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Observer-Embedded L_2-Gain Control

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Abstract—This paper demonstrates that the H_∞ controller of Doyle *et al* [1] can be reinterpreted as an observer-embedded L_2-gain controller The embedded observer is similar to the conventional Luenberger observer except having an additional state-dependent calibration term that compensates for exogenous inputs and model uncertainties Based on this fact, we derive mixed Linear Matrix Inequalities, Hamiltonian Matrix, and Linear Parameterization to provide solutions to feasible observer-embedded L_2-gain controllers

Keywords—Robust control, Observer-based control, Output feedback control, Linear matrix inequalities

INTRODUCTION

This paper presents synthesis of feasible observer-embedded L_2-gain controllers for the following generalized plant

$$\begin{aligned} x &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{1}$$

where the controller processes the measurable output signal y to generate the control signal u in presence of exogenous disturbances and modeling uncertainty w For comparative study, the assumptions used here are the same as those of the DGKF H_∞ controller [1] as follows

- $D_{11} = 0$, $C_1^\top D_{12} = 0$, $D_{12}^\top D_{12} = I$, $D_{21} B_1^\top = 0$, $D_{21} D_{21}^\top = I$, $D_{22} = 0$
- Both triples (A, B_1, C_1) and (A, B_2, C_2) are stabilizable and detectable

The objective of robust performance is specified as

$$\int_0^T \|z\|^2 \, dt \le \gamma^2 \int_0^T \|w\|^2 \, dt, \qquad \forall T > 0, \qquad \forall w \in L_2[0,T] \tag{2}$$

under zero initial conditions The scalar parameter $\gamma > 0$ signifies the specified performance related to energy gain of the closed-loop system

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SYNTHESIS OF OBSERVER-EMBEDDED L_2-GAIN CONTROL

The controller is postulated to have an embedded observer similar to the conventional Luenberger observer except having an additional state-dependent calibration term The structure of the observer-imbedded control law is delineated below

$$\begin{aligned} \hat{x} &= (A + W_{\text{cal}})\,\hat{x} + B_2 u + L\,(y - C_2\hat{x})\,, \\ u &= K\hat{x}, \end{aligned} \tag{3}$$

where K is the controlling matrix, L is the observing matrix, and W_{cal} is the calibration matrix Unlike H_2 optimal control, the resulted closed-loop system under the control structure (3) is calibrated to reduce sensitivity to exogenous inputs and modeling uncertainties w This is achieved by introducing the calibration term W_{cal} that is determined simultaneously with the controller matrix K and the observer matrix L The task is to synthesize a control law based on the generalized plant model (1) and the control objective (2) in two coupled stages of State-Feedback and State-Estimation

In the state feedback stage, we choose a quadratic Lyapunov functional $V(x) = x^{\top}Xx$, $X = X^{\top} > 0$ along the trajectory specified in the generalized plant in (1) The resulting dissipation rate is obtained as

$$\begin{aligned} V &= -\|z\|^2 + \gamma^2\|w\|^2 + x^{\top}\left(A^{\top}X + XA + C_1^{\top}C_1 - F_2^{\top}F_2 + \gamma^2 F_1^{\top}F_1\right)x \\ &\quad - \gamma^2\left\|w - F_1 x\right\|^2 + \left\|u - F_2 x\right\|^2 \\ &\equiv -\|z\|^2 + \gamma^2\|w\|^2 + x^{\top}Q_X x - \gamma^2\left\|\tilde{w}\right\|^2 + \left\|\tilde{u}\right\|^2, \end{aligned} \tag{4}$$

where $F_1 \equiv \gamma^{-2}B_1^{\top}X$, $F_2 \equiv -B_2^{\top}X$, $\tilde{w} \equiv w - F_1 x$, $\tilde{u} \equiv u - F_2 x$ and the first Riccati matrix

$$Q_X \equiv A^{\top}X + XA + C_1^{\top}C_1 - XB_2B_2^{\top}X + \gamma^{-2}XB_1B_1^{\top}X \tag{5}$$

The controller gain matrix is chosen as $K = F_2$

The dynamics of estimation error $\tilde{x} \equiv x - \hat{x}$ is governed by

$$\tilde{x} = x - \hat{x} = \left(A - ZC_2^{\top}C_2\right)\tilde{x} + \left(B_1 - ZC_2^{\top}D_{21}\right)\tilde{w} + B_1F_1x - W_{\text{cal}}\hat{x}$$

This leads to derivation of the calibration matrix

$$W_{\text{cal}} = B_1F_1 \tag{6}$$

Consequently, the dynamics of estimation error becomes

$$\tilde{x} = \left(A + B_1F_1 - ZC_2^{\top}C_2\right)\tilde{x} + \left(B_1 - ZC_2^{\top}D_{21}\right)\tilde{w}, \tag{7}$$

where the observer gain matrix has been set as $L = ZC_2^{\top}$ The estimation accuracy decreases (increases) as the desired γ becomes smaller (larger) The estimation accuracy is also increased if $(A - LC_2)$ has eigenvalues with large negative real parts

In the state estimation stage, we choose the Lyapunov function $W = \tilde{x}^{\top}Z^{-1}\tilde{x}$, $Z = Z^{\top} > 0$ along the dynamical trajectory of the estimation error in (7) The resulting dissipation rate is obtained as

$$\begin{aligned} W &= -\frac{1}{\gamma^2}\left\|\tilde{u}\right\|^2 + \left\|\tilde{w}\right\|^2 - \left\|\tilde{w} - \left(B_1 - ZC_2^{\top}D_{21}\right)^{\top}Z^{-1}\tilde{x}\right\|^2 \\ &+ \tilde{x}^{\top}\left[Z^{-1}\left(A + B_1F_1\right) + \left(A + B_1F_1\right)^{\top}Z^{-1} - C_2^{\top}C_2 + \gamma^{-2}F_2^{\top}F_2 + Z^{-1}B_1B_1^{\top}Z^{-1}\right]\tilde{x} \end{aligned} \tag{8}$$

Similar to the first Riccati matrix in (5), we define the second Riccati matrix as

$$Q_Z \equiv \left(A + B_1F_1\right)Z + Z\left(A + B_1F_1\right)^{\top} + B_1B_1^{\top} - ZC_2^{\top}C_2Z + \gamma^{-2}ZF_2^{\top}F_2Z \tag{9}$$

If the two coupled Riccati inequalities

$$Q_X \le 0 \qquad \text{and} \qquad Q_Z \le 0 \tag{10}$$

have positive symmetric solutions X and Z, then a new candidate Lyapunov functional

$$U = V + \gamma^2 W$$

satisfies the inequality

$$U \le -\|z\|^2 + \gamma^2 \|w\|^2$$

The above inequality is equivalent to $\int_0^T \|z\|^2 \, dt \le \gamma^2 \int_0^T \|w\|^2 \, dt$, $\forall T > 0$, $\forall w \in L_2(0, T)$, arriving at the preset control objective as specified in (2) Therefore, any observer-embedded L_2-gain controller in (3) is represented as a triple

$$(K, L, W_{\text{cal}}) = \left(-B_2^{\mathsf T} X, Z C_2^{\mathsf T}, \gamma^{-2} B_1 B_1^{\mathsf T} X\right) \tag{11}$$

THE CENTRAL OBSERVER-EMBEDDED L_2-GAIN CONTROLLER

Referring to (5) and (9), the central L_2-gain controller in (10) is governed by two Riccati equations, $Q_X = 0$ and $Q_Z = 0$, as

$$A^{\mathsf T} X + XA + C_1^{\mathsf T} C_1 - X B_2 B_2^{\mathsf T} X + \gamma^{-2} X B_1 B_1^{\mathsf T} X = 0, \tag{12}$$

$$(A + B_1 F_1) Z + Z (A + B_1 F_1)^{\mathsf T} + B_1 B_1^{\mathsf T} - Z C_2^{\mathsf T} C_2 Z + \gamma^{-2} Z F_2^{\mathsf T} F_2 Z = 0 \tag{13}$$

Defining the Hamiltonian matrix as

$$H_X = \begin{bmatrix} A & -B_2 B_2^{\mathsf T} + \gamma^{-2} B_1 B_1^{\mathsf T} \\ -C_1^{\mathsf T} C_1 & -A^{\mathsf T} \end{bmatrix}, \tag{14}$$

we can solve the Riccati equation (12) Similarity of the matrices H_X and $-H_X^{\mathsf T}$ implies that H_X must have n stable eigenvalues and another n unstable eigenvalues provided that the H_X does not have purely imaginary eigenvalues A $(2n \times n)$ matrix is formed by stacking the stable column eigenvectors, and then partitioned as $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, where $X_1, X_2 \in \mathbf{R}^{n \times n}$ If X_1 is nonsingular, then (12) is solved by setting $X = X_2 X_1^{-1}$ From the above definition of X, one can easily conclude that the image of $\begin{bmatrix} I \\ X \end{bmatrix}$ is the invariant space of H_X This stable solution is denoted as $X = \operatorname{Ric}(H_X)$ Similarly, a Hamiltonian matrix defined as

$$H_Z \equiv \begin{bmatrix} (A + B_1 F_1)^{\mathsf T} & \gamma^{-2} F_2^{\mathsf T} F_2 - C_2^{\mathsf T} C_2 \\ -B_1 B_1^{\mathsf T} & -(A + B_1 F_1) \end{bmatrix} \tag{15}$$

can solve (13) $Z \in \operatorname{Ric}(H_Z)$, $Z > 0$

We define an H_X-dual Hamiltonian matrix as

$$H_Y \equiv \begin{bmatrix} A^{\mathsf T} & \gamma^{-2} C_1^{\mathsf T} C_1 - C_2^{\mathsf T} C_2 \\ -B_1 B_1^{\mathsf T} & -A \end{bmatrix} \tag{16}$$

and let $Y \in \operatorname{Ric}(H_Y)$, $Y > 0$, which corresponds to the dual Riccati equation (12)

$$AY + YA^{\mathsf T} + B_1 B_1^{\mathsf T} - Y C_2 C_2^{\mathsf T} Y + \gamma^{-2} Y C_1^{\mathsf T} C_1 Y = 0$$

Equivalently,

$$Q_Y \equiv Y^{-1} A + A^{\mathsf T} Y^{-1} + Y^{-1} B_1 B_1^{\mathsf T} Y^{-1} - C_2 C_2^{\mathsf T} + \gamma^{-2} C_1^{\mathsf T} C_1 = 0 \tag{17}$$

A relationship between H_Z and H_Y can be found as

$$H_Z = \begin{bmatrix} I & -\gamma^{-2}X \\ 0 & I \end{bmatrix} H_Y \begin{bmatrix} I & -\gamma^{-2}X \\ 0 & I \end{bmatrix}^{-1}, \tag{18}$$

where $X \in \mathrm{Ric}(H_X)$, $X > 0$ Since the images of the operators $\begin{bmatrix} I \\ Y \end{bmatrix}$ and $\begin{bmatrix} I \\ Z \end{bmatrix}$ are the stable eigenspaces of H_Y and H_Z, respectively, it follows from (18) that

$$\mathrm{Im}\begin{bmatrix} I \\ Z \end{bmatrix} = \begin{bmatrix} I & -\gamma^{-2}X \\ 0 & I \end{bmatrix} \mathrm{Im}\begin{bmatrix} I \\ Y \end{bmatrix} \mathrm{Im}\begin{bmatrix} I - \gamma^{-2}XY \\ Y \end{bmatrix} = \mathrm{Im}\begin{bmatrix} I \\ Y\left(I - \gamma^{-2}XY\right)^{-1} \end{bmatrix} \tag{19}$$

Therefore, given $X \in \mathrm{Ric}(H_X)$, $X > 0$, we obtain the following condition

$$\left(Z \in \mathrm{Ric}\left(H_Z\right),\ Z > 0\right) \Leftrightarrow \left(Y \in \mathrm{Ric}\left(H_Y\right),\ Y > 0 \text{ and } \rho(XY) < \gamma^2\right), \tag{20}$$

where it is implied that

$$Z = Y\left(I - \gamma^{-2}XY\right)^{-1} \tag{21}$$

Based on (3) and (11), the central observer-based L_2-gain controller becomes

$$\begin{aligned} \dot{\hat{x}} &= \left(A + \gamma^{-2}B_1B_1^\top X - B_2B_2^\top X\right)\hat{x} + Y\left(I - \gamma^{-2}XY\right)^{-1} C_2^\top \left(y - C_2\hat{x}\right), \\ u &= -B_2^\top X\hat{x}, \end{aligned} \tag{22}$$

where $X \in \mathrm{Ric}(H_X)$, $X > 0$, $Y \in \mathrm{Ric}(H_Y)$, $Y > 0$, and $\rho(XY) < \gamma^2$

This controller is identical to the DGKF H_∞ controller [1] This shows the following

The DGKF H_∞ controller is an observer-embedded L_2-gain controller, and it can be realized from the H_2 optimal controller by incorporating a calibration term into its embedded Luenberger observer to obtain robust estimation

FEASIBLE OBSERVER-EMBEDDED L_2-GAIN CONTROLLERS

We pose the following question *With the same order of the central controller (22), can all feasible observer-based L_2-gain controllers be solved by formulating the Riccati inequalities as $Q_X \leq 0$, $Q_Y \leq 0$, and $\rho(XY) \leq \gamma^2$?*

The answer to the above question is negative If the answer was positive, then the LMI version of H_∞ controller synthesis [2] could be obtained by directly extending the central controller governed by Riccati inequalities (20) We derive the feasible observer-embedded L_2-gain controllers to clarify these issues and arrive at the correct answer

Substitution of (21) into $Q_Z \leq 0$ in (9) yields

$$\begin{aligned} &\left(Y^{-1}\gamma^{-2}X\right)\left(A + \gamma^{-2}B_1B_1^\top X\right) + \left(A + \gamma^{-2}B_1B_1^\top X\right)\left(Y^{-1}\gamma^{-2}X\right) \\ &- C_2^\top C_2 + \gamma^{-2}XB_2B_2^\top X + \left(Y^{-1} - \gamma^{-2}X\right)B_1B_1^\top\left(Y^{-1} - \gamma^{-2}X\right) \leq 0 \\ &\Leftrightarrow \\ &Y^{-1}A + \gamma^{-2}Y^{-1}B_1B_1^\top X - \gamma^{-2}XA - \gamma^{-4}XB_1B_1^\top X + A^\top Y^{-1} - \gamma^{-2}A^\top X \\ &+ \gamma^{-2}XB_1B_1^\top Y^{-1} - \gamma^{-4}XB_1B_1^\top X - C_2^\top C_2 + \gamma^{-2}XB_2B_2^\top X + Y^{-1}B_1B_1^\top Y^{-1} \\ &- \gamma^{-2}Y^{-1}B_1B_1^\top X - \gamma^{-2}XB_1B_1^\top Y^{-1} + \gamma^{-4}XB_1B_1^\top X \leq 0 \\ &\Leftrightarrow \\ &\left(Y^{-1}A + A^\top Y^{-1} + Y^{-1}B_1B_1^\top Y^{-1} - C_2^\top C_2 + \gamma^{-2}C_1^\top C_1\right) \\ &- \gamma^{-2}\left(XA + A^\top X - XB_2B_2^\top X + \gamma^{-2}XB_1B_1^\top X + C_1^\top C_1\right) \leq 0 \\ &\Leftrightarrow \\ &Q_Y - \gamma^{-2}Q_X \leq 0, \end{aligned} \tag{23}$$

which is the correct answer to the above-posed problem

The central controller implied in (23) is solved by setting $Q_X = 0$, $Q_Y - \gamma^{-2}Q_X = 0$, and $\rho(XY) \le \gamma^2$, which is equivalent to $Q_X = 0$, $Q_Y = 0$, and $\rho(XY) \le \gamma^2$, and that is identical to the DGKF H_∞ control law [1] Instead, by erroneously setting ($Q_X \le 0$, $Q_Y \le 0$, and $\rho(XY) \le \gamma^2$) and using to (4) and (8), one may arrive at

$$V + \gamma^2 W + \|z\|^2 - \gamma^2\|w\|^2 \le x^\top Q_X x + \tilde{x}^\top \left(\gamma^2 Q_Y - Q_X\right)\tilde{x} \le \tilde{x}^\top \left(-Q_X\right)\tilde{x} \tag{24}$$

It follows from (24) that the control objective of energy gain in (2) is not guaranteed because $Q_X \le 0$, implying that there is no robust estimation On the other hand, for the output feedback control (where the measurement does not contain full information of the plant state) of LTI systems, we would like to postulate the following two criteria

There is no optimal control if there is no optimal observer embedded
There is no robust control if there is no robust observer embedded

Therefore, the answer to the above-posed question is stated as the following theorem

THEOREM 1 *Solutions of feasible observer-embedded L_2-gain controllers, having the same order as that of the generalized plant model, are obtained from the Riccati inequalities*

$$Q_X \le 0, \qquad Q_Y - \gamma^{-2}Q_X \le 0, \qquad \text{and} \qquad \rho(XY) \le \gamma^2$$ ∎

Next, we normalize feasible observer-embedded L_2-gain controllers by replacing X with γX in (12), and Y with γY in (17) The resulting formulation in terms of Riccati inequalities becomes

$$\begin{aligned} A^\top X + XA + \gamma^{-1}C_1^\top C_1 - \gamma X B_2 B_2^\top X + \gamma^{-1} X B_1 B_1^\top X &\le 0, \\ Y^{-1}A + A^\top Y^{-1} + \gamma^{-1}Y^{-1}B_1B_1^\top Y^{-1} - \gamma C_2^\top C_2 &\le A^\top X + XA - \gamma X B_2^\top B_2 X \\ &\quad + \gamma^{-1} X B_1 B_1^\top X, \\ X - Y^{-1} &\le 0, \end{aligned}$$

and the triple representing the controller (3) becomes

$$\{K, L, W_{\text{cal}}\} = \left\{-\gamma B_2^\top X, \gamma Y(I - XY)^{-1}C_2^\top, \gamma^{-1}B_1B_1^\top X\right\} \tag{25}$$

With no loss of generality [3], the generalized plant model (1) can be normalized by setting $\gamma = 1$ and replacing w by $\gamma^{-1}w$ We arrive at the final normalized formulation of feasible observer-embedded L_2-gain controllers from solutions of Riccati inequalities as

$$\begin{aligned} A^\top X + XA + C_1^\top C_1 - XB_2B_2^\top X + XB_1B_1^\top X &\le 0, \\ Y^{-1}A + A^\top Y^{-1} + Y^{-1}B_1B_1^\top Y^{-1} - C_2^\top C_2 &\le A^\top X + XA - XB_2^\top B_2 X + XB_1B_1^\top X, \\ X - Y^{-1} &\le 0, \end{aligned} \tag{26}$$

and the triple representing the controller in (3) is

$$(K, L, W_{\text{cal}}) = \left(-B_2^\top X, Y(I - XY)^{-1}C_2^\top, B_1B_1^\top X\right) \tag{27}$$

To compare to the linear fractional transformation (LFT) parameterization of H_∞ controllers, (26) can be formulated by (Q, S) parameterization as follows

COROLLARY 1 TO THEOREM 1 *Any free pair of parameters (Q, S), with $Q \ge 0$ and $S \ge 0$, such that*

$$\begin{aligned} A^\top X + XA + C_1^\top C_1 - XB_2B_2^\top X + XB_1B_1^\top X + Q &= 0, \\ Y^{-1}A + A^\top Y^{-1} + Y^{-1}B_1B_1^\top Y^{-1} - C_2^\top C_2 + S &= 0, \\ X - Y^{-1} &\le 0, \\ Q - S &\le 0, \end{aligned} \tag{28}$$

determines a feasible observer-embedded L_2-gain (normalized) controller ∎

The trade-off between control and estimation in the synthesis is achieved by the matrices Q and S in the sense that control is enhanced by increasing Q and estimation is enhanced by increasing S.

To utilize the efficient numerical procedures in Hamiltonian Matrix and Linear Matrix Inequalities, (26) can be relaxed as

$$\begin{aligned} X^{-1}A^{\top} + AX^{-1} + X^{-1}C_1^{\top}C_1X^{-1} - B_2B_2^{\top} + B_1B_1^{\top} &= 0, \\ Y^{-1}A + A^{\top}Y^{-1} + Y^{-1}B_1B_1^{\top}Y^{-1} - C_2^{\top}C_2 + C_1^{\top}C_1 &\le 0, \\ X - Y^{-1} &\le 0, \end{aligned} \tag{29}$$

and the results are summarized below.

COROLLARY 2 TO THEOREM 1. *Given a normalized performance level of energy gain (i.e., $\gamma = 1$), if*

$$X \in \mathrm{Ric}\begin{bmatrix} A & -B_2B_2^{\top} + B_1B_1^{\top} \\ -C_1^{\top}C_1 & -A^{\top} \end{bmatrix},$$

$$\begin{bmatrix} Y^{-1}A + A^{\top}Y^{-1} - C_2^{\top}C_2 & Y^{-1}B_1 & C_1^{\top} \\ B_1^{\top}Y^{-1} & -I & 0 \\ C_1 & 0 & -1 \end{bmatrix} \le 0; \tag{30}$$

$$\begin{bmatrix} Y^{-1} & I \\ I & X^{-1} \end{bmatrix} \ge 0,$$

then $(K, L, W_{\mathrm{cal}}) \equiv (-B_2^{\top}X,\ Y(I - XY)^{-1}C_2^{\top},\ B_1B_1^{\top}X)$ represents a feasible observer-embedded L_2-gain controller. ∎

The feasible controller in (30) contains a central control and a feasible estimation.

At the expense of more involved numerical computations, Theorem 1 can be reformulated in terms of two linearly coupled matrix inequalities.

COROLLARY 3 TO THEOREM 1. *All feasible observer-embedded L_2-gain controllers, with a normalized performance level of energy gain (i.e., $\gamma = 1$), can be formulated as the following two linearly coupled matrix inequalities:*

$$\begin{bmatrix} X^{-1}A^{\top} + AX^{-1} - B_2B_2^{\top} & X^{-1}C_1^{\top} & B_1 \\ C_1X^{-1} & -I & 0 \\ B_1^{\top} & 0 & -I \end{bmatrix} \le 0, \tag{31}$$

and

$$\begin{bmatrix} Z^{-1}A_Z + A_Z^{\top}Z^{-1} - C_2^{\top}C_2 & Z^{-1}B_1 & F_2^{\top} \\ B_1^{\top}Z^{-1} & -I & 0 \\ F_2 & 0 & -I \end{bmatrix} \le 0, \tag{32}$$

where $A_Z = A + B_1F_1$, $F_1 = B_1^{\top}X$, and $F_2 = -B_2^{\top}X$, and the resulting observer-embedded L_2-gain controller becomes

$$\begin{aligned} \dot{\hat{x}} &= \left(A + B_1F_1 + B_2F_2 - ZC_2^{\top}C_2\right)\hat{x} + ZC_2^{\top}y, \\ u &= -B_2^{\top}X\hat{x}. \end{aligned} \tag{33}$$

SUMMARY AND CONCLUSIONS

This paper presents formulation of mixed Linear Matrix Inequalities, Hamiltonian Matrix, and Linear Parameterization to provide solutions to feasible observer-embedded L_2-gain controllers that are capable of explicitly estimating plant states. Such observer-embedded L_2-gain controller can be realized from the H_2 optimal controller by introducing a calibration term in the Luenberger observer to obtain robust state estimation. It is also shown that the DGKF H_∞ controller is the observer-embedded central L_2-gain controller. Since the L_2-gain control law is not restricted to LTI systems, the proposed approach is applicable to synthesis of robust linear parameter varying (LPV) systems [4].

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