain matrix:

$$\begin{aligned}
&P_{k+1} - \Phi_k P_k \Phi_k^\top - F_k^1 V_k F_k^{1\top} \\
&= -K_k^2 \begin{bmatrix} \left(\alpha_{k+1}\tilde{C}_k + (1-\alpha_{k+1})\tilde{C}_{k+1}\Phi_k\right) P_k \Phi_k^\top \\ +(1-\alpha_{k+1})\tilde{C}_{k+1} F_k^1 V_k F_k^{1\top} \end{bmatrix} \\
&\quad - \begin{bmatrix} \Phi_k P_k \left(\alpha_{k+1}\tilde{C}_k^\top + (1-\alpha_{k+1})\Phi_k^\top \tilde{C}_{k+1}^\top)\right) \\ +(1-\alpha_{k+1}) F_k^1 V_k F_k^{1\top} \tilde{C}_{k+1}^\top \end{bmatrix} K_k^{2\top} \\
&\quad + K_k^2 \begin{bmatrix} \left(\alpha_{k+1}\tilde{C}_k + (1-\alpha_{k+1})\tilde{C}_{k+1}\Phi_k\right) P_k \left(\alpha_{k+1}\tilde{C}_k + (1-\alpha_{k+1})\tilde{C}_{k+1}\Phi_k\right)^\top \\ +E\left\{C_{k+1}^e A_k \tilde{X}_k A_k^\top C_{k+1}^{e\;\top}\right\} + (1-\alpha_{k+1})\tilde{C}_{k+1} F_k^1 V_k F_k^{1\top} \tilde{C}_{k+1}^\top \\ +\alpha_{k+1} F_k^2 W_k F_k^{2\top} + (1-\alpha_{k+1}) F_{k+1}^2 W_{k+1} F_{k+1}^{2\;\top} \end{bmatrix} K_k^{2\top} \\
&:= -K_k^2 N_k^\top - N_k K_k^{2\top} + K_k^2 M_k K_k^{2\top} \\
&= \left(K_k^2 - K_k^o\right) M_k \left(K_k^2 - K_k^o\right)^\top - K_k^o M_k K_k^{o\top},
\end{aligned} \tag{16}$$

where we define

$$K_k^o := N_k M_k^{-1}. \tag{17}$$

Equation (16) is obtained by using definitions in (3) and the existence of the inverse in (17) is guaranteed because $F_k^2 W_k F_k^{2\top} > 0$, $\forall\, k \geq 0$. Substituting (17) into (16) and rearranging yield:

$$\begin{aligned}
L_k L_k^\top &= P_{k+1} - \Phi_k P_k \Phi_k^\top - F_k^1 V_k F_k^{1\top} + N_k M_k^{-1} N_k^\top \\
&= \left(K_k^2 - N_k M_k^{-1}\right) M_k \left(K_k^2 - N_k M_k^{-1}\right)^\top
\end{aligned} \tag{18}$$

with $L_k \in R^{n\times p}$ because the right-hand side in (18) is positive semi-definite and of rank not exceeding p. The optimal gain in the minimum variance sense is obviously given by (17) from the above discussion and its use in (16) gives rise to the expression (18) for minimal estimation error covariance with $L_k = 0$, $\forall\, k \geq 0$. Alternatively,

$$P_{k+1} = \Phi_k P_k \Phi_k^\top + F_k^1 V_k F_k^{1\top} - N_k M_k^{-1} N_k^\top. \tag{19}$$

The covariance matrix in (19) is iteratively found by starting with $P_o = E\{e_o e_o^\top\}$. The optimal value of K_k^2, i.e., K_k^o, is computed based on P_o, and then K_k^1 is computed from K_k^2 by using (13). For a suboptimal gain, in general, (18) yields

$$L_k L_k^\top = \left(K_k^2 - N_k M_k^{-1}\right) M_k \left(K_k^2 - N_k M_k^{-1}\right)^\top. \tag{20}$$

This equation can be decomposed as in [15] into:

$$L_k U_k = \left(K_k - K_k^o\right)\sqrt{\tilde{\Sigma}_k} \tag{21}$$

for a nonsingular square root of $\tilde{\Sigma}_k$ and some orthogonal matrix sequence U_k. Solving for suboptimal gains yields:

$$K_k^2 = L_k U_k \sqrt{M_k^{-1}} + N_k M_k^{-1}. \tag{22}$$

REMARK. There is a dimensional reduction not only in the estimator in (8) and error covariance matrix P_k in (18) but also in the second moment $\tilde{X}_k$ of the state that is not necessary to iterate in full because the term in M_k containing $\tilde{X}_k$ reduces to:

$$\begin{aligned}
&E\left\{C_{k+1}^e A_k \tilde{X}_k A_k^\top C_{k+1}^{e\;\top}\right\} = \left(\alpha_{k+1} - \alpha_{k+1}^2\right) \\
&\times \left(\tilde{C}_{k+1}\Phi_k X_k^1 \Phi_k^\top \tilde{C}_{k+1}^\top - \tilde{C}_{k+1}\Phi_k X_k^1 \tilde{C}_k^\top - \tilde{C}_k X_k^1 \Phi_k^\top \tilde{C}_{k+1}^\top + \tilde{C}_{k+1} X_k^1 \tilde{C}_{k+1}^\top\right),
\end{aligned} \tag{23}$$

where $X_k^1 = E\{x_k x_k^\top\} \in R^{n\times n}$ is evaluated by using (1) from the following recursive relation:

$$X_{k+1}^1 = \Phi_k X_k^1 \Phi_k^\top + F_k^1 V_k {F_k^1}^\top. \tag{24}$$

The results are now summarized as follows.

THEOREM 1. *Given the models (3)–(6) and the linear unbiased reduced order estimator (8)–(10), all assignable estimation covariances are given by (18) where N_k and M_k are defined in (16). The gains that guarantee an assignable covariance matrix are given by (22) where the term dependent on the second moment of the state vector is given by (23) and (24). The minimum possible error covariance is given by (19) and the corresponding optimal estimator gain is found from (17).*

INFINITE-TIME (STEADY-STATE) ESTIMATION

Let us assume that all deterministically time-varying quantities are constant and all randomly varying disturbances are weakly stationary (with constant first two moments). Under these assumptions, it is possible to obtain steady-state results for asymptotically stable systems because P_k and K_k can converge to respective constant matrices only if the X_k^1 sequence converges. This can be achieved if the spectral radius $\rho(\Phi) < 1$ so that there exists a unique $X^1 = {X^1}^\top \geq 0$ that solves:

$$X^1 = \Phi X^1 \Phi^\top + F^1 V {F^1}^\top = \sum_{k=0}^{\infty} \Phi_k F^1 V {F^1}^\top \Phi_k^\top. \tag{25}$$

The proof of convergence of the proposed steady-state estimator is a subject of future research. New algorithms are being tested for realistic applications [2] and the results will be presented in a forthcoming publication. Extension of this delay compensation approach is under consideration for solving the complete problem of distributed random delays in both sensors and actuators [18].

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