On State-Space Modeling and Signal Localization in Dynamical Systems

Asok Ray
Fellow ASME
Departments of Mechanical Engineering and Mathematics,
Pennsylvania State University,
University Park, PA 16802
e-mail: axr2@psu.edu

This letter focuses on two topics in engineering analysis, which are (1) degree-of-freedom (DOF) in modeling of dynamical systems and (2) simultaneous time and frequency localization of signals. These issues are explained from the perspectives of decision and control by making use of concepts from applied mathematics and theoretical physics. Specifically, a new definition is proposed to clarify the notion of "DOF," which is consistent with the dimension of the state space of the dynamical system model. Relevant examples are presented on (finite-dimensional) vector spaces over the real field $\mathbb{R}$ *and/or the complex field* $\mathbb{C}$. [DOI: 10.1115/1.4051142]

Keywords: dynamical systems modeling, degree-of-freedom, time-frequency analysis of signals

1 Introduction

In many engineering applications, mathematical models are constructed to represent the behavior of dynamical systems [1], and tools of time-frequency analysis [2] are applied for processing the measured signals (e.g., sensor time series). To this end, mathematical structures of dynamical system models and signal waveforms are built upon both finite-dimensional and infinite-dimensional vector spaces that are defined over a (complete[1]) algebraic field such as the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. The properties (e.g., dimension) of a vector space are significantly dependent on the choice of the algebraic field over which the vector space is constructed. In this context, interpretations of commonly used terms in modeling of dynamical systems, such as degree-of-freedom (DOF), may often become ambiguous. Another example of ambiguity in signal processing is the uncertainty of simultaneous time and frequency localization.

This letter elaborates the above two ambiguities, namely, *DOF* and *time-frequency localization*, which are relevant in physics-based modeling of dynamical systems and data-driven analysis of sensor signals, respectively. The underlying issues are explained from the perspectives of decision and control by making use of concepts from applied mathematics and theoretical physics. Relevant engineering applications of these concepts in dynamic system modeling [3] and signal processing [4–6] exist; and such applications are constructed on vector spaces defined over the real field $\mathbb{R}$ and/or the complex field $\mathbb{C}$.

The letter is organized in four sections. Section 2 points out the reasons for possible ambiguities in the relationships between DOF and dimension of the state space, and it provides explanations with well-known examples of mechanical systems. Section 3 addresses the issue of uncertainty in simultaneous time localization and frequency localization of signals. Section 4 summarizes and concludes the letter.

2 State-Space Representation of Degree-of-Freedom

This section first introduces the notion of DOF from the perspectives of classical mechanics [7,8] and vibration theory [9–11]. Then, the DOF is compared with the dimension of the state space that represents the dynamical system model as a vector space constructed over the complex field $\mathbb{C}$.

In the context of classical mechanics [7], the notion of DOF of a dynamical system is presented as follows:

> Let a dynamical system of (mutually non-interacting) N particles be free from constraints. Then, the system has $3N$ DOFs, which implies that the dynamical system is represented by $3N$ independent coordinates. However, if there exist K holonomic constraints, where $0 \leq K \leq 3N$, then the dynamical system is said to have $(3N - K)$ DOFs.

The above notion of DOF has been widely used for vibration analysis [9] in various engineering disciplines as follows:

> The minimum number of independent coordinates needed to determine completely the positions of all parts of a mechanical system at any instant of time defines its DOF.

In state-variable representations of dynamical systems, the role of an algebraic field is very critical in the construction of vector spaces, because a given Abelian group of vectors generates two different vector spaces when they are defined over two different algebraic fields [12,13]. For example, the Abelian group of complex numbers $\mathbb{C}$ forms a two-dimensional vector space over the real field $\mathbb{R}$, which is isomorphic to the two-dimensional real vector space $\mathbb{R}^2$. The same Abelian group of complex numbers $\mathbb{C}$ forms a one-dimensional vector space over the complex field $\mathbb{C}$. The implications of this fact on modeling of dynamical systems are illustrated below by well-known physical examples.

Figure 1(*a*) presents an unforced linear time-invariant mass-spring-damper system with discrete passive [14] components and having non-zero initial conditions. Such a system is said to have

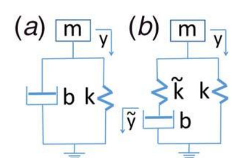

Fig. 1 Schematic diagrams of unforced linear time-invariant mass-spring-damper systems, where the model parameters are mass *m*; viscous damping constant *b*; and spring constants *k* and $\tilde{k}$: (*a*) single degree-of-freedom (SDOF) system and (*b*) augmentation of the SDOF system with an additional spring

[1]A metric space is called complete if every Cauchy sequence converges in the space. In this sense, an example of an incomplete field is the field of rationals $\mathbb{Q}$ with the usual metric, i.e., $d(x, y) = |x - y|\ \forall x, y \in \mathbb{Q}$.

Manuscript received November 6, 2020; final manuscript received May 5, 2021; published online May 31, 2021. Assoc. Editor: Reza Tafreshi.

SDOF [9] and is governed by the following second-order differential equation:

$$\frac{d^2y}{dt^2} + 2\,\zeta\,\omega_n\frac{dy}{dt} + \omega_n^2\,y = 0 \tag{1}$$

where, referring to Fig. 1(*a*), y is the time-dependent displacement; $\omega_n \triangleq \sqrt{k/m}$ is the natural frequency; and $\zeta \triangleq b/2\sqrt{mk}$ is the damping coefficient that has a range of $(0 \le \zeta < \infty)$, in general, and a range of $(0 \le \zeta < 1)$ for underdamped systems.

A state-space representation of Eq. (1) for an underdamped system in the two-dimensional vector space $\mathbb{R}^2$ (over the real field $\mathbb{R}$) is given as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\zeta\,\omega_n & \sqrt{1-\zeta^2}\,\omega_n \\ -\sqrt{1-\zeta^2}\,\omega_n & -\zeta\,\omega_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1a}$$

where the two state variables are chosen as $x_1 = \zeta\omega_n\,y + (dy/dt)$ and $x_2 = -\sqrt{1-\zeta^2}\,\omega_n y$, and the associated state matrix is

$$\begin{bmatrix} -\zeta\,\omega_n & \sqrt{1-\zeta^2}\,\omega_n \\ -\sqrt{1-\zeta^2}\,\omega_n & -\zeta\,\omega_n \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

which maps $\mathbb{R}^2$ into $\mathbb{R}^2$.

Next, let us construct a vector space of the same physical system (see Fig. 1(*a*) and Eq. (1)) over the complex field $\mathbb{C}$ instead of the real field $\mathbb{R}$. To this end, let us define a complex-valued state variable as $z(t) \triangleq x_1(t) + i\,x_2(t)$, where $i = \sqrt{-1}$ and x_1 and x_2 are as defined above. Then, it follows that

$$\frac{dx_1}{dt} + i\frac{dx_2}{dt} = \left(-\zeta\omega_n\,x_1 + \sqrt{1-\zeta^2}\,\omega_n\,x_2\right) + i\left(-\sqrt{1-\zeta^2}\,\omega_n\,x_1 - \zeta\omega_n\,x_2\right)$$

which reduces to

$$\frac{d}{dt}[x_1 + ix_2] = \left[-\left(\zeta + i\sqrt{1-\zeta^2}\right)\omega_n\right]\,[x_1 + ix_2]$$

Consequently, the state-space representation of the above underdamped system in the one-dimensional vector space $\mathbb{C}^1$ (defined over the complex field $\mathbb{C}$) is

$$\frac{d}{dt}[z] = \left[-\left(\zeta + i\sqrt{1-\zeta^2}\right)\,\omega_n\right]\,[z] \tag{2}$$

where the state matrix $[-(\zeta + i\sqrt{1-\zeta^2})\omega_n] \in \mathbb{C}^{1\times 1}$ maps $\mathbb{C}^1$ into $\mathbb{C}^1$. It is noted that, in the disciplines of mechanical engineering and classical mechanics, such a second-order underdamped system is often referred to as a SDOF system. Here, we have shown that an underdamped system is equivalently represented on a vector space of dimension 2 over $\mathbb{R}$ or on a vector space of dimension 1 over $\mathbb{C}$ as an SDOF system. In both representations, the system models are linear.

Next, we consider an overdamped system, i.e., the damping coefficient $\zeta > 1$ in Eq. (1). A state-space representation of the overdamped system in the two-dimensional vector space $\mathbb{R}^2$ (over the real field $\mathbb{R}$) is given as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\zeta\,\omega_n & \sqrt{\zeta^2-1}\,\omega_n \\ \sqrt{\zeta^2-1}\,\omega_n & -\zeta\,\omega_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1b}$$

where a choice for the states is $x_1 = \zeta\omega_n y + (dy/dt)$ and $x_2 = \sqrt{\zeta^2-1}\,\omega_n y$, and the associated state matrix is

$$\begin{bmatrix} -\zeta\,\omega_n & \sqrt{\zeta^2-1}\,\omega_n \\ \sqrt{\zeta^2-1}\,\omega_n & -\zeta\,\omega_n \end{bmatrix} \in \mathbb{R}^{2\times 2} \text{ mapping } \mathbb{R}^2 \text{into } \mathbb{R}^2$$

Similar to what was done for the underdamped system, the vector space is now constructed over the complex field $\mathbb{C}$ instead of the real field $\mathbb{R}$. The complex-valued state is defined as $z(t) \triangleq x_1(t) + i\,x_2(t)$, where $i = \sqrt{-1}$ and x_1 and x_2 are as defined in Eq. (1*b*). Then it follows that

$$\frac{dx_1}{dt} + i\frac{dx_2}{dt} = \left(-\zeta\omega_n\,x_1 + \sqrt{\zeta^2-1}\,\omega_n\,x_2\right) + i\left(\sqrt{\zeta^2-1}\,\omega_n\,x_1 - \zeta\omega_n\,x_2\right)$$

which reduces to

$$\frac{d}{dt}[x_1 + ix_2] = \left(\left(-\zeta x_1 + \sqrt{\zeta^2-1}\,x_2\right) + i\left(\sqrt{\zeta^2-1}\,x_1 - \zeta\,x_2\right)\right)\,\omega_n$$

Consequently, the state-space representation of the above overdamped system in the one-dimensional vector space $\mathbb{C}^1$ (over the complex field $\mathbb{C}$) is given as

$$\frac{d}{dt}[z] = \left[-\left(\zeta - i\sqrt{\zeta^2-1}\right)\,\omega_n\right][z] - \left[i\,\sqrt{\zeta^2-1}\,\omega_n\right][(z-\bar{z})] \tag{3}$$

where $\bar{z}$ is the complex conjugation of $z \in \mathbb{C}^1$. It is noted that the operation of complex conjugation in Eq. (3) is not linear because, given any $z, \tilde{z}, \gamma \in \mathbb{C}$, it follows that $\overline{z + \gamma\tilde{z}} = \bar{z} + \bar{\gamma}\bar{\tilde{z}}$ instead of having the linear relationship of complex conjugation, i.e., $\overline{z + \gamma\tilde{z}} = \bar{z} + \gamma\bar{\tilde{z}}$, which is false because $\bar{\gamma} \ne \gamma$, in general.

For a critically damped system (i.e., $\zeta = 1$), the results cannot be obtained simply by substituting $\zeta = 1$ in Eq. (2) or Eq. (3), because the two states (in the real-field representation) are coupled, which gives rise to a state matrix in the Jordan form [12] as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\omega_n & \omega_n \\ 0 & -\omega_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1c}$$

where a choice for the states is $x_1 = \omega_n\,y$ and $x_2 = \omega_n\,y + (dy/dt)$ and the associated state matrix $\begin{bmatrix} -\omega_n & \omega_n \\ 0 & -\omega_n \end{bmatrix} \in \mathbb{R}^{2\times 2}$ maps $\mathbb{R}^2$ into $\mathbb{R}^2$. Letting $z(t) \triangleq x_1(t) + ix_2(t)$, it follows that

$$\frac{d}{dt}[z] = [-\omega_n][z] - \left[i\frac{\omega_n}{2}\right][(z-\bar{z})] \tag{4}$$

which, although one-dimensional, is not a linear representation of the physical system in Fig. 1(*a*). It is noticed that Eq. (4) is structurally similar to Eq. (3).

In view of the above observations, we examine how the DOF of the dynamical system can be related to the dimension of the state space, i.e., the vector space over $\mathbb{C}$. Now, we delineate the following notions regarding DOF and linearity in the vector space over $\mathbb{C}$.

(1) Following Eq. (2), if the system in Fig. 1(*a*) is underdamped (i.e., $0 \le \zeta < 1$), then it can be represented as a first-order linear system in the vector space $\mathbb{C}^1$. Therefore, the notion of SDOF (i.e., DOF = 1) is in agreement with that of the one-dimensional linear system in Eq. (2). By extending this concept to finite-dimensional (linear) underdamped multi-DOF systems in the usual sense (e.g., see Ref. [9]), the notion of DOF can be made equivalent to that of the dimension of the state space, constructed over $\mathbb{C}$.

(2) If the system in Fig. 1(*a*) is critically damped (i.e., $\zeta = 1$) or overdamped (i.e., $\zeta > 1$), then the vector space representation over $\mathbb{C}$ may not be represented as a linear system, while the linearity in the $\mathbb{R}^2$ space is still retained (see Eqs. (1*b*) and (1*c*)). Apparently, there is no clear notion of DOF, including that of SDOF, for $\zeta \ge 1$.

2.1 Extension to a State-Space of Higher Dimension. This subsection investigates how the notion of SDOF could be affected if the dynamical system in Fig. 1(*a*) is augmented with an additional (passive) component. In Fig. 1(*b*), an additional spring, having spring constant $\tilde{k}$, is inserted in tandem with the dashpot. Following Fig. 1(*b*), the governing equations are obtained as

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \tilde{k}(y - \tilde{y}) + ky = 0 \quad \text{and} \quad b\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = \tilde{k}(y - \tilde{y}) \tag{5}$$

where the parameters m, b, k, and $\tilde{k}$ are as defined in the caption of Fig. 1.

Now let us first construct a vector space over $\mathbb{R}$ by choosing the three state variables as $x_1 = y$, $x_2 = (\mathrm{d}y/\mathrm{d}t)$, and $x_3 = \tilde{y}$. Then, a (three-dimensional) state-space representation (over $\mathbb{R}$) of the mass-spring-damper system in Fig. 1(*b*) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -(k+\tilde{k})/m & 0 & \tilde{k}/m \\ \tilde{k}/b & 0 & -\tilde{k}/b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{5a}$$

It is obvious that the system, represented by Eq. (5*a*), is *not* SDOF [9]. Then, what is the number of DOFs of this discrete-parameter linear system—should it be called a two degrees-of-freedom (TDOF) system? Apparently, the literature on vibration analysis, such as Refs. [9–11], would not classify such a system as TDOF.

Before delving further into the above discussion, let us examine the role of the parameter $\tilde{k}$ in Fig. 1(*b*), which has apparently changed the SDOF property of the dynamical system in Fig. 1(*a*). The following observations are made in this context:

- As the parameter $\tilde{k} \to \infty$ in Fig. 1(*b*), an application of the principle of singular perturbation [14], reduces the dimension of the state space (over $\mathbb{R}$) from 3 to 2 and degenerates the system in Fig. 1(*b*) into that in Fig. 1(*a*). A physical interpretation is that, as $\tilde{k} \to \infty$, the spring in tandem with the dashpot in Fig. 1(*b*) becomes a rigid (massless) link. Similarly, as $\tilde{k} \to 0$, the system in Fig. 1(*b*) degenerates into a (non-dissipative and SDOF) mass-spring system.
- The examination of a state-space representation (over $\mathbb{C}$) of the system in Fig. 1(*b*) would reveal that the resulting state vector belongs to $\mathbb{C}^2$, in general. Such a state-space can be constructed by two linearly independent combinations (over $\mathbb{C}$) of the three states in Eq. (5*a*). Moreover, in Fig. 1(*b*), if the parameter $\tilde{k} \to \infty$ or if $\tilde{k} \to 0$, then applications of singular perturbation [14] would reduce the dimension of the (complex) state space from 2 to 1 in each case.

As a conclusion of this section, the notion of state-space representation over the real field $\mathbb{R}$ should be emphasized when (first-time) introducing the topic of dynamic system modeling, by avoiding the concept of DOF; the rationale is that although the notion of DOF provides nice physical intuitions in some cases, it is not precisely defined in the author's opinion. An alternative precise definition of DOF, which may not always be in agreement with the conventional notion of DOF as reported in the literature (e.g., see Ref. [9]), is proposed as follows:

Definition 2.1. *The DOF of a (finite-dimensional) dynamical system is the dimension of the state space when the underlying vector space is constructed over the complex field* $\mathbb{C}$.

Remark 2.1. The dynamical system in Definition 2.1 could be time-varying and/or nonlinear (i.e., not restricted to be linear time-invariant as in Fig. 1). An example of a time-varying dynamical system is a rocket launched into the outer space from the planet earth, where the total mass of the rocket may significantly change with time because the rate of fuel depletion in the rocket engine is very high. Examples of nonlinear dynamics are (i) cubic force–displacement characteristics of a spring and (ii) Coulomb friction in a damper.

3 Signal Analysis in Time and Frequency Domains

This section addresses the topic of time-frequency analysis of signals in data-driven dynamical systems, which often complements that of dynamic modeling in engineering analysis of decision and control systems. The topic is discussed here from the perspectives of theoretical physics, where the signals are often analyzed using vector spaces, constructed over the complex field $\mathbb{C}$, as explained below.

Let a particle of (constant) mass m, constrained to move along the x-axis, be subjected to a time-dependent force $F(x, t)$, where the vector space is defined over the real field $\mathbb{R}$. One of the objectives in classical mechanics [7] is to determine the position $x(t)$ of the particle at any given time t. Then, we obtain the velocity $v(t) \triangleq \mathrm{d}x/\mathrm{d}t$, the momentum $p(t) \triangleq m\, v(t)$, and the kinetic energy $T(t) \triangleq (1/2)m\, v^2(t)$. On the other hand, the wave function $\psi(x, t)$ in quantum mechanics [15,16] is a complex-valued function of the position x of a particle at time t, which belongs to a Hilbert space as defined over the complex field $\mathbb{C}$. A statistical interpretation of the (complex-valued) wave function $\psi(x, t)$ is that $|\psi(x, t)|^2$ is the probability density of finding the particle at a given position x at time t, i.e., $\int_{-\infty}^{+\infty} \mathrm{d}x\, |\psi(x, t)|^2 = 1$ for all time t.

The state of a particle in classical Hamiltonian mechanics [7], where the vector space is defined over the real field $\mathbb{R}$, is specified by both position x and momentum p. In contrast, the state of a particle in quantum mechanics, where the vector space is defined over the complex field $\mathbb{C}$, depends on only one of the two variables, namely, either x or p [16]. The information on momentum p is contained in the wave function $\psi(x, t)$. A convenient way to extract this information on p as a Fourier transform of $\psi(x, t)$ is to construct the *momentum wave function* at any given time t:

$$\begin{aligned}\hat{\psi}(p, t) &\triangleq \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(\frac{-ipx}{\hbar}\right)\psi(x, t) \\ &= \frac{1}{\sqrt{h}}\int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(\frac{-i2\pi px}{h}\right)\psi(x, t)\end{aligned}$$

where the Planck constant $\hbar = 1.054572 \times 10^{-34}$ Joule s and the original Plank constant $h = 2\pi\hbar$.

The above Fourier transform can be inverted to generate the information on position x from $\hat{\psi}(p, t)$ at any given time t as

$$\begin{aligned}\psi(x, t) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \mathrm{d}p\, \exp\left(\frac{+ipx}{\hbar}\right)\hat{\psi}(p, t) \\ &= \frac{1}{\sqrt{h}}\int_{-\infty}^{\infty} \mathrm{d}p\, \exp\left(\frac{+i2\pi px}{h}\right)\hat{\psi}(p, t)\end{aligned}$$

Now let us fix the time parameter t and, with a slight abuse of notation, let us denote $\psi(x, t)$ and $\hat{\psi}(p, t)$ as $\psi(x)$ and $\hat{\psi}(p)$, respectively. Therefore, the position waveform $\psi(x)$ and the momentum waveform $\hat{\psi}(p)$ contain precisely the same information about a quantum state at a given time instant. By Plancherel theorem [1], in the inner product space L_2 defined over the complex field $\mathbb{C}$, the same information is obtained in either position domain or momentum domain as

$$\left(\langle \varphi, \psi \rangle = \langle \hat{\varphi}, \hat{\psi} \rangle\right) \Leftrightarrow \left(\int_{-\infty}^{\infty} \mathrm{d}x\, \overline{\psi}(x)\varphi(x) = \int_{-\infty}^{\infty} \mathrm{d}p\, \overline{\hat{\psi}}(p)\hat{\varphi}(p)\right) \tag{6}$$

In essence, the information about the momentum of a quantum particle can be obtained from the momentum wave function in the same way that information about its position can be obtained from the position wave function as explained later.

Let the position wave function have a compact support, i.e., $\psi(x)$ vanishes outside a finite interval $x_1 \le x \le x_2$; similarly, let the momentum wave function have a compact support, i.e., $\hat{\psi}(p)$ vanishes outside a finite interval $p_1 \le p \le p_2$. Therefore, it is safe to say that the quantum particle does not lie outside a position interval and a momentum interval. However, we may not be able to specify precise (i.e., deterministic) values of the position and momentum within these intervals. This issue is further examined in the next section from the perspectives of both quantum mechanics and signal processing.

3.1 Uncertainty Principle and Signal Analysis. In the discipline of signal processing, it is a well-known mathematical fact [2] that

> A narrow (time-domain) waveform yields a wide-band frequency spectrum and a wide (time-domain) waveform yields a narrow-band frequency spectrum.

It is possible that both time and frequency bands can be narrow, but the probability densities of time and frequency localization cannot be made simultaneously arbitrarily narrow, i.e., their variances cannot be made simultaneously arbitrarily small.

In both quantum mechanics and signal processing, the uncertainty principle applies to a pair of variables whose associated operators *do not* commute (i.e., their commutator is not identically equal to zero) [15,16].

In the physics literature,

$$\text{Position operator:} \quad \chi \equiv (x)$$

$$\text{Momentum operator:} \quad \mathcal{P} \equiv \left(-i\hbar \frac{\partial}{\partial x}\right) \text{or} \left(-i\frac{h}{2\pi}\frac{\partial}{\partial x}\right)$$

The commutator of χ and $\mathcal{P}$ on an arbitrary C^1 (i.e., continuously differentiable) test function $f(x)$ is defined as

$$[\chi, \mathcal{P}] \triangleq \chi\mathcal{P} - \mathcal{P}\chi$$

$$\Rightarrow [\chi, \mathcal{P}]f(x) = x\left(-i\hbar\frac{\partial}{\partial x}\right)f(x) + i\hbar\frac{\partial}{\partial x}(xf(x))$$

$$= -i\hbar\left(x\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial x} - f(x)\right) = i\hbar f(x)$$

$$\Rightarrow [\chi, \mathcal{P}] = i\hbar \quad \text{or} \quad [\chi, \mathcal{P}] = i\frac{h}{2\pi}$$

Similarly, in the signal processing literature,

$$\text{Time operator:} \quad \mathfrak{I} \equiv (t) \triangleq \left(\frac{i}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)$$

$$\text{Frequency operator:} \quad \Xi \equiv (\xi) \triangleq \left(\frac{-i}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\right)$$

The commutator of $\mathfrak{I}$ and Ξ on an arbitrary C^1 (i.e., continuously differentiable) signal $s(t)$ is defined as

$$[\mathfrak{I},\ \Xi] \triangleq \mathfrak{I}\ \Xi - \Xi\ \mathfrak{I}$$

$$\Rightarrow [\mathfrak{I},\ \Xi]\ s(t) = (t)\left(\frac{-i}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\right)s(t) - \left(\frac{-i}{2\pi}\right)\frac{\mathrm{d}}{\mathrm{d}t}(ts(t))$$

$$= \frac{-i}{2\pi}\left(t\frac{\mathrm{d}s}{\mathrm{d}t} - t\frac{\mathrm{d}s}{\mathrm{d}t} - s(t)\right) = \frac{i}{2\pi}s(t)$$

$$\Rightarrow [\mathfrak{I},\ \Xi] = \frac{i}{2\pi}$$

Defining the commutator of $\mathfrak{I}$ and Ξ on the Fourier transform $\hat{s}(\xi)$ of the signal $s(t)$ yields the same result:

$$[\mathfrak{I},\ \Xi] \triangleq \mathfrak{I}\ \Xi - \Xi\ \mathfrak{I}$$

$$\Rightarrow [\mathfrak{I},\ \Xi]\ \hat{s}(\xi) = \left(\frac{i}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)(\xi)\hat{s}(\xi) - (\xi)\left(\frac{i}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\hat{s}(\xi)\right)$$

$$= \frac{i}{2\pi}\left(\hat{s}(\xi) + \xi\frac{\mathrm{d}\hat{s}}{\mathrm{d}\xi} - \xi\frac{\mathrm{d}\hat{s}}{\mathrm{d}\xi}\right) = \frac{i}{2\pi}\hat{s}(\xi)$$

$$\Rightarrow [\mathfrak{I},\ \Xi] = \frac{i}{2\pi}$$

The previous discussion evinces that the mathematical structure of the commutator of time and frequency operators is identical to that of the position and momentum operators in the discipline of quantum mechanics. This fact leads to the notion of simultaneous time-frequency localization in the next section.

3.2 Time-Frequency Localization for Signal Analysis. Let us consider a unit energy signal $s(t) \in L_2(\mathbb{R})$, i.e., $\int_{-\infty}^{\infty} \mathrm{d}t \mid s(t) \mid^2 = 1$ and $\int_{-\infty}^{\infty} \mathrm{d}\xi \mid \hat{s}(\xi) \mid^2 = 1$ by following Plancherel theorem [1]. In line with the concepts in quantum mechanics, $\mid s(t)\mid^2$ and $\mid \hat{s}(\xi) \mid^2$ are taken as the probability densities of time localization and frequency localization, respectively. By appropriate time translation and frequency modulation, it follows that

$$\int_{-\infty}^{\infty} \mathrm{d}t \mid s(t) \mid^2 t = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{d}\xi \mid \hat{s}(\xi) \mid^2 \xi = 0$$

without loss of generality. Let us define the so-called variances of time localization and frequency localization as

$$\sigma_t^2 \triangleq \int_{-\infty}^{\infty} \mathrm{d}t \mid s(t) \mid^2 t^2 \quad \text{and} \quad \sigma_\xi^2 \triangleq \int_{-\infty}^{\infty} \mathrm{d}\xi \mid \hat{s}(\xi) \mid^2 \xi^2$$

THEOREM 3.1 (Uncertainty of time-frequency localization). *If the signal $s(t) \in L_2(\mathbb{R})$ (equivalently, $s(t)$ decays to zero faster than $|t|^{-(1/2)}$ as $t \to \pm\infty$), then $\sigma_t\sigma_\xi \ge 1/4\pi$ and the equality holds for Gaussian signals $s(t) \triangleq \sqrt{\alpha/\pi}\exp(-\alpha t^2)$ for arbitrary $\alpha \in (0, \infty)$.*

Proof. By Cauchy–Schwarz inequality [1], it follows that

$$\left|\int_{-\infty}^{\infty} \mathrm{d}t \left(t\ s(t)\frac{\mathrm{d}s}{\mathrm{d}t}\right)\right|^2 \le \int_{-\infty}^{\infty} \mathrm{d}t\ |t\ s(t)|^2 \int_{-\infty}^{\infty} \mathrm{d}t \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|^2 \tag{7}$$

The first integral on the right-hand side of Eq. (7) is equal to σ_t^2. Taking Fourier transform of the signal $\mathcal{F}(\mathrm{d}s/\mathrm{d}t) = i2\pi\xi\hat{s}(\xi)$ and using Plancherel theorem (i.e., equality of signal energy in the time-domain and frequency domain as $\int_{-\infty}^{\infty} \mathrm{d}t|s(t)|^2 = \int_{-\infty}^{\infty} \mathrm{d}\xi|\hat{s}(\xi)|^2$), the second integral on the right-hand side of Eq. (7) is expressed as

$$\int_{-\infty}^{\infty} \mathrm{d}t\left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|^2 = \int_{-\infty}^{\infty} \mathrm{d}\xi\ |i2\pi\xi\hat{s}(\xi)|^2 = (2\pi)^2\sigma_\xi^2 \tag{8}$$

Therefore, $\left|\int_{-\infty}^{\infty} \mathrm{d}t\big(t\ s(t)(\mathrm{d}s/\mathrm{d}t)\big)\right| \le 2\pi\sigma_t\sigma_\xi$. Now, integration by parts yields

$$\int_{-\infty}^{\infty} \mathrm{d}t\ \left(t\ s(t)\frac{\mathrm{d}s}{\mathrm{d}t}\right) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\left(t\ \frac{\mathrm{d}|s(t)|^2}{\mathrm{d}t}\right)$$

$$= \frac{1}{2}t\ |s(t)|^2\Big|_{-\infty}^{\infty} - \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t|s(t)|^2 = 0 - \frac{1}{2} = -\frac{1}{2}$$

$$\Rightarrow \sigma_t\sigma_\xi \ge \frac{1}{4\pi} \tag{9}$$

The Cauchy–Schwarz inequality in Eq. (7) becomes an equality if $t\, s(t)$ and $\mathrm{d}s/\mathrm{d}t$ are collinear. For Gaussian signals,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\frac{\alpha}{\pi}}\exp(-\alpha t^2)\right)$$
$$= -2\alpha t\sqrt{\frac{\alpha}{\pi}}\exp(-\alpha t^2) = -2\alpha t s(t)$$

Hence, $\sigma_t \sigma_\xi = 1/4\pi$ for $s(t) = \sqrt{\alpha/\pi}\exp(-\alpha t^2)$ for arbitrary $\alpha \in (0, \infty)$. ∎

The results of Theorem 3.1 form a key concept for selection of filter parameters in time-frequency analyses, such as windowed Fourier transform and wavelet transform [4], of signals. Such problems are encountered in data-driven analysis of dynamical systems in the discipline of machine learning [17,18].

4 Summary and Conclusions

This letter is intended to clarify two ambiguities that are frequently encountered in decision and control of dynamical systems. The first ambiguity lies in the relationship between the dimension of the state space and the DOF for state-space modeling of dynamical systems. The second ambiguity is related to the uncertainty associated with time localization and frequency localization of signals for filter design in data-driven analysis of dynamical systems. These two ambiguities are addressed by applying elementary concepts of abstract algebra [12,13] and Heisenberg uncertainty principle in quantum mechanics [15,16], respectively.

A new definition is proposed to clarify the concept of DOF, which is consistent with the dimension of the state space when the dynamical system is modeled in the state-variable setting on a vector space defined over the complex field $\mathbb{C}$. In this definition, the dynamical system is not restricted to be linear time-invariant, i.e., the underlying state-space model could be nonlinear and/or time-varying. However, the proposed definition of DOF may not always be in agreement with the conventional notion of DOF, as reported in the literature [9]. Therefore, in this regard, the following topics of future work are recommended:

- Extension of the analysis on DOF to various dynamical systems (e.g., with holonomic constraints).
- Evaluation of the proposed definition of DOF (Definition 2.1) for acceptance by the scientific community in dynamical systems and control.

Acknowledgment

The reported work has been supported in part by U.S. Air Force Office of Scientific Research (AFOSR) under Grant No. FA9550-15-1-0400 in the area of dynamic data-driven application systems (DDDAS). Any opinions, findings, and conclusions in this paper are those of the author and do not necessarily reflect the views of the sponsoring agencies.

Conflict of Interest

There are no conflicts of interest.

References

[1] Naylor, A., and Sell, G., 1982, *Linear Operator Theory in Engineering and Science*, 2nd ed., Springer-Verlag, New York.
[2] Cohen, L., 1995, *Time-Frequency Analysis*, Prentice-Hall PTR, Upper Saddle River, NJ.
[3] Joshi, S., 1991, "Design of Optimal Second-Order State Estimators," J. Guid. Control, **14**(2), pp. 466–468.
[4] Rajagopalan, V., and Ray, A., 2006, "Symbolic Time Series Analysis Via Wavelet-Based Partitioning," Signal Process., **86**(11), pp. 3309–3320.
[5] Subbu, A., and Ray, A., 2008, "Space Partitioning Via Hilbert Transform for Symbolic Time Series Analysis," Appl. Phys. Lett., **92**(8), p. 084107 (3 pages).
[6] Mukherjee, K., and Ray, A., 2014, "State Splitting and Merging in Probabilistic Finite State Automata for Signal Representation and Analysis," Signal Process., **104**, pp. 105–119.
[7] Goldstein, H., Poole, C., and Safko, J., 2002, *Classical Mechanics*, 3rd ed., Addison Wesley, San Francisco, CA.
[8] Sinha, S., 2009, *Classical Mechanics*, Alpha Science, Oxford.
[9] Rao, S., 1990, *Mechanical Vibrations*, 2nd ed., Addison-Wesley, Reading, MA.
[10] Kelly, S., 2012, *Mechanical Vibrations: Theory and Applications*, Cengage Learning, Stamford, CT.
[11] Mori, Y., 2017, *Mechanical Vibrations: Applications to Equipment*, Wiley, London.
[12] Hoffman, K., and Kunze, R., 1971, *Linear Algebra*, 2nd ed., Springer-Verlag, Englewood Cliffs, NJ.
[13] Gill, A., 1976, *Applied Algebra for the Computer Sciences*, Prentice-Hall PTR, Upper Saddle River, NJ.
[14] Khalil, H., 2002, *Nonlinear Systems*, 3rd ed., Chapter 11, Prentice-Hall, Upper Saddle River, NJ.
[15] Griffiths, D., 2005, *Introduction to Quantum Mechanics*, 2nd ed., Pearson Prentice-Hall PTR, Upper Saddle River, NJ.
[16] Griffiths, R., 2002, *Consistent Quantum Theory*, Cambridge University Press, Cambridge.
[17] Bishop, C., 2007, *Pattern Recognition and Machine Learning*, Springer, New York.
[18] Murphy, K., 2012, *Machine Learning: A Probabilistic Perspective*, The MIT Press, Cambridge, MA.