

# Stochastic modeling of fatigue crack damage for information-based maintenance

Asok Ray<sup>a</sup> and Shashi Phoha<sup>b</sup>

<sup>a</sup>*Mechanical Engineering Department, The Pennsylvania State University, University Park, PA 16802, USA*

<sup>b</sup>*Information Systems Department, Applied Research Laboratory, The Pennsylvania State University, University Park, PA 16802, USA*

E-mail: axr2@psu.edu

The concept of *information-based maintenance* is that of updating decisions for inspection, repair, and maintenance scheduling based on evolving knowledge of operation history and anticipated usage of the machinery as well as the physics and dynamics of material degradation in critical components. This paper presents a stochastic model of fatigue crack damage in metallic structures for application to information-based maintenance of operating machinery. The information on operation history allows the stochastic model to predict the current state of damage, and the information on anticipated usage of the machinery facilitates forecasting the remaining service life based on the stress level to which the critical components are likely to be subjected. The Karhunen–Loève expansion for nonstationary processes is utilized for formulating the stochastic model which generates the crack length statistics in the setting of a two-parameter lognormal distribution. Hypothesis tests are built upon the (conditional) probability density function of crack damage that does not require the solution of stochastic differential equations in either Wiener integral or Itô integral settings. Consequently, structural damage and remaining life of stressed components can be predicted to make maintenance decisions in real time. The damage model is verified by comparison with experimental data of fatigue crack statistics for 2024-T3 and 7075-T6 aluminum alloys. Examples are presented to demonstrate how this concept can be applied to hypothesis testing and remaining life prediction.

**Keywords:** stochastic modeling, predictive maintenance, fatigue crack damage

## 1. Introduction

Decision systems for maintenance of operating machinery are synthesized by taking mission objectives, (e.g., productivity and performance), service life, and overall cost into consideration [14]. The current state-of-the-art in synthesizing decision systems for maintenance of operating machinery focuses on enhancement of reliability and diagnostic capabilities under constraints that often do not adequately represent

the material degradation aspects of critical plant components [8]. The reason is that traditional design methodologies are usually based upon the assumption of invariant characteristics of structural materials. However, in reality, since structural integrity of critical components monotonically degrades with age and cycles of operation, the maintenance strategies for new and old machinery are likely to be significantly different even if they are identically operated. Therefore, it is highly desirable to update operation and maintenance strategies as new information on history of machinery operation and anticipated usage becomes available.

The overall concept of *information-based maintenance* is that of updating decisions for inspection, repair, and maintenance scheduling based on evolving knowledge of operation history and anticipated usage of the machinery, as well as the physics and dynamics of material degradation in critical components. The key steps in the synthesis of maintenance strategies are formulations of

- Physics-based dynamic models of material degradation including identification of failure precursors.
- Statistical models of hypothesis tests for risk analysis and remaining life prediction under different operating conditions.
- Decision algorithms for maintenance scheduling based on the information derived from operation history (e.g., sensor data and expert knowledge) and anticipated usage of the machinery.

This paper addresses the issues of reliability and maintenance in operating machinery where the major source of failures is the fatigue crack damage of metallic structures. Specifically, the statistics of (nonstationary) fatigue crack damage in metallic materials are generated from

- the information on operation history that allows the fatigue crack model to predict the current state of damage; and
- the information on anticipated usage of the machinery that facilitates forecasting the remaining service life based on the (time-dependent) stress level to which the critical components are likely to be subjected.

For example, at any given time, mechanical stress on a critical component of the machinery can be calculated based on the (recorded and anticipated) load profile using either an operational model of the plant or archived operational data. The resulting stress profile can, in turn, be used to excite a stochastic damage model for predicting the current damage state of the critical component and also its remaining service life.

The specific objectives of this paper are (i) to formulate a (real-time) stochastic model of the fatigue crack propagation in metallic structures that is based on the physics of material degradation, and (ii) to make use of the fatigue crack model along with the information on operation history and anticipated usage of the machinery for risk assessment, remaining life prediction and maintenance scheduling.

This paper is organized into five sections including the introduction. Section 2 presents an overview of the state-of-the-art of stochastic modeling of fatigue crack propagation. Section 3 provides details of model formulation. Section 4 shows how the statistics of fatigue crack damage can be utilized for failure prognosis, risk analysis, and decision-making for maintenance. Section 5 summarizes the paper and concludes the major benefits and applications of this maintenance concept.

## **2. Stochastic modeling of fatigue crack propagation**

Stochastic modeling of fatigue crack propagation in metallic materials is a relatively new area of research. An extensive list of technical literature representing the state-of-the-art is cited by Sobczyk and Spencer [19], and the special issue of *Engineering Fracture Mechanics* [18] presents recent developments in this field. One approach to stochastic modeling of fatigue crack growth is to randomize the coefficients of an established deterministic model to represent material inhomogeneity [4]. Another approach is to generate the necessary stochastic information by multiplying the deterministic dynamics of fatigue crack growth with a non-negative random process [2,20]. The process of fatigue crack propagation is thus modeled by nonlinear stochastic differential equations in the Itô setting [11]. Specifically, Kolmogorov forward and backward diffusion equations, which require solutions of nonlinear partial differential equations, have been proposed to generate the statistical information required for risk analysis of mechanical structures [1,9]. These nonlinear partial differential equations can only be solved numerically; the computational procedures, however, are computationally intensive as they rely on fine-mesh models using finite-element or combined finite-difference and finite-element methods [19]. Therefore, although this numerical approach might be useful for making off-line decisions for design analysis and predictive maintenance, it is not sufficiently fast for on-line damage monitoring, failure prognosis, and prediction of remaining service life. Casciati et al. [3] have analytically approximated the solution of Itô equations by Hermite moments to obtain a probability distribution function of the crack length. To enhance the computational efficiency for on-line execution of the damage estimation and life prediction algorithms, Ray and Tangirala [16] have developed an algorithm for real-time estimation of fatigue crack damage based on the underlying principle of extended Kalman filtering. In this approach, the first two moments of the stochastic damage state are computed on-line by constructing the stochastic differential equations in the Wiener setting as opposed to the Itô setting. This damage estimation algorithm follows the two-state model structure of Spencer et al. [20], where the shaping filter is constructed with additive white Gaussian noise. The concept of extended Kalman filtering has been used without any sensor(s) for continuous measurements of the crack length. The absence of sensor data is equivalent to having the inverse of the intensity of measurement noise covariance tend to zero, implying that the filter gain approaches zero. Consequently, the conditional density function generated by the filter becomes

identical to the prior density function whose evolution is governed by the Kolmogorov forward equation [10].

The dynamic model of fatigue crack damage presented in this paper is formulated based upon the underlying principle of Karhunen–Loève expansion of nonstationary processes, and does not require solutions of the extended Kalman filter equation in the Wiener integral setting or the Kolmogorov forward equation in the Itô integral setting for identification of the probability density function of crack damage. Therefore, structural damage and remaining life of stressed components are then predicted in real time by utilizing up-to-date knowledge of operation history and anticipated load profile. As such, maintenance decisions can be updated as new information becomes available. The damage model has been verified by comparison with experimental data of fatigue crack statistics for 2024-T3 and 7075-T6 aluminum alloys. Examples are presented to illustrate how the stochastic damage model can be used to generate hypothesis tests for reliability analysis of critical components.

### 3. Modeling of fatigue crack damage for real-time applications

The stochastic model of fatigue crack damage presented in this paper is built upon a deterministic model of fatigue crack growth [13], which is based on the principle of short crack growth. The Newman model represents the mean value of the fatigue crack growth process down to micro-cracks of the order of material defect size and has the following form:

$$d\mu_c(t) = \tilde{C}(\Delta K_{eff})^{\tilde{m}} dt; \text{ given } \mu_c(t_0) = \mu_{c0} > 0 \text{ and } t > t_0, \quad (1)$$

$$\Delta K_{eff} = (S_{max} - S_0) \sqrt{\pi \mu_c} F, \quad (2)$$

where  $\mu_c$  is the estimated mean of the (time-dependent) crack length process  $c(\omega, t)$  conditioned on the initial crack length  $c(\omega, t_0)$ ; a sample of the stressed component is indicated by  $\omega$ ; the time  $t$  is expressed in units of number of cycles and  $t_0$  is the initial time;  $d\mu_c$  is the so-called differential of  $\mu_c$ , as commonly used in the fracture mechanics literature [21];  $\Delta K_{eff}$  is the effective stress intensity factor range; the constants  $\tilde{C}$  and  $\tilde{m}$  are material-dependent;  $S_{max}$  is the maximum applied remote stress;  $S_0$  is the crack opening stress; and  $F$  is the correction factor for geometrical configuration. Details of this model are reported by Newman [13]. It should be noted, however, that any deterministic fatigue crack growth law can be used in this formulation provided that the average characteristics of the crack growth profile are accurately represented.

Ray and Tangirala [17] have modeled the crack length process  $c(\omega, t)$  to be implicitly dependent on the (discrete) time parameter,  $t$ , and the (estimated) variance  $\sigma_c^2(t)$  to be explicitly dependent on the (estimated) mean  $\mu_c(t)$ , which is directly obtained by solving equations (1) and (2). Our approach is to model  $c(\omega, t)$  with a continuous function of  $\mu_c(t)$  as the independent variable in lieu of time,  $t$ . To this effect, we introduce a continuous function of the (estimated) mean crack length,  $\mu_c(t)$ , as

$$\tau(t) := \left( \frac{\mu_c(t)}{\mu_c(t_0)} - 1 \right) \quad \forall t \in [t_0, t_f] \text{ for } t_f < \infty. \quad (3)$$

The dimensionless parameter  $\tau$ , which is a monotonically non-decreasing continuous function of  $t$ , is used as the independent variable in the sequel. In view of the definition in (3), the stochastic process  $\{c(\omega, t) : t \geq t_0\}$  is denoted as  $\{c_\tau(\omega) : \tau \geq 0\}$ .

Assuming that the initial crack length,  $c_0(\omega) = c_\tau(\omega)|_{\tau=0}$ , of a critical component can be directly or indirectly measured by inspection, the normalized crack length process conditioned on the initial crack length is defined as

$$\psi_\tau(\omega) := \left\{ \frac{c_\tau(\omega)}{c_0(\omega)} \Big|_{c_0} \right\} \quad \text{for } \tau \geq 0. \quad (4)$$

(Note that  $\psi_0 = 1$ .)

Since  $\{c(\omega, t) : t \geq t_0\}$  is a continuous function of  $t$  (and hence  $\{c_\tau : \tau \geq 0\}$  is a continuous function of  $\tau$ ) in the mean square sense,  $\psi_\tau$  has a continuous covariance function [10] and, therefore, can be expressed via the Karhunen–Loève expansion [5]. Let the continuous process  $\psi_\tau$  be discretized at  $m$  points beyond  $\tau = 0$  as an  $m$ -dimensional random vector  $\Psi = [\psi_1, \psi_2, \dots, \psi_m]^T$ , where  $\psi_j := \psi_{\tau_j}$ ,  $j = 1, 2, \dots, m$ . The covariance matrix of the random vector  $\Psi$  can be expressed as  $K_{\Psi\Psi} = \Phi\Lambda\Phi^T$ , with  $\Phi^T\Phi = 1$  and  $\Lambda = \text{diag}[\lambda_1, \dots, \lambda_m]$  ordered as  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ , which are the  $m$  eigenvalues of  $K_{\Psi\Psi}$ . The deterministic matrix  $\Phi$  is composed of columns,  $\phi^j$ , which are the eigenvectors of  $K_{\Psi\Psi}$ . Therefore, the random vector  $\Psi$  can be defined as

$$\Psi(\omega) := E[\Psi] + \Phi Z(\omega) \quad \text{and} \quad \text{Cov}(Z) = \Lambda = \text{diag}[\lambda_1, \dots, \lambda_m]. \quad (5)$$

Orthogonality of the Karhunen–Loève expansion ensures that the random vector  $Z = [z_1, z_2, \dots, z_m]$  is a set of zero-mean independent random variables. This leads to

$$\psi_k = E[\psi_k] + \sum_{j=1}^m (\phi_k^j z_j), \quad k = 1, 2, \dots, m, \quad (6)$$

where  $\psi_k := \psi_{\tau_k}$  and  $E[\psi_k] = E[c_{\tau_k}/c_0|_{c_0=\mu_{c_0}}] = \mu_c(\tau_k)/\mu_{c_0}$ , and  $\phi_k^j := \phi^j(\tau_k)$  is the  $k$ th element of the eigenvector  $\phi^j$ . If the first  $M$  eigenvalues are dominant, then the random variable  $\psi_k$  can be approximated as  $\hat{\psi}_k(M)$  by truncation of the last  $(m - M)$  terms in (6) as

$$\hat{\psi}_k(M) := E[\psi_k] + \sum_{j=1}^M (\phi_k^j z_j), \quad k = 1, 2, \dots, m, \quad 1 \leq M < m. \quad (7)$$

Consequently, because of the orthogonality property of the Karhunen–Loève expansion, the covariance of the continuous process,  $\hat{\psi}_\tau$ , can be expressed as

$$K_{\hat{\psi}\hat{\psi}}(\tau_k, \tau_\ell) = \text{Cov}(\hat{\psi}_k, \hat{\psi}_\ell) = \sum_{j=1}^M \lambda_j \phi_k^j \phi_\ell^j \quad (8)$$

having the associated minimum mean square error

$$\bar{\varepsilon}^2(M) = E[(\Psi - \hat{\Psi}(M))^T (\Psi - \hat{\Psi}(M))] = \sum_{j=M+1}^m \lambda_j. \quad (9)$$

Statistical information generated from each of four sets of fatigue test data [7, 22] shows that  $\lambda_1$  is the dominant eigenvalue. The mean square error in (9) is in the range of 1.0 to 3.5% if  $M$  is chosen to be 1, i.e., if only the principal eigenvector,  $\phi^1(\tau)$ , associated with the dominant eigenvalue,  $\lambda_1$ , is used to model the process  $\psi_\tau$ . A constitutive relationship is proposed for  $\phi^1(\tau)$  as a continuous function of  $\tau$ :

$$\phi^1(\tau) = \left( \sqrt{Q_c/\lambda_1} \right) \left( \frac{e^{\xi\tau} - 1}{\xi} \right). \quad (10)$$

The covariance in (8) is modified by (10) as

$$K_{\hat{\psi}\hat{\psi}}(\tau + \theta, \tau) = Q_c \left( \frac{e^{\xi(\tau+\theta)} - 1}{\xi} \right) \left( \frac{e^{\xi\tau} - 1}{\xi} \right), \quad (11)$$

and  $K_{\psi\psi}(\tau, \tau) \approx K_{\hat{\psi}\hat{\psi}}(\tau, \tau) = \sigma_{\hat{\psi}}^2(\tau)$  is readily obtained from (11). Based on statistical fatigue test data, the model parameters  $\xi$  and  $Q_c$  are found to have the following characteristics:

- $\xi$  is material dependent but it is independent of the peak stress and stress ratio to which the specimens were subjected.
- $Q_c$  is both material dependent and stress dependent.

**Remark 1.** Separation of the stochastic process  $(\psi_\tau - E[\psi_\tau])$  into (deterministic)  $\tau$ -dependent and (random)  $\omega$ -dependent variables follows by choosing  $M = 1$  in (7). If  $M > 1$  is chosen, then the additional terms to be included in the model of (11) will act as small perturbations.

Standard statistical tests (e.g., chi-square and Kolmogorov–Smirnov) conducted on the fatigue test data revealed that the crack length is approximately lognormal distributed [17, 23]. Based on this information, a lognormal random variable is defined as

$$\eta(\omega) := \left( \sqrt{Q_c/\lambda_1} \right) z_1(\omega) \quad (12)$$

with  $E[\eta|_{c_0=\mu_{c_0}}] = 0$  and  $Var[\eta|_{c_0=\mu_{c_0}}] = Q_c$ . Setting  $M = 1$  in (7), and following (3), (6), (10) and (12), the fatigue-induced crack length is modeled as

$$\begin{aligned} \psi_\tau(\omega) &= \frac{c_\tau(\omega)}{c_0(\omega)} \Big|_{c_0(\omega)=\mu_{c_0}} = \left( \tau + 1 + \left( \frac{e^{\xi\tau} - 1}{\xi} \right) \eta(\omega) \right) \\ \Rightarrow d\psi_\tau(\omega) &= \frac{dc_\tau(\omega)}{c_0(\omega)} \Big|_{c_0(\omega)=\mu_{c_0}} = (1 + e^{\xi\tau} \eta(\omega)) d\tau. \end{aligned} \quad (13)$$

**Remark 2.** The dynamics of fatigue crack growth are represented in (13) in terms of the dimensionless parameter  $\tau$ , which is a monotonically increasing function of time (or cycles) as defined in (3).

The physical phenomena,  $c_0(\omega) > 0$  and  $dc_\tau(\omega) \geq 0, \forall \omega \forall \tau \geq 0$ , satisfy the conditions  $\mu_{c0} > 0$  and  $d\mu_c(\tau) \geq 0$ , and imply that the model in (13) must satisfy the inequality constraint  $d\psi_\tau(\omega)/d\tau \geq 0, \forall \omega \forall \tau \geq 0$ . Therefore, the inequality

$$\inf_{\omega} \left( \frac{d\psi_\tau(\omega)}{d\tau} \right) \geq 0 \Rightarrow \inf_{\omega} \eta(\omega) \geq -e^{-\xi\tau} \quad (14)$$

must be satisfied for all  $\tau \leq \tau_f$  up to the critical crack length,  $\mu_{cf}$ , at which the service life of the stressed structure is considered to be completely expended. Following (3), the dimensionless parameter,  $\tau_f$ , can be defined as

$$\tau_f := \frac{\mu_{cf}}{\mu_{c0}} - 1. \quad (15)$$

Following (13), if  $(d\psi_\tau(\omega)/d\tau)_{\tau=\tau_f} = 0$  for some  $\omega$ , then  $d\psi_\tau(\omega)/d\tau \geq 0, \forall t \in [0, \tau_f)$ , for the same  $\omega$ . Therefore, setting  $\inf_{\omega} (d\psi_\tau(\omega)/d\tau)_{\tau=\tau_f} = 0$  guarantees a non-negative crack growth rate for all cracks less than the critical crack length  $\mu_{cf}$ .

**Remark 3.** In general,  $\mu_{cf}$  represents the critical crack length beyond which the crack growth rate becomes very large, rapidly leading to complete rupture [9]. Therefore,  $\mu_{cf}$  (and hence  $\tau_f$ ) depends on several factors, including the geometry and dimensions of the stressed structure, the allowable factor of safety, and sensitivity and resolution of the inspection equipment.

The zero-mean lognormal-distributed random variable  $\eta(\omega)$  in (12) is expressed in terms of a Gaussian variable,  $y(\omega)$ , as

$$\eta(\omega) = e^{y(\omega)} - E \left[ e^{y(\omega)} \Big|_{c_0=\mu_{c0}} \right]; \quad y \sim N \left( m, \sigma^2 \Big|_{c_0=\mu_{c0}} \right). \quad (16)$$

The following two coupled algebraic equations are derived from (12), (14) and (16):

$$\inf_{\omega} \eta(\omega) = -\exp(-\xi\tau_f) \Rightarrow m + \frac{\sigma^2}{2} = -\xi\tau_f, \quad (17)$$

$$\exp \left( 2 \left( m + \frac{\sigma^2}{2} \right) \right) (\exp(\sigma^2) - 1) = Q_c. \quad (18)$$

Equations (17) and (18) are solved simultaneously to yield

$$\sigma^2 = \ln(1 + Q_c \exp(2\xi\tau_f)); \quad m = - \left( \xi\tau_f + \frac{\sigma^2}{2} \right). \quad (19)$$

Given  $\xi$ ,  $Q_c$  and  $\tau_f$ , the unknowns  $m$  and  $\sigma^2$  in (19) can be evaluated to specify the probability density function (pdf) of the Gaussian random variable  $y(\omega)$  in (16). This information, in turn, prescribes the lognormal (conditional) pdf of  $\eta(\omega) = e^{y(\omega) - e^{m+\sigma^2/2}}$  as

$$f_{\eta|c_0}(\theta | \mu_{c_0}) = \begin{cases} \frac{\exp\left[\frac{1}{-2\sigma^2}(\ln(\theta + e^{m+\sigma^2/2}) - m)^2\right]}{(\theta + e^{m+\sigma^2/2})\sqrt{2\pi\sigma^2}} & \theta \geq -e^{m+\sigma^2/2}, \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

Using (3), (13) and (20), the conditional pdf of

$$\psi_\tau(\omega) = \left( \frac{c_\tau(\omega)}{c_0(\omega)} \Big|_{c_0=\mu_{c_0}} \right) = \left( 1 + \tau + \left( \frac{e^{\xi\tau} - 1}{\xi} \right) \eta(\omega) \right)$$

becomes

$$f_{\psi|c_0}(\theta, \tau | \mu_{c_0}) = \begin{cases} \frac{\exp\left(\frac{\ln\left(\left(\frac{(\theta - \tau - 1)\xi}{e^{\xi\tau} - 1}\right) + e^{m+\sigma^2/2}\right) - m}{-2\sigma^2}\right)^2}{\left(\left(\frac{(\theta - \tau - 1)\xi}{e^{\xi\tau} - 1}\right) + e^{m+\sigma^2/2}\right)\left(\frac{e^{\xi\tau} - 1}{\xi}\right)\sqrt{2\pi\sigma^2}} & \text{for } \theta \geq \tau + 1 - \frac{e^{m+\sigma^2/2}(e^{\xi\tau} - 1)}{\xi}, \\ 0 & \text{otherwise.} \end{cases} \quad (21)$$

The (unconditional) statistics of crack length  $c_\tau(\omega)$  can be obtained in terms of the (conditional) statistics of  $\psi_\tau(\omega)$  in (22) and the (unconditional) statistics of the initial crack length as

$$f_c(\cdot, \tau) = f_{\psi|c_0}(\cdot, \tau | \mu_{c_0}) \times f_{c_0}(\mu_{c_0}). \quad (22)$$

Hence, if  $f_{c_0}(\cdot)$  is available, (22) provides an analytical expression for  $f_c(\cdot, \tau)$ .

#### 4. Usage of the fatigue crack model for information-based maintenance

Models of damage statistics are integrated with the available information on operation history of the machinery and its anticipated usage to formulate viable alternatives for maintenance strategies. Traditionally, the risk index and remaining service life [1] of mechanical structures and machine components are calculated off-line, based on statistical models of material degradation, operating history and anticipated disruptions in the plant operation (i.e., the stress levels to which critical components are expected to be subjected). Since the service life of a critical component is finite (as



the damage accumulation is monotone with time), scheduling of plant operation and maintenance is based on the fact that damage accumulation is likely to be accelerated or retarded in the event of unscheduled operations. Therefore, on-line computation of damage statistics allows refinements of the risk index and residual service life prediction [1, 2] as time progresses. Specifically, on-line damage prediction supports decision making that influences safety procedures, mission accomplishments, and the time interval between major maintenance actions. In the absence of any change in plant operations after the current time, the maintenance strategy may remain unaltered. However, under unscheduled changes in operation, statistics of fatigue crack damage in critical components become different due to different stress levels and hence the maintenance strategy may have to be updated. Since the crack damage is modeled as a Markov process, its statistics are dependent on the initial state of damage (e.g., the initial crack length  $c_0$ ) and the stress profile to which the cracked component is subjected. As new information on operation history and anticipated usage becomes available, the damage statistics (and hence the failure hypotheses) are updated in real time. In this context, we illustrate how the fatigue crack statistics can be used to estimate the current state of damage and remaining service life of critical components to issue warnings and alert and influence maintenance strategy.

Let  $(M + 1)$  hypotheses be defined based on a partition of the crack length in the range  $[\bar{c}_0, \infty)$ , where  $\bar{c}_0$  is the (known) minimum threshold of the initial crack length. The first  $M$  hypotheses are defined on the range  $[\bar{c}_0, \bar{c}_M]$ , where  $\bar{c}_M = \mu_{cf}$  is the critical crack length beyond which the crack growth rate becomes very large, rapidly leading to complete rupture (see remark 3 in section 3), as

$$\begin{aligned} H_0: & \quad c_\tau \in [\bar{c}_0, \bar{c}_1), \\ H_1: & \quad c_\tau \in [\bar{c}_1, \bar{c}_2), \\ & \quad \vdots \\ H_{M-1}: & \quad c_\tau \in [\bar{c}_{M-1}, \bar{c}_M), \quad \text{where } \bar{c}_i = \bar{c}_0 + i \frac{(\bar{c}_M - \bar{c}_0)}{M}, \quad i = 1, 2, \dots, (M - 1). \end{aligned} \quad (23)$$

The last, i.e., the  $M$ th, hypothesis is defined as  $H_M: c_\tau \in [\bar{c}_M, \infty)$ , which is popularly known as the unstable crack region in the fracture mechanics literature [21]. Each of these  $(M + 1)$  hypotheses represents a distinct range in the entire space of crack lengths from an initial value of  $\mu_{c_0}$  until rupture occurs, and together they form an exhaustive set of mutually exclusive regions in the damage space. For the special case of the initial crack length being measured precisely, (i.e.,  $\sigma_{c_0}^2 \approx 0 \Rightarrow c_\tau|_{\tau=0} = c_0 \approx \mu_{c_0}$ ), the minimum threshold is set as  $\bar{c}_0 = \mu_{c_0}$ , and the first  $M$  hypotheses are expressed as

$$c_\tau \in H_i = [\bar{c}_i, \bar{c}_{i+1}) \Rightarrow \frac{c_\tau}{c_0} \Big|_{c_0 = \mu_{c_0}} = \psi_\tau \in \left[ \frac{\bar{c}_i}{\mu_{c_0}}, \frac{\bar{c}_{i+1}}{\mu_{c_0}} \right); \quad i = 0, 1, 2, \dots, M - 1. \quad (24)$$

The probability that the  $j$ th hypothesis,  $H_j$ , holds at any instant of time,  $\tau$ , is then obtained from the instantaneous (conditional) probability distribution function (PDF)  $F_{\psi|c_0}(\cdot; \tau | \mu_{c_0})$  of  $\psi_\tau$  as

$$P[H_j(\tau)] = F_{\psi|c_0}\left(\frac{\bar{c}_{j+1}}{\mu_{c_0}}; \tau\right) - F_{\psi|c_0}\left(\frac{\bar{c}_j}{\mu_{c_0}}; \tau\right), \quad j = 0, 1, 2, \dots, M-1, \quad (25)$$

$$P[H_M(\tau)] = 1 - \sum_{j=0}^{M-1} P[H_j(\tau)],$$

where  $F_{\psi|c_0}(\cdot; \tau | \mu_{c_0})$  is generated from the probability density function (pdf)  $f_{\psi|c_0}(\cdot; \tau | \mu_{c_0})$  in (21) without (computationally expensive) integration. Since  $f_{\psi|c_0}$  is lognormal-distributed, conversion of the range of integration in the log scale allows evaluation of the error function via table-lookup. (These details are straightforward and are not presented in this paper.) Following (25), probabilities of each of the  $(M+1)$  hypotheses can thus be computed in real time as a function of the dimensionless parameter  $\tau$ , which is a monotonically increasing function of time (or cycles) as defined in (3). From the perspective of operation and maintenance, it is more convenient to express probabilities of these hypotheses as functions of time rather than functions of  $\tau$ . Therefore, in the sequel we present these results as functions of time (or cycles) by making use of (1) and (2) in (3) and (15).

To elucidate the concept of hypothesis testing for maintenance decision support, we present examples based on the fatigue test data sets [7, 22]. The probability that the random crack length  $\{c(\omega, t): t \geq t_0\}$  at a given time  $t$  is located in one and only one of these segments is computed in real time by using (25). For each data set,  $\mu_{c_0} = 9.0$  mm and  $\sigma_{c_0}^2 = 0$ , implying that the minimum threshold of the initial crack length is  $\bar{c}_0 = \mu_{c_0} = 9.0$  mm. The critical crack length is chosen based on the geometry of the test specimens: For the Virkler experiment (in which the specimen half-width is 76.2 mm),  $\bar{c}_M = 45.0$  mm; for the Ghonem experiments (in which the specimen half-width is 50.4 mm),  $\bar{c}_M = 27.0$  mm. The damage state space  $[\bar{c}_0, \infty)$  is partitioned into  $(M+1)$ . In these examples, we have chosen eleven hypotheses (i.e.,  $M = 10$ ) for both the Virkler and Ghonem data sets. The range of each hypothesis is defined as depicted in tables 1 and 2, respectively. The time evolution of probability of the hypotheses for the simulation of the Virkler experiment is shown in figure 1 and for the three Ghonem experiments in figures 2, 3, and 4. In each case, the plot of  $H_0$  begins with a probability equal to 1 at time  $t = t_0$  and later diminishes as the crack grows with time (i.e., number of load cycles applied). The probability of each of the remaining hypotheses  $H_1$  to  $H_9$  is initially zero, and then increases to a maximum and subsequently decreases as the crack growth process progresses with time. The probability of the last hypothesis  $H_{10}$  (on the extreme right in figures 1–4) of unstable crack growth beyond the critical crack length,  $\bar{c}_M$ , initially remains at zero and increases rapidly only when the specimen is close to rupture. At this stage, the probability of each of the remaining

Table 1

Crack length hypotheses for the Virkler data.

Description	Range
Hypothesis $H_0$	9.00 mm < $c(t)$ < 12.6 mm
Hypothesis $H_1$	12.6 mm < $c(t)$ < 16.2 mm
⋮	⋮
Hypothesis $H_9$	41.4 mm < $c(t)$ < 45.0 mm
Hypothesis $H_{10}$	45.0 mm < $c(t)$ (unstable crack growth)

Table 2

Crack length hypotheses for the three Ghonem data sets.

Description	Range
Hypothesis $H_0$	9.00 mm < $c(t)$ < 10.8 mm
Hypothesis $H_1$	10.8 mm < $c(t)$ < 12.6 mm
⋮	⋮
Hypothesis $H_9$	25.2 mm < $c(t)$ < 27.0 mm
Hypothesis $H_{10}$	27.0 mm < $c(t)$ (unstable crack growth)

hypotheses is either zero or rapidly diminishes to zero. For example, the space of crack lengths, defined by  $[\bar{c}_0, \infty)$ , can be partitioned into four hypotheses, denoting three regions of green, yellow and red alert conditions for the first three hypotheses and catastrophic conditions for the fourth hypothesis.

While alerts and warnings are useful for operational support and safety enhancement, operations planning and maintenance scheduling require remaining service life prediction. Equipment readiness assessment and failure prognosis based on current condition and projected usage of the machinery are important tools for operations and maintenance planning, especially in an information-based maintenance environment where access to all pertinent information is enabled. Remaining life prediction can be obtained based on the stochastic fatigue damage model in section 3.

Using pdf  $f_{\psi|e_0}$  in (21), the remaining service life  $T(t, Y_d(t), \varepsilon)$  can be computed at any specified time instant,  $t$ , based on a desired plant operational profile  $Y_d(t) = \{y(\theta): \theta \geq t\}$  and a confidence level  $(1 - \varepsilon)$ . The algorithm for predicting the remaining service life is obtained as

$$T(t; Y_d(t); \varepsilon) = \sup\{\theta \in [0, \infty): P[c_{t+\theta} \leq \bar{c}_M] > (1 - \varepsilon)\}. \quad (26)$$

Equation (26) implies that if the plant operation is scheduled to yield the desired output, then  $T(t, Y_d(t), \varepsilon)$  is the least upper bound of the time of operation such that the probability of the crack length  $c_{t+T}$  not exceeding  $\bar{c}_M$  is greater than  $(1 - \varepsilon)$ . The

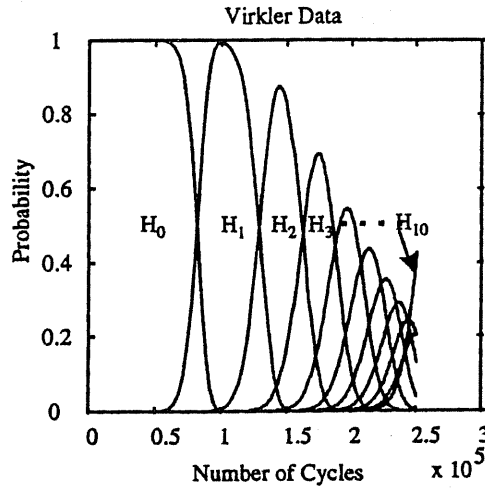


Figure 1. Probabilities of hypotheses.

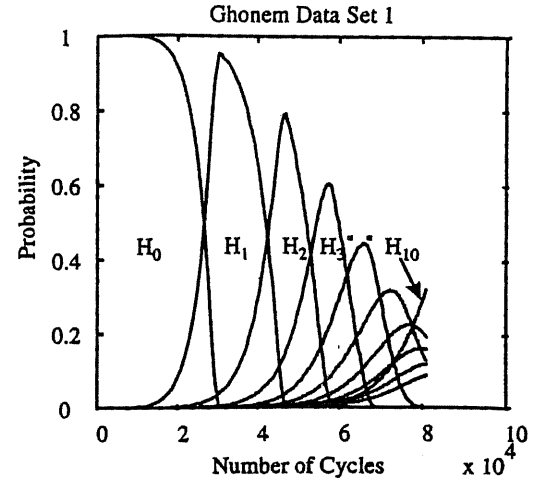


Figure 2. Probabilities of hypotheses.

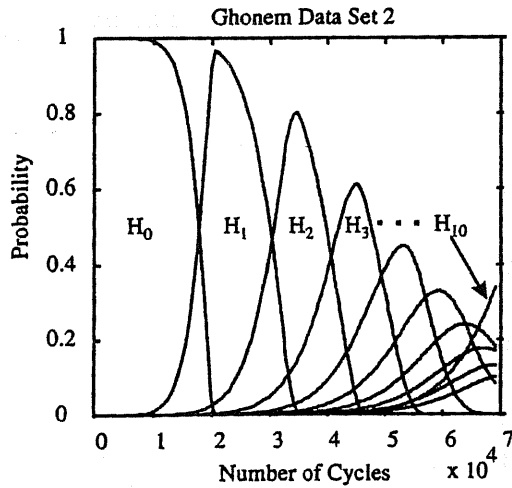


Figure 3. Probabilities of hypotheses.

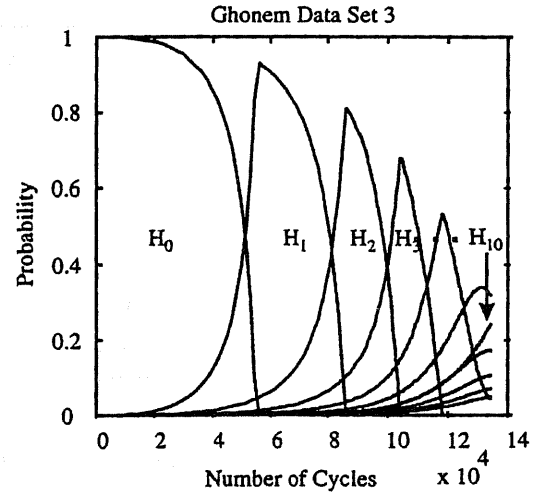


Figure 4. Probabilities of hypotheses.

prediction algorithm in (26) is executed in real time based on the current information. The generated results are then conveyed to a decision-making module (for example, a discrete-event supervisor [6]) at a higher level for failure prognosis, life extending control, and maintenance scheduling, or simply for generation of warnings and alerts. (These results may also be displayed as a decision support tool for human operators.) The objective is to determine the statistical confidence with which plant operations can be planned for a specified period of time or to evaluate alternative operational scenarios. This is also of considerable importance in the scheduling of maintenance to avoid untimely shutdowns since failure prognostic information is inherent in remaining life prediction. Some of these issues have been addressed by Ray and Phoha [15] in the context of information-based maintenance.

## 5. Summary and conclusions

The overall concept of *information-based maintenance* is that of updating decisions for inspection, repair, and maintenance scheduling based on evolving knowledge of operation history and anticipated usage of the machinery. In this context, this paper presents a formulation and verification of a stochastic model of fatigue crack damage in metallic structures. The stochastic model allows updating of the damage statistics in real time based on the recent information to synthesize decision policies for risk assessment and maintenance. The information on operation history and anticipated usage of the machinery allows the stochastic model (i) to predict the current state of damage, and (ii) to forecast the remaining service life based on the stress level to which the critical components are likely to be subjected. The generated statistical information is essential for failure prognosis and reliability analysis of critical machinery components and also for development of a maintenance strategy. Maintenance decision algorithms can be formulated by the use of failure hypotheses that are generated based on the probability density function of crack damage, which does not require solutions of stochastic differential equations in either Wiener integral or Itô integral settings. The two-parameter lognormal-distribution of fatigue crack statistics, developed in this paper, has been verified by comparison with experimental data of 2024-T3 and 7075-T6 aluminum alloys at different levels of stress excitation. Examples are presented to illustrate how the stochastic damage model can be used to generate and update failure hypotheses based on current information.

Potential applications of information-based maintenance include (i) formulation of decision policies for maintenance scheduling in real time based on up-to-date information of machinery operation history and anticipated usage, (ii) generation of alerts and warnings for operational support and safety enhancement, (iii) equipment readiness assessment and failure prognosis based on current condition and projected usage, and (iv) remaining life prediction of machinery components.

The condition monitoring and maintenance strategies presented here are suitable for machinery operated under constant load excitation. Extension of the stochastic crack damage model to varying amplitude load excitation is a subject of current research. While the stochastic modeling approach presented in this paper focuses on inherent material uncertainties, there are two other major sources of uncertainties, namely, random loading and unknown initial conditions. Although random loading has not been considered here, uncertain initial conditions can be incorporated into this formulation. A unified model that accounts for all three primary sources of uncertainties in crack growth needs to be developed before practical applications become viable.

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