Multivariable Nonadaptive Controller Design

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Abstract—**Although proportional-integral-derivative control remains one of the most common control schemes used in industry, its tuning still remains inadequately understood in many applications. This task becomes much more challenging when applied to multi-input multi-output (MIMO) systems. This article presents the design of a discrete-time robust multivariable (nonadaptive) tracking controller that comes with a simple structure, requires very limited information on the plant model, and is relatively easy to tune. In addition to being easy to tune and implement, an objective of this controller is to deal with a class of large-scale systems with complex dynamics. We analytically demonstrate the robustness and convergence of the closed-loop system for a class of MIMO linear time-varying systems. The overall superiority of the proposed controller is experimentally validated on a Barrett robot arm in a laboratory environment. The article also provides a stochastic framework of the general setting of the controller. Within this framework, two minimum mean square error optimal solutions of the controller are provided; one is designed for the case where the number of inputs is not greater than the number of outputs, and the other is for the antithesis.**

Index Terms—**Multivariable control, robot manipulator.**

I. Introduction

OPTIMAL control design generally requires a dynamical system model of the underlying plant. This requirement also applies to identification of optimal gains of the multivariable output-feedback controller(s) (e.g., see [1]–[3]). In general, an output-feedback control law requires a state estimator including a model of the plant dynamics, which may degrade robustness of the control system. On the other hand, among the common control strategies, standard proportional-integral-derivative (PID)

controllers have shown to be sufficiently robust and are widely used in industry. From the perspectives of industrial applications, it is desirable to adopt a control strategy that is robust, yet simple enough to implement while achieving the specified tracking performance.

It is noted that tuning the parameters of a PID controller may pose a challenging task, especially in multivariable systems. Several PID tuning methods have been developed for single-input single-output (SISO) systems, where the gains are systematically or adaptively tuned [4]. A large amount of work has also been reported for tuning multivariable controllers. These methods include multiobjective optimization tuning design for nonlinear systems (e.g., see [5]), PID tuning using techniques based on Lyapunov stability analysis (e.g., see [6] and [7]), and constraint optimization (e.g., see [8]). A sampling-rate-dependent controller is proposed in [9], which selects gains by stabilizing an augmented system where a solution is obtained by treating the augmented system as static output feedback.

There are many methods available to control uncertain systems. A precise tracking estimator-based controller with an observer-based estimator is developed for a class of uncertain nonlinear systems with mismatched uncertainties [10]. This method involves rigorous tuning due to parameters related to the estimator gains and the control gains. In addition, its implementation, which is based on the measurement of the states, requires high quality sensors with minor measurement noise. Adaptive tracking control is proposed for a class of constrained Euler–Lagrange systems with unknown linearities [11]. However, this controller is restricted to one specific class of systems, and it involves many parameters; e.g., in their implementation on a two-joint planar robotic manipulator, there are fifty parameters that require some tuning. In the presence of unknown kinematics and dynamics, an adaptive robot control/identification scheme with enhanced convergence rate is developed [12]; however, periodic and band-limited excitation joint reference trajectories are required. Another method that does not require any knowledge of the plant model and works without using system identification is proposed in [13] for two-input two-output systems. However, the design of the PID controller aims at stabilizing the close-loop system based on open-loop frequency data of the transfer matrix.

Another approach that deals with uncertain systems is time-delayed control (TDC), which employs a time-delayed estimation (TDE) technique to cancel out complex uncertainties. The performance of TDC is affected by the errors in TDE. For example, such errors become substantial in cases, where Coulomb friction is significant. Different approaches have been

proposed to suppress the contribution of TDE errors by integrating TDC with auxiliary controls such as adaptive sliding-mode control (e.g., see [14] and [15]). A fast adaptive law is developed [16] that better treats the undesirable side effects of TDE without imposing restrictions on the control gains. Another PID tuning method is implemented on an industrial plant without any knowledge of the plant model [17] by perturbing the initial conditions of the servocompensator; however, the system under consideration is linear time invariant (LTI) and is assumed to be asymptotically stable with constant reference and disturbance signals. A modular-based method for a class of uncertain nonlinear systems with the treatment of intermittent actuator failures is provided in [18] by estimating the bounds of uncertain jumping parameters. This adaptive fault-tolerant control method requires the boundary information and signs of the unknown parameters; it is limited to single-output systems, and its effectiveness is justified numerically. In [19], a linear parameter varying, continuous-time PID controller is proposed with unknown control direction and actuation failures for a class of uncertain feedback linearizable nonlinear systems. However, the design requires some partial known information of the plant, and the effectiveness of the proposed controller is only illustrated numerically.

Data-driven algorithms use only the input/output data from the process in order to compute, systematically in an algorithmic manner, the tuning parameters of the controller [20] using indirect or direct controller design methods. The former group of techniques first identifies a model, then a controller is tuned based on such model, which is usually erroneous in practice. For example, in [21], a multiple adaptive observer is used to approximate online the original unknown multi-input multi-output (MIMO) nonlinear process from the input/output data. Next, the controller is obtained from the identified observer model by the inverse control law. However, the system under consideration should satisfy a global Lipschitz condition with a bounded-input bounded-output like assumption. The effectiveness of this approach is justified numerically on a two-input two-output plant. In some applications, the physical system could be too complex to use adaptation-based controllers, while achieving precise control. To overcome this problem, the idea behind the latter group of techniques is to map the experimental data directly on to the controller, without any model to be identified in between. For example, a PID tuning [22] method using an iterative learning control (ILC) approach is proposed where the local controls are designed by using PID, and the references for PID control were optimized by ILC. However, this approach requires the input of a periodic reference signal, typically used to execute a repetitive sequence with repetitive dynamics. Another method of iterative feedback tuning and ILC in combination with the extended symmetrical optimum method and fuzzy control for PI-fuzzy controllers is proposed in [23]. However, this approach is limited to a discrete-time LTI SISO system, its implementation is rather systematically involved, and it is computationally complex. A neural network composite learning control approach with friction compensation is specifically designed for robotic systems and implemented on a relatively small robot arm constraint to its

2-degrees-of-freedom (DoF) planar joints using a 17-b absolute encoder. Such encoders are considered rather expensive when implemented on industrial arm [24]. Naturally, during learning phase, the tracking transient errors are large and after adaptation of 100 s, the controller exhibits better tracking results.

The literature is swamped with sound control schemes that deal with uncertainties derived from complex systems. However, the approach adopted in the corresponding literature is very much based on different direct or indirect data-driven adaptation techniques. Most of the sophisticated multivariable controllers are tailored for a specific class of systems and/or reference trajectories, lacks experimental justification, and/or may not be adequately tuned by novice engineers.

It follows that the presence of such challenges in dealing with uncertain complex systems raises an important question: *Is there a nonadaptive multivariable tracking controller that can deal with a class of large-scale systems with complex dynamics that is robust, computationally inexpensive, easy to tune, and simple to implement?*

To address this issue, the article proposes a discrete-time multivariable (nonadaptive) PI-like controller that is recursively updated based on the returned error. Two gains are associated with the controller to essentially weigh the current and previous errors. We provide two different methods for selecting the controller gains. One method is designed for systems that require very limited information about the physical plant model and that do not employ any adaptive scheme. The controller architecture is composed of a single gain matrix and a scalar, making the controller implementation easy to tune, and computationally inexpensive. The other method is designed for systems, where the two controller-gain matrices are automatically and optimally generated. The optimal gains are synthesized by making use of the available limited information on the plant model and is based on a recursive algorithm that minimizes a stochastic cost functional in the presence of erroneous initial conditions, white measurement noise, and white process noise.

The proposed controller is described as: $u(k + 1) = u(k) + \gamma K e(k + 1) - K e(k)$, where k is the discrete time index, $u(k)$ is the control input, $e(k)$ is the output error, and $1 < \gamma \ll 2$. An augmented system representative of the input and state errors is formulated using the proposed control law, including output error and state disturbances. For a class of linear time-varying systems, we show that there exists a (controller) matrix K such that the closed-loop system is stable. In addition, for discretized stable plants, we show that the steady-state error can be made arbitrarily small for sufficiently small sampling period. Furthermore, for square MIMO systems, we show that a diagonal constant gain matrix, K, can guarantee the aforementioned results.

In order to examine the potential of the proposed controller beyond the analytical setup considered in this article, we test it experimentally on a 4-DoF Barrett robot arm while using a diagonal gain matrix and Euler method to estimate the angular rates. Robot manipulators are known to have nonlinear dynamics and are prone to chattering problems. Such problems can excite unmodeled high-frequency plant dynamics, resulting in either

severe vibrations in the arm or instability of the control system. The proposed control scheme is validated by comparing its performance to four other recent control schemes, namely a deterministic PD controller [25], a stochastic PD controller [24], a proportional-double derivative (PDD) controller [26], and a sliding mode controller [27]. In this way, the proposed controller is implemented with a decaying scalar gain, K, to a quadruple tank process. We compare the performance of the proposed controller to that of three optimal PID controllers that use full knowledge of the system model. We study the transient response of the controllers and their performance in the presence of measurement errors while reflecting the advantages of the proposed controller.

To go one step further, we consider the general structure of the proposed two-gain control law, i.e., $u(k + 1) = u(k) + K_1(k)e(k + 1) - K_2(k)e(k)$ and derive the optimal gains that minimize the covariance of the augmented error system, which makes use of the system model and the statistics of the measurement and process noise. The convergence characteristics of the proposed controller is shown to be similar to those of the optimal PID controllers in [1] and [2], where the control law in [1] and [2] is $u(k + 1) = u(k) + K_1(k)e(k + 1) + K_2(k)e(k) + K_3(k)e(k - 1)$. We also build on the results, provided in [1] and [2], to present similar necessary and sufficient conditions for convergence of all trajectories. The latter results provide a framework aimed at rejecting measurement noise without incorporating any external filters.

The main contribution *points* of this article are in the developments that follow.

1) *New PI-like discrete-time multivariable tracking control of square and nonsquare systems:* A simplified, robust and easy-to-tune, version of the controller is suitable for uncertain dynamical systems.
2) *Controller design with specifications of boundedness and zero-error convergence in probability:* A framework consisting of necessary and sufficient conditions of the specifications is established.
3) *Recursive algorithms for automatic generation of optimal controller gains for certain systems:* One algorithm addresses the case when the number of outputs is not less than the number of inputs, and the other for the antitheses.
4) *Vanishing steady-state error of the control system:* The steady-state errors can be made arbitrarily small for sufficiently small sampling periods even in the presence of measurement noise.

The remainder of the article is organized as follows. Section II defines the systems and formulates the problems under consideration for linear time-varying (LTV) and LTI systems. The proposed controllers and their convergence characteristics are presented in Section III. Section IV provides our numerical and experimental studies. Lastly, Section V concludes this article.

Nomenclature: The expectation operator is denoted by $\mathrm{E}[\cdot]$, $\prod_{i=0}^{k} M_i = M_0 M_1 \dots M_k$, and $\prod_{i=k+1}^{k} M_i = I$, $I_m \in \mathbb{R}^{m \times m}$ is the identity matrix, $0_{n \times m} \in \mathbb{R}^{n \times m}$ is the zero matrix, $\lambda(M)$ denotes eigenvalues of M, $\rho(M)$ denotes the maximum eigenvalue of M, and $\mathrm{tr}(\cdot)$ is the trace operator.

II. PROBLEM STATEMENT AND SYSTEM DESCRIPTIONS

This section presents two scenarios for the control system under consideration. The first scenario considers a discrete-time linear time-varying system with the number of outputs greater than or equal to the number of inputs. In the second scenario, a discrete-time LTI system is considered where the number of inputs is greater than or equal to the number of outputs. The problem formulations associated with the model-dependent controller adopted in the first scenario and the second scenario follow similar formulations as in [1] and [2], respectively.

A. Linear Time-Varying Systems

The system under consideration is a discrete time-varying system described by the following state-space equation:

$$x(k + 1) = A(k)x(k) + B(k)u(k) + w(k)$$
$$y(k) = C(k)x(k) + v(k) \quad (1)$$

where $k \in \mathbb{N}$ is the discrete time index, $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^n$ is the state disturbance, $y(k) \in \mathbb{R}^p$ is the system output, $v(k) \in \mathbb{R}^p$ is the output error, and $u(k) \in \mathbb{R}^q$ is the system input. The system is assumed to be discretized with sampling rate $T_s > 0$. Here $p \geq q$, i.e., the number of outputs is greater-than-or-equal-to the number of inputs.

Assumptions:

(AI1) The system matrices $A(k), B(k), C(k)$ are assumed bounded $\forall k$ and as $k \to \infty$.

(AI2) The desired reference trajectory has a solution. That is, for any desired trajectory or reference signal, $y_d(k) = [y_{1,d}(k), y_{2,d}(k), \dots, y_{p,d}(k)]^T$ and an appropriate initial condition $x(0)$, there exists a control input $u_d(k)$ generating the desired output trajectory, $y_d(k)$, for the nominal plant. That is $x_d(k + 1) = A(k)x_d(k) + B(k)u_d(k)$ and $y_d(k) = C(k)x_d(k)$, where $x_d(k)$ is the state response due to $u_d(k)$ with a given $x_d(0)$ such that $y_d(0) = C(0)x_d(0)$.

(AI3) $w(k)$ and $v(k)$ are zero-mean white noise processes, mutually uncorrelated with each other and with $x(0)$. $\mathrm{E}[w(k)w(k)^T] = Q(k) \geq 0$, and $\mathrm{E}[v(k)v(k)^T] = R(k) > 0$.

(AI4) The system in (1) is stable with relative degree 1 with no more inputs than outputs.

The general setting of the proposed control law is given by

$$u(k + 1) = u(k) + K_1(k)e(k + 1) - K_2(k)e(k) \quad (2)$$

where the matrices $K_{1,2}(k) \in \mathbb{R}^{q \times p}$ are the learning gains, and $e(k) \equiv y_d(k) - y(k)$ is the output measurement error due to control action $u(k)$. For compactness, we will denote $A \equiv A(k)$, $B \equiv B(k)$, $C \equiv C(k)$, $C^{\pm} \equiv C(k \pm 1)$, $A^- \equiv A(k - 1)$, $B^- \equiv B(k - 1)$, $K_{1,2} \equiv K_{1,2}(k)$, $K_{1,2}^{\pm} \equiv K_{1,2}(k \pm 1)$.

Define the state and input errors as $\delta x(k) = x_d(k) - x(k)$ and $\delta u(k) = u_d(k) - u(k)$, respectively. The input error model corresponding to the control law is derived as

$$\delta u(k + 1) = \delta u(k) - K_1 e(k + 1) - K_2 e(k) + \Delta u_d(k) \quad (3)$$

where $\Delta u_d(k) = u_d(k + 1) - u_d(k)$. In addition,

$$\delta x(k + 1) = A \delta x(k) + B \delta u(k) - w(k). \quad (4)$$

Furthermore, the errors can be expanded as follows:

$$e(k) = CA^-\delta x(k-1)$$
$$+ CB^-\delta u(k-1) - Cw(k-1) - v(k) \quad (5)$$
$$e(k+1) = C^+B\delta u(k) - C^+w(k) - v(k+1)$$
$$+ C^+A(A^-\delta x(k-1)$$
$$+ B^-\delta u(k-1) - w(k-1)). \quad (6)$$

Inserting (5) and (6) in (3), we get

$$\delta u(k+1) = \left(I - K_1 C^+ B\right)\delta u(k)$$
$$- \left(K_1 C^+ AB^- - K_2 CB^-\right)\delta u(k-1)$$
$$- \left(K_1 C^+ AA^- - K_2 CA^-\right)\delta x(k-1)$$
$$+ K_1 C^+ w(k) + \left(K_1 C^+ A - K_2 C\right)w(k-1)$$
$$+ K_1 v(k+1) - K_2 v(k) + \Delta u_d(k). \quad (7)$$

Equation (7) can be reformulated as an augmented system by first defining

$$X \equiv \begin{bmatrix} \delta u(k) \\ \delta u(k-1) \\ \delta x(k-1) \end{bmatrix}, \quad X^+ \equiv \begin{bmatrix} \delta u(k+1) \\ \delta u(k) \\ \delta x(k) \end{bmatrix}, \quad \Omega \equiv \begin{bmatrix} \Delta u_d(k) \\ 0 \\ 0 \end{bmatrix}$$

$$V \equiv \begin{bmatrix} v(k+1) \\ v(k) \\ v(k-1) \end{bmatrix}, \quad W \equiv \begin{bmatrix} w(k) \\ w(k-1) \end{bmatrix}.$$

The resulting augmented system is then derived by combining (4) and (7) to yield

$$X^+ = \Phi X + \bar{I}\bar{K}V + \left(\bar{I}\bar{K}\Gamma_1 + \Gamma_2\right)W + \Omega \quad (8)$$

where $\Phi = H - \bar{I}\bar{K}E$, $\bar{K} \equiv [K_1, K_2]$

$$H \equiv \begin{bmatrix} I_q & 0_q & 0_{qn} \\ I_q & 0_q & 0_{qn} \\ 0_{nq} & B^- & A^- \end{bmatrix}, \quad \bar{I} \equiv \begin{bmatrix} I_q \\ 0_q \\ 0_{nq} \end{bmatrix}, \quad \Gamma_1 \equiv \begin{bmatrix} C^+ & C^+A \\ 0_{pn} & C \end{bmatrix}$$

$$E \equiv \begin{bmatrix} C^+B & C^+AB^- & C^+AA^- \\ 0_{pq} & CB^- & CA^- \end{bmatrix}, \quad \Gamma_2 \equiv \begin{bmatrix} 0_{qn} & 0_{qn} \\ 0_{qn} & 0_{qn} \\ 0_n & I_n \end{bmatrix}.$$

The covariance of the augmented system is $P^+ = \mathrm{E}[X^+ X^{+T}]$ and $P = \mathrm{E}[XX^T]$, which statistically represents the input error and state error covariances.

We present three different problems under consideration for the system in (1) employing the controller (2) under Assumptions (A1)–(A4).

Problem I1 Statement: Establish a framework for the design of the control law in (2) for boundedness, and zero convergence of the error covariance, $P(k)$.

Problem I2 Statement: Develop optimal gains that satisfy the framework established in Problem I1. In addition show that $\lim_{k\to\infty} ||P(k)||$ can be made arbitrary small for sufficiently small sampling period in presence of measurement noise and initialization errors.

Problem I3 Statement: Simplify (2) so that the tuning is made easy. The simplified control law should not use the knowledge of the system model or implementation of any adaptation scheme. In addition, show that such a controller is robust, and small steady-state errors for sufficiently small sampling period can be achieved.

B. Linear Time-Invariant Systems

The system under consideration in this scenario is a discrete time-invariant system.

Assumptions:

(AII1) $w(k)$ and $v(k)$ are zero-mean white noise processes mutually uncorrelated with each other and with $x(0)$. $\mathrm{E}[w(k)w(k)^T] = Q(k) \geq 0$, and $\mathrm{E}[v(k)v(k)^T] = R(k) > 0$.

(AII2) CB is full-row rank (implies that the number of inputs is greater than or equal to the number of outputs).

The plant is not assumed to be stable. Assumptions (AII1) and (AII2) hold throughout the parts of the article addressing the scenario with LTI systems.

The same control law in (2) is proposed for the LTI system. With the output measurement error as $e(k) \equiv y_d(k) - y(k)$, we define the output error as $\hat{e}(k) \equiv C(x_d(k) - x(k))$ (i.e., $e(k) = \hat{e}(k) - v(k)$).

Denote $\eta(k) \equiv x(k) - x(k-1)$, $\Delta y_d(k) \equiv y_d(k) - y_d(k-1)$, $\Delta u(k) \equiv u(k) - u(k-1)$, $\Delta w(k) \equiv w(k) - w(k-1)$, and $\Delta v(k) \equiv v(k) - v(k-1)$. From (2), we can write

$$\Delta u(k) = K_1^-[\hat{e}(k+1) - v(k+1)] + K_2^-[\hat{e}(k) - v(k)]. \quad (9)$$

From (9), we get

$$\hat{e}(k+1) = (I - CBK_1^-)\hat{e}(k) - CBK_2^-\hat{e}(k-1) - CA\eta(k)$$
$$+ \Delta y_d(k+1) - C\Delta w(k) + v(k+1)$$
$$+ (CBK_1^- - I)v(k) + CBK_2^- v(k-1) \quad (10)$$
$$\eta(k+1) = A\eta(k) + BK_1^-\hat{e}(k) + BK_2^-\hat{e}(k-1) + \Delta w(k)$$
$$- BK_1^- v(k) + BK_2^- v(k-1). \quad (11)$$

To formulate the augmented system of difference equations, we first let $\tilde{\Omega} \equiv \left[\begin{smallmatrix} \Delta y_d(k+1) \\ w(k)-w(k-1) \end{smallmatrix}\right]$,

$$\tilde{X} \equiv \begin{bmatrix} \hat{e}(k) \\ \hat{e}(k-1) \\ \eta(k) \end{bmatrix}, \quad \tilde{X} \equiv \begin{bmatrix} \hat{e}(k+1) \\ \hat{e}(k) \\ \eta(k+1) \end{bmatrix}, \quad \tilde{V} \equiv \begin{bmatrix} v(k+1) \\ v(k) \\ v(k-1) \end{bmatrix}$$

The resulting augmented system is then derived by combining (4) and (7) to yield

$$\tilde{X}^+ = \Psi\tilde{X} + \tilde{\Gamma}_1\tilde{V} + \tilde{\Gamma}_2\tilde{\Omega} \quad (12)$$

where $\Psi = \tilde{H} - \tilde{E}\tilde{K}^- \tilde{\bar{I}}, \tilde{K}^- \equiv [K_1^-, K_2^-]$

$$\tilde{H} \equiv \begin{bmatrix} I_p & 0_p & -CA \\ I_p & 0_p & 0_{pn} \\ 0_{np} & 0_{np} & A \end{bmatrix}, \tilde{E} \equiv \begin{bmatrix} CB \\ 0_{pq} \\ -B \end{bmatrix}, \tilde{\Gamma}_2 \equiv \begin{bmatrix} I & -C \\ 0 & 0 \\ 0 & I \end{bmatrix}$$

$$\tilde{\Gamma}_1 \equiv \begin{bmatrix} I_p & CBK_1^- - I_p & CBK_2^- \\ 0 & 0 & 0 \\ 0 & -BK_1^- & BK_2^- \end{bmatrix}, \tilde{\bar{I}} \equiv \begin{bmatrix} I_p & 0_p & 0_{pn} \\ 0_p & I_p & 0_{pn} \end{bmatrix}.$$

Consequently, we get

$$\tilde{X}^+ = \tilde{H}\tilde{X} - \tilde{E}\tilde{K}^- \tilde{\bar{I}}\tilde{X} + \tilde{\Gamma}_1 \tilde{V} + \tilde{\Gamma}_2 \tilde{\Omega}. \tag{13}$$

We present the following problem statement under consideration for the LTI system employing the controller (2) under Assumptions (AII1) and (AII2).

Problem II1 Statement: Find the optimal gains for $K_{1,2}(k)$ that minimize the mean-square state error of the augmented system at every time-step.

Problem II2 Statement: Develop binding conditions associated with the optimal control law pertaining to the boundedness and convergence of $\mathrm{E}[\tilde{X}\tilde{X}^T]$.

III. MAIN RESULTS

This section addresses the problem statements defined above.

A. Linear Time-Varying Systems

In order to address Problem I1, we first formulate the covariance $P(k)$ of the system in (8). Since the augmented system in (8) follows the same augmented system of the PID control law in [1], with E being also full-row rank, then all results in [1] apply while following the same proofs. For convenience, we only list the main results.

From assumption (AI3) we have that $w(0), v(0)$, and $\delta x(0)$ uncorrelated and are zero-mean white noise, then (8) leads to $P^+ = \Phi P \Phi^T + \bar{I}\bar{K}\bar{R}(\bar{I}\bar{K})^T + (\bar{I}\bar{K}\Gamma_1 + \Gamma_2)\bar{Q}(\bar{I}\bar{K}\Gamma_1 + \Gamma_2)^T + E[\Omega\Omega^T]$ where

$$\bar{R} = \begin{bmatrix} R(k+1) & 0 \\ 0 & R(k) \end{bmatrix} \text{ and } \bar{Q} = \begin{bmatrix} Q(k) & 0 \\ 0 & Q(k-1) \end{bmatrix}.$$

The augmented system state covariance matrix can be rewritten as $P(k) = \bar{P}_0 + \bar{P}_T$ for $k > 0$, where

$$\bar{P}_0 = \left(\prod_{i=0}^{k-1} \Phi(k-1-i)\right) P(0) \left(\prod_{i=0}^{k-1} \Phi(k-1-i)\right)^T \tag{14}$$

$$\bar{P}_T = \sum_{i=0}^{k-1} \left(\prod_{j=0}^{k-1-i} \Phi(k-j)\right) T(i) \left(\prod_{j=0}^{k-1-i} \Phi(k-j)\right)^T \tag{15}$$

with $T(i) = \bar{I}\bar{K}\bar{R}(\bar{I}\bar{K})^T + (\bar{I}\bar{K}\Gamma_1 + \Gamma_2)\bar{Q}(\bar{I}\bar{K}\Gamma_1 + \Gamma_2)^T + E[\Omega\Omega^T]$.

Theorem A.1: Let the system (1) satisfy Assumptions $(AI1) - (AI4)$ with $\bar{R}(k) > 0, \bar{Q}(k) = 0, \forall k \geq 0$ and $P(0) > 0$, with $\bar{K}$ full-row rank. The boundedness of all trajectories is

guaranteed if and only if $\exists c_{pi} > 0, c_\Sigma > 0$ such that $\forall k > 0$

$$\left\| \left(\prod_{i=0}^{k-1} \Phi(k-1-i)\right) \left(\prod_{i=0}^{k-1} \Phi(k-1-i)\right)^T \right\| \leq c_{pi} \tag{16}$$

$$\left\| \sum_{i=0}^{k-1} \left(\prod_{j=0}^{k-1-i} \Phi(k-j)\right) \bar{I}\bar{K}(\bar{I}\bar{K})^T \left(\prod_{j=0}^{k-1-i} \Phi(k-j)\right)^T \right\|$$
$$\leq c_\Sigma. \tag{17}$$

Proof: Same proof as the one of Theorem 1 in [1]. ∎

Theorem A.2: Let the system (1) satisfy Assumptions $(AI1) - (AI4)$ with $\bar{R}(k) > 0, \bar{Q}(k) = 0, \forall k \geq 0$ and $P(0) > 0$, with $\bar{K}$ full-row rank. If $\lim_{k\to\infty} \Delta u_d(k) = 0$, then $\lim_{k\to\infty} P(k) = 0$ if and only if

$$\lim_{k\to\infty} \prod_{i=0}^{k-1} \Phi(k-1-i) = 0 \tag{18}$$

$$\lim_{k\to\infty} \sum_{i=0}^{k-1} \left[\left(\prod_{j=0}^{k-1-i} \Phi(k-j)\right) \bar{I}\bar{K}(i)(\bar{I}\bar{K}(i))^T \right.$$
$$\left. \cdot \left(\prod_{j=0}^{k-1-i} \Phi(k-j)\right)^T \right] = 0. \tag{19}$$

Proof: Same proof as the one of Theorem 2 in [1]. ∎

Corollary A.1: If (18) and (19) hold, then $\lim_{k\to\infty} \bar{K}(k) = 0$.

Proof: Same proof as the one of Corollary 1 in [1]. ∎

Theorems A.1 and A.2, along with Corollary A.1, establish the framework of the design of the gain matrices in order to ensure the boundedness of all trajectories, and zero convergence of $P(k)$ in presence of measurement noise and initialization errors. In order to develop optimal gains that satisfy the framework established in Problem I1, we provide the following theorem that presents a recursive algorithm for generating the optimal gains.

Theorem A.3: The gains, represented in $\bar{K}(k)$ that minimize the mean-square of the input and state errors, i.e., minimizing the $\mathrm{tr}(P(k+1))$, at each kth instant are given in the following recursive formulas for all $k > 0$:

$$\bar{K} = \bar{I}^T P E^T \left(EPE^T + \bar{R} + \Gamma_1 \bar{Q}\Gamma_1^T\right)^{-1} \tag{20}$$

$$P^+ = (H - \bar{I}\bar{K}E)P((H - \bar{I}\bar{K}E))^T + \bar{I}\bar{K}\bar{R}\left(\bar{I}\bar{K}\right)^T$$
$$+ \left(\bar{I}\bar{K}\Gamma_1 + \Gamma_2\right)\bar{Q}\left(\bar{I}\bar{K}\Gamma_1 + \Gamma_2\right)^T + E[\Omega\Omega^T]. \tag{21}$$

The optimality of (20) is based on the minimization of $\mathrm{tr}(P(k+1))$ at every sampling instance. The following theorem addresses the first part of Problem I1.

Proof: Same proof as the one of Theorem 3 in [1]. ∎

Theorem A.4: Let $\bar{R}(k) > 0, \forall k \geq 0$ and $P(0) > 0$. If $C^+ B$ is full-column rank and the system in (1) is asymptotically stable, then the optimal recursive algorithms (20) and (21) guarantee the following:

1) $\bar{K}$ is full-row rank and $0 \leq \rho(\Phi) < 1, \forall k \geq 0$;

2) The conditions (16) and (17) of Theorem 1 hold, and $P(k)$ is bounded $\forall k$, while $\bar{Q}(k) \geq 0, \forall k \geq 0$;

3) The conditions (18) and (19) of Theorem 2 are satisfied, $\lim_{k\to\infty} P(k) = 0$ and $\lim_{k\to\infty} \bar{K}(k) = 0$ whenever $\bar{Q}(k) = 0, \forall k \geq 0$ and $\lim_{k\to\infty} \Delta u_d(k) = 0$.

Proof: Same proof as the one of Theorem 4 in [1].   ∎

Definition 1: A trajectory $u_d(k)$ is said to be smooth if for any given sampling period T_s and any consistent norm $|| \cdot ||, \exists c_u > 0$ such that $\forall k \geq 0, ||u_d(k + 1) - u_d(k)|| \leq c_u T_s$.

The subsequent theorem addresses the second part of Problem I2 without assuming that $\lim_{k\to\infty} \Delta u_d(k) = 0$.

Theorem A.5: Consider the optimal recursive algorithm presented in (20) and (21). If $C^+ B$ is full-column rank, the system in (1) is asymptotically stable, the trajectory of $u_d(k)$ is smooth, $\bar{R}(k) > 0$, and $\bar{Q}(k) = 0, \forall k \geq 0$, then $\exists c_p(c_d T_s)$ such that $\lim_{k\to\infty} ||P(k)|| \leq c_p(c_d T_s)$ where $c_p(c_d T_s)$ decreases as $c_d T_s$ decreases, and $\lim_{c_d T_s \to 0} c_p(c_d T_s) = 0$.

Proof: Same proof as the one of Theorem 5 in [1].   ∎

Remark 1: Although the assumption that $\lim_{k\to\infty} \Delta u_d(k) = 0$ may be considered restrictive, we have $\lim_{k\to\infty} K(k) = 0$ and $\lim_{k\to\infty} P(k) = 0$ (Theorem A.4). Therefore, the latter indicates the capability of the controller rejecting random measurement noise. If $\lim_{k\to\infty} \Delta u_d(k) \neq 0$, then $\lim_{k\to\infty} E[\Omega\Omega^T] \neq 0$, and from (20) and (21), we have $\lim_{k\to\infty} K(k) \neq 0$ and $\lim_{k\to\infty} P(k) \neq 0$. On the other hand, for the case where $\lim_{k\to\infty} \Delta u_d(k) \neq 0$, Theorem A.5 implies that $\lim_{k\to\infty} ||P(k)||$ decreases as the sample period T_s is made smaller. That is, the error can be made arbitrary small (in probability) for sufficiently small sample period.

To address Problem I3, we begin by reducing the number of gain matrices to one by setting $K_1 = \gamma K_2$. We then show that there exists a gain matrix that can still achieve bounded trajectories (Theorem A.6) while achieving arbitrarily small errors in the absence of measurement and process noise (Theorem A.7).

Theorem A.6: Let the system (1) satisfy Assumptions $(AI1), (AI2)$, and $(AI4)$ with bounded measurement errors and bounded process noise. Consider the control law

$$u(k + 1) = u(k) + \gamma K(k)e(k + 1) - K(k)e(k) \quad (22)$$

with $1 < \gamma \ll 2$. There exists $K(k)$ such that X in (8) is bounded $\forall k \geq 0$.

Proof: We choose $K(k) = \alpha D$ with $D = ((C^+ B)^T (C^+ B))^{-1}(C^+ B)^T$, we get

$$\Phi = H - \bar{I}\bar{K}E = \begin{bmatrix} (1 - \alpha\gamma)I_q & D_q & D_{qn} \\ I_q & 0_q & 0_{qn} \\ 0_{nq} & B^- & A^- \end{bmatrix} \quad (23)$$

where $D_q = -\alpha\gamma DC^+ AB^- - \alpha DCB^-$ and $D_{qn} = -\alpha\gamma DC^+ AA^- - \alpha DCA^-$.

Boundedness of $X(k)$ requires that Φ is Schur stable. The eigenvalues of Φ can be found by computing the roots of $\det(\lambda I - \Phi)$ as follows:

$$\det(\lambda I - \Phi) = \det\left((1 - \alpha\gamma)I_q\right)$$

$$\times \det\begin{bmatrix} \lambda I_q - \frac{D_q}{\lambda - 1 + \alpha\gamma} & -\frac{D_{qn}}{\lambda - 1 + \alpha\gamma} \\ -B^- & \lambda I_n - A^- \end{bmatrix}$$

$$= \det\begin{bmatrix} \lambda(\lambda - 1 + \alpha\gamma)I_q - D_q & -D_{qn} \\ -B^- & \lambda I_n - A^- \end{bmatrix}$$

$$= \det\left(\lambda(\lambda - 1 + \alpha\gamma)I_q - D_q\right)$$

$$\times \det\left(\lambda I_n - A^- - B^-\left(\lambda(\lambda - 1 + \alpha\gamma)\right)^{-1} D_{qn}\right).$$

For sufficiently small α we have $D_q = -\alpha\gamma DC^+ AB^- - \alpha DCB^- \approx -(\alpha\gamma + \alpha)I_q$. With $D_{qn} = -\alpha\gamma DC^+ AA^- - \alpha DCA^-$, we have $A^- - B^-(\lambda(\lambda - 1 + \alpha\gamma)^{-1} D_{qn} = A^- + \alpha B^-(\lambda \times (\lambda - 1 + \alpha\gamma)^{-1} \times (\gamma DC^+ AA^- - DCA^-)$.

For $0 < \alpha \ll 1$ we get $A^- - B^-(\lambda(\lambda - 1 + \alpha\gamma)^{-1} D_{qn} \approx A^-$. This means

$$\det(\lambda I - \Phi) = \det\left(\lambda(\lambda - 1 + \alpha\gamma)I_q \right.$$
$$\left. + (\alpha\gamma + \alpha)I_q\right) \times \det\left(\lambda I_n - A^-\right).$$

Therefore, the eigenvalues of $\lambda(\Phi)$ are the roots of $(\lambda^2 - (1 - \alpha\gamma)\lambda + (\alpha\gamma + \alpha))^q \cup \lambda(A^-)$. Since the system in (1) is asymptotically stable, the eigenvalues of A^- are inside the open unit circle. The roots of $\lambda^2 - (1 - \alpha\gamma)\lambda + (\alpha\gamma + \alpha)$ are $\lambda_{1,2} = \frac{1}{2}(1 - \alpha\gamma \pm \sqrt{(1 - \alpha\gamma)^2 - 4(\alpha\gamma + \alpha)})$.

For $0 < \alpha \ll 1$ we get $|1 - \alpha\gamma \pm \sqrt{(1 - \alpha\gamma)^2 - 4(\alpha\gamma + \alpha)}| < 2$. Then since $1 < \gamma < 2$ and $0 < \alpha \ll 1$, we have $\sqrt{(1 - \alpha\gamma)^2 - 4(\alpha\gamma + \alpha)} = \sqrt{1 + (\alpha\gamma)^2 - 6\alpha\gamma - 4\alpha} \approx 1 + \frac{1}{2}((\alpha\gamma)^2 - 6\alpha\gamma - 4\alpha)$.

Therefore, $|1 - \alpha\gamma \pm \sqrt{(1 - \alpha\gamma)^2 - 4(\alpha\gamma + \alpha)}| \approx |1 - \alpha\gamma \pm 1 \pm \frac{1}{2}((\alpha\gamma)^2 - 6\alpha\gamma - 4\alpha)| < 2$, thus Φ is Schur stable. Next, consider the augmented system in (8). Using Assumptions (AI1) and (AI4), then $\bar{K}$ is bounded. Since Φ is Schur stable and disturbances are assumed to be bounded, then X is bounded $\forall k$.   ∎

In what follows we show that, if the number of inputs is equal to the number of outputs, then there exists a positive and constant diagonal matrix, such that the results in both Theorems A.6 and A.7 are applicable.

For example, we consider the state-space representation of the Euler–Lagrange model describing the dynamics of a rigid n-link robot manipulator, with all actuated revolute joints. Denoting $\dot{x}_1 \triangleq q$, $\dot{x}_2 \triangleq \dot{q}$, and $u \triangleq \tau$, where $q \in \mathbb{R}^n$ represents the joint angles, and $M(q) \in \mathbb{R}^{n\times n}$ is a symmetric positive definite inertia matrix. It can be shown that the plant dynamics can be written in as $\dot{x}_1 = x_2, \dot{x}_2 = -M^{-1}(x_1)[C(x_1, x_2)x_2 + b(x_2) + g(x_1)] + M^{-1}(x_1)u$, and $y = x_2$. Therefore, the state vector becomes $[q, \dot{q}]^T$ and the input coupling matrix, $B(q) = [0, M^{-1}(q)]^T$. In addition, if we have the system output as $\dot{q}$, then the output coupling matrix becomes $C = [0, I]$, hence $CB(q) = M^{-1}(q)$.

It can be noted that one of the model properties is as follows [24]: there exist positive scalars, β_1 and β_2 such that $0 < \beta_1 < \lambda_m(M(q)) \leq ||M(q)|| \leq \lambda_M(M(q)) < \beta_2 < \infty$, where $\lambda_m(M)$ and $\lambda_M(M)$ are the minimum and maximum eigenvalues of the matrix M, respectively, where $||M||$ is the induced Frobenius norm. Equivalently, there exist positive scalars, α_1 and α_2 such that $0 < \alpha_1 < \lambda_m(M^{-1}(q)) \leq ||M^{-1}(q)|| \leq \lambda_M(M^{-1}(q)) < \alpha_2 < \infty$.

Corollary A.2: Assume that C^+B is a symmetric positive bounded matrix, then there exists a positive diagonal gain matrix $K = K(k)$ such that X in (8) is bounded $\forall k \geq 0$.

Proof: Since C^+B is positive and bounded for all k, there exist positive scalars c_1 and c_2 such that $c_1 I \prec C^+B \prec c_2 I$, where $\prec$ means that $U \prec V \iff V - U$ is a positive definite matrix.

If K is a diagonal positive matrix, then $c_1 K \prec KC^+B \prec c_2 K$ or $I - c_2 K \prec I - KC^+B \prec I - c_1 K$. Let K_{ii} be the ith diagonal entry of K. By selecting $\max_i K_{ii} \leq \frac{1}{c_1}$, one has $I - KC^+B \prec I - c_1 K \prec I$. Therefore, $0 < \lambda(I - KC^+B) < 1$ and $0 < \lambda(I - \alpha KC^+B) < 1$, where $0 < \alpha \ll 1$. The rest of the proof follows similar steps as in Theorem A.6. ∎

Theorem A.7. Let the system (1) satisfy Assumptions $(AI1), (AI2), (AI4)$ and consider (22). If $u_d(k)$ is smooth, then there exists a $K(k)$ and sufficiently small T_s such that arbitrarily small steady-state errors can be achieved in the absence of measurement and process noise.

Proof: In absence of measurement and process noise we have $X^+ = \Phi X + \Omega$. Thus, $X(k) = (\prod_{i=0}^{k-1} \Phi(k-1-i))X(0) + \sum_{i=0}^{k-1}(\prod_{j=0}^{k-1-i} \Phi(k-j))\Omega(i)$. Given $\epsilon > 0$

$$\|X(k)\| \leq \left\| \left(\prod_{i=0}^{k-1} \Phi(k-1-i) \right) X(0) \right\|$$
$$+ \left\| \sum_{i=0}^{k-1} \left(\prod_{j=0}^{k-1-i} \Phi(k-j) \right) \right\| \max_i \|\Omega(i)\|.$$

From Theorem A.6, we have Φ is Schur stable. As $k \to \infty$, $\|(\prod_{i=0}^{k-1} \Phi(k-1-i))X(0)\| \to 0$ and $\|\sum_{i=0}^{k-1}(\prod_{j=0}^{k-1-i} \Phi(k-j))\| \to C$, where $C > 0$ is a positive scalar. Since u_d is smooth, then there exists a sufficiently small T_s such that $\max_i \|\Omega(i)\| \leq \frac{\epsilon}{C}$. Thus, as $k \to \infty, \|X(k)\| \leq \epsilon$. ∎

Corollary A.3: Assume that C^+B is a symmetric and positive bounded matrix and u_d is smooth, then there exists a positive diagonal gain matrix $K = K(k)$ and sufficiently small T_s such that arbitrarily small steady-state errors can be achieved in the absence of measurement and process noise.

Proof: Follows same steps as Theorem A.7, thus omitted. ∎

Remark 2: In order to motivate the selection of $\gamma > 1$, we consider the single-loop control (22) with gain, $K = K(k)$, being a positive scalar, i.e., $u(k+1) = u(k) + K[\gamma e(k+1) - e(k)]$. We map the controller to the time domain with T_s being the sampling period. We have $\frac{u(k+1)-u(k)}{T_s} = \frac{(\gamma-1)K}{T_s}e(k+1) + K[\frac{e(k+1)-e(k)}{T_s}]$. We can approximate the latter with $\dot{u} \approx \frac{(\gamma-1)K}{T_s}[e(t) + T_s\dot{e}(t)] + K\dot{e}(t)$. The transfer function of the controller becomes $\frac{U(s)}{E(s)} \approx \frac{\gamma K(s+\frac{(\gamma-1)}{\gamma T_s})}{s}$. The proposed controller can be approximated by a PI-type control where its zero should be relatively close to the origin or, practically, $\gamma < \frac{1}{1-T_s}$ where $0 < T_s \ll 1$, or γ should be selected in the neighborhood of $1 + T_s$ otherwise the relative stability or stability of the system can be compromised.

The controller (22), $u(k+1) = u(k) + \gamma K(k)e(k+1) - K(k)e(k)$, where $K(k) = \mu(k)\hat{K}(k)$, involves adequate selection of $\gamma, \hat{K}(k)$, and $\mu(k)$. In what follows, we suggest few tips for the selection of $\gamma, \hat{K}(k)$, and $\mu(k)$.

Selection of γ: As suggested in Remark 2, the value of γ should be selected in the neighborhood of $1 + T_s$, where T_s is the sample period.

Selection of $\hat{K}(k)$:
  1) If a rough estimate of the product of the input/output coupling matrices, C^+B, say $\hat{C}^+\hat{B}$, is available, then, $\hat{K}(k) = \beta D$, where D is the Moore–Penrose inverse of $\hat{C}^+\hat{B}$, and $0 < \beta \ll \hat{\beta} < 1$ (as in the proof of Theorem A.6). The less accurate the estimate $\hat{C}^+\hat{B}$ is available the smaller $\hat{\beta}$ is chosen.
  2) If C^+B is a symmetric positive bounded matrix, then $\hat{K}(k)$ can be set to a positive diagonal gain matrix, $\hat{K}(k) = K$, as shown in Corollary A.2. The values of the diagonal entries can be intuitively selected based on the capacity of the plant (e.g., see, experimental implementation on a robot arm of Section IV-B).
  3) In a black-box type of applications, simply set $\hat{K}(k) = c$, where c is a positive scalar (e.g., see the quadruple tank process example of Section IV-A). Some tuning is needed to select the proper magnitude of c.

Selection of $\mu(k)$:
  1) In case measurement noise is negligible, set $\mu(k) \equiv 1$.
  2) Otherwise, $\mu(k)$ should be a decreasing function of positive scalars starting with $\mu(0) = 1$ and decreasing to a positive lower bound; e.g., $\mu(k) = \{ \begin{smallmatrix} \frac{1}{k^\alpha} & k \leq \bar{k} \\ \frac{1}{k^\alpha} & k > \bar{k} \end{smallmatrix}$, where $0 < \alpha \leq 1$, and $\frac{1}{k^\alpha} \approx 0.1$. The latter is aligned with the comments in Remark 1.

B. Linear Time-Invariant Systems

In order to address Problem II1 and Problem II2, we first formulate the error covariance $\tilde{P}(k)$ of the system in (12). Since the augmented system in (12) follows the same augmented system of the PID control law in [2], with E being also full-column rank where the latter follows from (AII2), then all results in [2] apply while following the same proofs. For convenience, we only list the main results.

We consider the augmented system (12) and we find the optimal $\tilde{K}$ by minimizing $\text{tr}(\tilde{P}(k+1))$, where $\tilde{P}(k+1) \equiv \text{E}[\tilde{X}^+ \tilde{X}^{+T}]$.

Denote $\tilde{P} \equiv \tilde{P}(k)$ and $\tilde{K}^- \equiv \tilde{K}(k-1)$. From Assumption (AII1) we have that $w(0), v(0)$, and $\delta x(0)$ are mutually uncorrelated and zero-mean white noise, then (12) leads to

$$\tilde{P}(k+1) = \Psi \tilde{P} \Psi^T + \tilde{\Gamma}_1 \tilde{R} \tilde{\Gamma}_1^T + \tilde{\Gamma}_2 \text{E}[\tilde{\Omega}\tilde{\Omega}^T]\tilde{\Gamma}_2^T. \quad (24)$$

The results in the subsequent theorem provides a recursive algorithm that automatically generates the optimal gains of (2).

Theorem B.1: Let the system satisfy Assumptions (AII1) and (AII2), and consider the update law in (2). The gains $K_{1,2}(k)$ that minimize the mean-square output errors and $\eta(k)$, i.e., minimize $\text{tr}(\tilde{P}(k+1))$, at each kth instant are given in the following recursive formula $\forall k > 0$:

$$\tilde{K}^- = (\tilde{E}^T \tilde{E})^{-1} \tilde{E}^T \tilde{H} \tilde{P} \tilde{I}^T \left(\tilde{I} \tilde{P} \tilde{I}^T + \tilde{R} \right)^{-1} \quad (25)$$

$$\tilde{P}(k+1) = \Psi \tilde{P} \Psi^T + \tilde{E} K^{-} \tilde{R} (\tilde{E} K^{-})^T + E[\tilde{\Omega} \tilde{\Omega}^T] \quad (26)$$

where $\tilde{P}(1)$ is selected to be a positive-definite matrix.

Proof: Same proof as the one of Theorem 1 in [2]. ∎

The following theorem provides necessary and sufficient conditions for convergence of the error covariance $\tilde{P}(k)$.

Theorem B.2: Consider the update law given in (2) and the recursive algorithm presented by (25) and (26). Then, $\lim_{k \to \infty} \tilde{P}(k)$ exists if and only if $|\lambda(\Theta_1)_{\lambda(\Theta_1) \neq 1}| < 1$, where

$$\Theta_1 \equiv BMB^T + (I_n - BMB^T M^T C^T C - BMB^T) A$$

$$M \equiv (E^T E)^{-1} = [(CB)^T CB + B^T B]^{-1}.$$

Proof: Same proof as the one of Theorem 2 in [2]. ∎

The next theorem provides necessary and sufficient conditions for bounding the steady-state error, in absence of process noise where the bound decreases as the measurement error decreases and the sample rate increases. These characteristics are justified numerically in [2].

Theorem B.3: Consider the update law given in (2) and the recursive algorithm presented by (25) and (26). Assume that $y_d(k)$ is a smooth function and $w(k) = 0, \forall k > 0$. Therefore, $|\lambda(\Theta_1)_{\lambda(\Theta_1) \neq 1}| < 1$ if and only if $\lim_{k \to \infty} \tilde{P}(k) \leq c(R, T_s) c_{\Sigma} I$, where $c_{\Sigma} > 0$ and $c(R, T_s) \equiv \sup_k (\rho(R(k)) \rho(U \tilde{H} \tilde{H}^T U) + c_y T_s$.

Proof: Same proof as the one of Theorem 3 in [2]. ∎

IV. NUMERICAL AND EXPERIMENTAL RESULTS

In this section we illustrate the performance capabilities of the proposed simplified controller given by (22).

A. Numerical Implementation on a Quadruple Tank Process

In this example we compare the performance of the proposed controller (22) with the optimal stochastic PID controller in [1], [2], and [3]. The multivariable PID gains of the former two controllers are obtained by minimizing the error covariance matrix of different augmented systems, and the latter considers the problem of designing a multivariable PID controller via direct optimal linear quadratic regulator.

We consider the continuous-time system of a quadruple tank process with nonminimum phase setting as described in [3], with $\dot{x}(t) = A_c x(t) + B_c u(t)$ and $y(t) = C x(t)$, where

$$A_c = \begin{bmatrix} -0.0159 & 0 & 0.0256 & 0 \\ 0 & -0.011 & 0 & 0.00179 \\ 0 & 0 & -0.0256 & 0 \\ 0 & 0 & 0 & -0.0179 \end{bmatrix}$$

$$B_c^T = \begin{bmatrix} 0.0482 & 0 & 0 & 0.0178 \\ 0 & 0.035 & 0.0236 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix}.$$

TABLE I
TRANSIENT RESPONSE OF FOUR MULTIVARIABLE CONTROLLERS

	[3]		[1]		[2]		Proposed	
	$OS\%$	t_s	$OS\%$	t_s	$OS\%$	t_s	$OS\%$	t_s
y_1	50%	400	0	400	25%	36	0	5
y_2	20%	400	25%	200	7%	20	25%	5
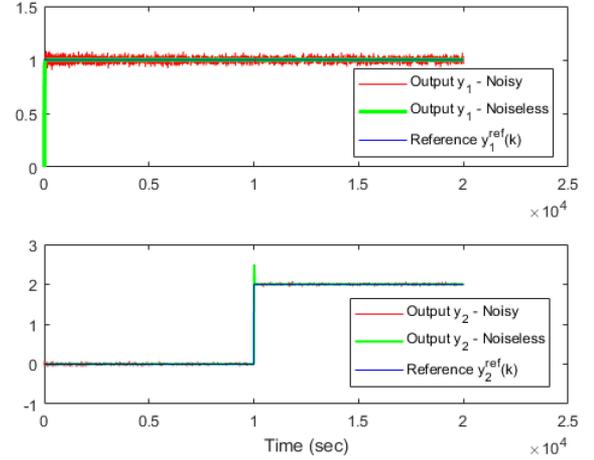
Fig. 1. Output response in the presence and absence of measurement noise.

TABLE II
PERFORMANCE IN PRESENCE OF NOISE, $N(0, 0.05^2)$

	$t \in [0, 100]$	$t \in [0, 2000]$	$t \in [1900, 2000]$
$\text{std}(y_1^{ref} - y_1)$	0.075	0.026	0.019
$\text{std}(y_2^{ref} - y_2)$	0.024	0.030	0.016

The reference signals are given by [3]

$$y_1^{\text{ref}}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}, \quad y_2^{\text{ref}}(t) = \begin{cases} 0, & t < 1000 \\ 2, & t \geq 1000 \end{cases}.$$

We discretize the system with sampling period $T_s = 0.1$ s. The proposed controller parameters used in both settings (with noise and without noise) are $K(k) = \frac{1}{k^{0.2}} 500 I$ and $\gamma = 1.1$. As per the tuning tips provided in Section III-A, we divide the gain by $k^{0.2}$ so that the controller can partly reject measurement noise.

The transient performance is illustrated in Table I where we compare the percentage overshoot, OS%, and the settling time, t_s (s), for the four different multivariable controllers. By examining Table I, it is observed that the general performance of the proposed controller is superior to that PID of [3], and superior to [1] and [2] in absence of measurement noise. The corresponding steady-state errors are about 10^{-10}. Unlike our simplified proposed controller, these three optimal PID controllers use the system model to obtain their gains. We also examine the performance of our proposed controller in the presence of measurement noise. We add zero-mean Gaussian noise to both outputs with standard deviation $\sigma = 0.05$. Fig. 1 shows the outputs in absence and presence of measurement noise. Table II lists the standard deviation of the error during the first 100 s, over the entire range, and during the last 100 s. One can observe that the error values during the last 100 s are significantly smaller than

the ones corresponding to the first 100 s. Although the proposed controller settles much faster than the one in [1], it does not completely reject measurement as in [1]. The reported results in [2] on the output error standard deviations, $\text{std}(y_1^{\text{ref}} - y_1) = 0.033 < \sigma$ and $\text{std}(y_2^{\text{ref}} - y_2) = 0.018 < \sigma$, which resembles the performance of the proposed controller are illustrated in Table II.

Remark 3: The magnitude of the control signals is not constrained to any threshold at which the signal must be truncated in practical application. Consequently, we implement the following saturation function to the controller output: $sat(u(k+1), \bar{u}) \triangleq sign(u(k+1))min(|u(k+1)|, \bar{u})$. For illustration, we set a threshold of $\bar{u} = 5$, and we find the corresponding settling time, t_s, to be about 12 s for the first output, and 25 s for the second output. In addition, the overshoot drops down to zero for both outputs.

B. Experimental Implementation on 4-DoF Robot Manipulator

We consider the Euler–Lagrange model describing the continuous-time dynamics of rigid n-link serial nonredundant robot manipulator, with all actuated revolute joints, which is given by $\tau = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + F(q, \dot{q})$, where q, $\dot{q}$, and $\ddot{q} \in \mathbb{R}^n$ represent joint angles, velocities, and accelerations, respectively. $M(q) \in \mathbb{R}^{n \times n}$ is the symmetric positive definite inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ denotes the Coriolis and centripetal matrix, $G(q) \in \mathbb{R}^n$ is the gravity vector, $F(q, \dot{q}) \in \mathbb{R}^n$ represents the joint friction, and $\tau \in \mathbb{R}^n$ denotes the joint torque input.

The experiments are conducted on a cable-driven 4-DoF Barret WAM, with a reach of 1 m and 4 kg payload with a 12-b position encoder for each joint; a picture of the same experimental setup is given in [24]. The Barrett WAM is operated using MATLAB-SIMULINK on a host PC through an external target PC, which is connected to the robotic arm through Barrett CAN bus. The two PCs are connected via an Ethernet cable. The target PC is a core i7. The host PC is a core i5. The target PC is booted using a MATLAB kernel.

The sampling rate of the controller is set at 1000 Hz.

The proposed controller gain in (22) is given by

$$K(k) = \frac{1}{k^\alpha} \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 2.5 \end{bmatrix}$$

and $\gamma = 1.1$. We choose the same value of γ used in the quadruple tank process example. The diagonal entries of $K(k)$ are intuitively selected based on the capacity of each joint and the limit torque limits of the arm. Simple values are chosen to show that not much tuning has been exercised. The tracking performance of our proposed method is illustrated in Fig. 2, where the actual outputs and reference trajectories are displayed.

We compare performances for three values of $\alpha \in \{0, 0.1, 0.2\}$. We estimate the velocity error using backward Euler method. That is, $e_i(k+1) = \frac{q_{i,d}(k) - q_{i,d}(k-1)}{T_s}$, where $q_{d,i}$
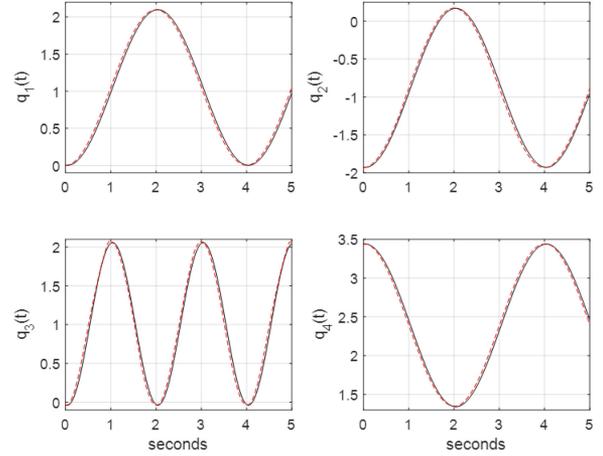
Fig. 2. Output response of the robot manipulator. The actual outputs are in black and the reference in dashed red.

TABLE III
PE AND MAXIMUM ABSOLUTE ERROR OF CONTROLLERS (C1)–(C5)

Metric	Control Law	J1	J2	J3	J4	Avg
PE	(C1)	0.52	0.4	2.29	0.66	**0.97**
	(C2)	2.03	1.19	8.1	3.33	**3.66**
	(C3)	5.88	2.93	12.1	4.13	**6.26**
	(C4)	4.75	2.94	12.84	4.07	**6.15**
	(C5) $\alpha=0$	0.21	0.49	3.56	0.19	**1.11**
	(C5) $\alpha=0.1$	0.45	0.68	4.99	0.33	**1.61**
	(C5) $\alpha=0.2$	0.62	1.13	5.19	0.53	**1.87**
Max Abs. Error	(C1)	0.01	0.005	0.026	0.019	**0.0150**
	(C2)	0.046	0.028	0.118	0.111	**0.0758**
	(C3)	0.091	0.057	0.21	0.113	**0.1178**
	(C4)	0.024	0.001	0.03	0.196	**0.0628**
	(C5) $\alpha=0$	0.0034	0.0219	0.0909	0.0108	**0.0318**
	(C5) $\alpha=0.1$	0.0077	0.0265	0.0905	0.0186	**0.0358**
	(C5) $\alpha=0.2$	0.0182	0.0391	0.1328	0.0598	**0.0625**

is the reference trajectory of the ith joint. Throughout this work, we use the percentage of error (PE), which is defined as $\text{PE}_i \triangleq \frac{\text{AVG}_{t \in [0,5]}|q_i - q_{d,i}|}{\text{AVG}_{t \in [0,5]}|q_{d,i}|}$; and the maximum absolute error over the duration of the reference trajectory.

We report experimental validation by comparing the performance of: (C1) the stochastic PDD controller in [26], (C2) the stochastic PD controller in [24], (C3) the deterministic PD controller in [25], and (C4) the sliding mode controller coupled with dirty derivative filter in [27], and (C5) our controller (22).

The description of each controller is presented in [26]. It is important to note that there are other sliding mode controllers such as the adaptive sliding mode controllers [14] and [15]. However, such controllers require extensive tuning. Although experimental results are reported in [27], without integrating a dirty filter to estimate angular rates, intolerable vibrations in the arm and torque saturation are realized and even with the use of dirty filter, the torque signals are not smooth. In addition, it is the opinion of the authors that better tracking and smoother torque signals can be achieved with better tuning using the adaptive sliding mode controllers. For the latter reasons we have not reflected on such controllers in this work.

We report the resulting percent error (PE) and maximum absolute error of our proposed controller for $\alpha \in \{0, 0.1, 0.2\}$, and of the controllers (C1)–(C5) in Table III, where the average

TABLE IV
PERFORMANCE COMPARISON OF CONTROLLERS (C1)–(C5) USING
METRICS (M1)–(M5)

Controller	(M1)	(M2)	(M3)	(M4)	(M5)	Average
(C1)	5	4	3	3	2	3.4
(C2)	3	5	5	3	3	3.8
(C3)	2	3	4	4	4	3.4
(C4)	1	2	2	4	4	2.6
(C5) $\alpha = 0.2$	4	4	1	5	5	3.8

performance across all four joints are presented in bold. By examining Table III one can conclude that smaller values of α lead to higher accuracy, yet experimentation revealed that the accuracy was at the cost of the smoothness of the torque signals. Although no arm vibrations are noticed during experiments, the torques corresponding to $\alpha = 0$ caused some audible noise in the arm, which may harm the arm in the long run.

We employ the following five ranking metrics to intuitively assess the pros and cons of each controller while assigning five points to the best performer, four points to the second, three points to the third, two points to the fourth, and one point to the fifth.

(M1) Overall output tracking performance: Evaluation is based on values in Table III where our controller comes second.

(M2) Ease of tuning: (C1) involves tuning of five covariance matrices and a Kalman filter. (C2) involves tuning of one covariance matrix. (C3) and (C4) involve tuning of its two controller gains and dirty filter parameters. (C4) also includes tuning of saturation function. (C5) involves tuning of one gain matrix.

(M3) Smoothness of torque signals: Evaluation of controllers are based on torques observed during experiments.

(M4) Model requirement: Unlike our controller (C5), the controllers (C1)–(C4) all use the knowledge of gravity vector and inertia matrix. (C1) and (C2) requires some knowledge of statistical errors.

(M5) Computation complexity: (C1)–(C4) compute and invert the inertia matrix and (C1) converts a larger innovation covariance matrix than the one in (C2).

Table IV lists the points attained by the five controllers corresponding to metrics (M1)–(M5). As a recap, we find the proposed controller and (C2) attaining the highest average; however, for a novice engineer, our controller is the easiest to implement.

V. CONCLUSION

This article proposed a nonadaptive multivariable tracking controller that required very limited information on the plant model and dealt with a class of large-scale systems with complex dynamics, which was robust and was easy to tune. The potential of the proposed controller was shown numerically on a quadruple tank process and displayed its transient response superiority over three different optimal PID controllers that used the system model. Furthermore, efficacy of the proposed controller was demonstrated on experimental data from a 4-DoF Barrett robot arm. The overall performance of the control scheme was compared with four recent controllers, namely, a deterministic PD controller, stochastic PD controller, stochastic PDD controller,

and sliding mode control (SMC) integrated with a dirty filter. The four controllers used knowledge of the gravity vector and inertia matrix. The proposed approach was shown to be the least computationally expensive, easiest to tune, easiest to implement, and did not require knowledge of the plant model. The proposed controller achieved uniform output tracking performance, outperforming three schemes and very comparable with the stochastic PDD controller. However, the resulting torque signals were not as smooth as the other four controllers. In addition, a framework for selecting optimal gain matrices was provided for two different discrete-time systems where knowledge of system model was required. In both cases, necessary and sufficient conditions were provided for boundedness of all trajectories and convergence of the error covariance matrices. The framework associated with the first system achieved complete rejection of measurement noise in probability. To extend the scope of this work, the authors suggested: 1) performance comparisons with other control schemes including adaptive control strategies that handle the constraints outlined in the article, and 2) experimental validation on real-life plants such as power generation systems.

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