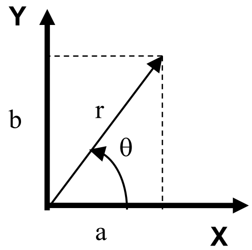
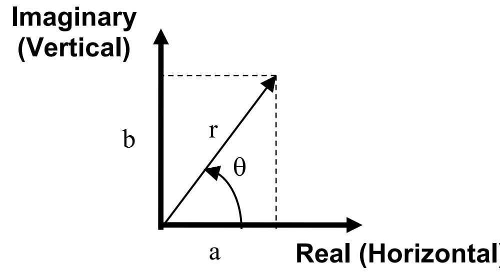
Complex Numbers for Planar Kinematics

Standard XY Notation



Complex Number Notation



$$\overline{R} = a\hat{i} + b\hat{j} = r\angle\theta \qquad \overline{R} = a + j\,b = r\,e^{j\theta}$$

$$r^2 = a^2 + b^2 \qquad e^{j\theta} = \cos\theta + j\sin\theta$$

$$\tan\theta = b/a \qquad j = \sqrt{-1}$$

Expanding: $\overline{R} = re^{j\theta} = r\cos\theta + j\,r\sin\theta \qquad a = r\cos\theta \qquad b = r\sin\theta$

r = magnitude, $e^{j\theta}$ = direction

Position: $\overline{R} = re^{j\theta}$

Velocity: $\dot{\overline{R}} = \dot{r}e^{j\theta} \quad + j\,r\,\dot{\theta}\,e^{j\theta} \qquad \dot{\theta} = \dfrac{d\theta}{dt} = \omega \qquad \ddot{\theta} = \dfrac{d^2\theta}{dt^2} = \alpha$

Extension Tangential

Acceleration: $\ddot{\overline{R}} = \ddot{r}\,e^{j\theta} \quad + j\,r\,\ddot{\theta}\,e^{j\theta} \quad - r\,\dot{\theta}^2\,e^{j\theta} \quad + j\,2\,\dot{r}\,\dot{\theta}\,e^{j\theta}$

Extension Tangential Normal Coriolis

Jerk: $\dddot{\overline{R}} = (\,\dddot{r} - 3\dot{r}\,\dot{\theta}^2 - 3r\,\dot{\theta}\,\ddot{\theta})e^{j\theta} \quad + (r\dddot{\theta} + 3\dot{r}\,\ddot{\theta} + 3\ddot{r}\,\dot{\theta} - r\,\dot{\theta}^3)\,j\,e^{j\theta}$

Snap:

$$\ddddot{\overline{R}} = (\,\ddddot{r} - 6\ddot{r}\,\dot{\theta}^2 - 12\dot{r}\,\dot{\theta}\,\ddot{\theta} - 4r\,\dot{\theta}\,\dddot{\theta} - 3r\,\ddot{\theta}^2 + r\,\dot{\theta}^4)e^{j\theta} + (r\ddddot{\theta} + 4\dot{r}\dddot{\theta} - 6r\,\dot{\theta}^2\ddot{\theta} + 6\ddot{r}\,\ddot{\theta} + 4\dddot{r}\,\dot{\theta} - 4\dot{r}\,\dot{\theta}^3)\,j\,e^{j\theta}$$

Note:

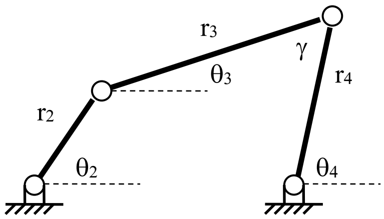
1. θ is always measured CCW from positive real axis. $\dot{\theta}$, $\ddot{\theta}$ and $\dddot{\theta}$ are positive CCW.
2. $j\,e^{j\theta} \perp e^{j\theta}$ (Try it using $\overline{R} = a + j\,b = r\,e^{j\theta}$)

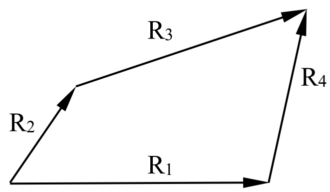
Matrix Solution: $[A]\{x\} = \{B\} \qquad \{x\}=[A]^{-1}\,\{B\}$

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad [A]^{-1} = \frac{1}{\det[A]}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \qquad \det[A] = a_{11}\,a_{22} - a_{12}\,a_{21}$$

Complex Number Analysis of Four Bar

Given: constants r_1 r_2 r_3 r_4 θ_1 and variable θ_2
Find: θ_3 and θ_4





Loop Equation: $\overline{R}_2 + \overline{R}_3 - \overline{R}_4 - \overline{R}_1 = 0$

Position: $r_2 e^{j\theta_2} + r_3 e^{j\theta_3} - r_4 e^{j\theta_4} - r_1 e^{j\theta_1} = 0$

Real Components: $r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_4 \cos\theta_4 - r_1 \cos\theta_1 = 0$

Imaginary Components: $j\, r_2 \sin\theta_2 + j\, r_3 \sin\theta_3 - j\, r_4 \sin\theta_4 - j\, r_1 \sin\theta_1 = 0$

<u>Use Newton-Raphson iterative solution for position.</u>

Given: Position solution and $\dot\theta_2$ and $\dot r_1 = \dot r_2 = \dot r_3 = \dot r_4 = \dot\theta_1 = 0$

Find: $\dot\theta_3$ and $\dot\theta_4$

Velocity: $j\, r_2\, \dot\theta_2\, e^{j\theta_2} + j\, r_3\, \dot\theta_3\, e^{j\theta_3} - j\, r_4\, \dot\theta_4\, e^{j\theta_4} = 0$

Real Components: $-r_2\dot\theta_2 \sin\theta_2 - r_3\dot\theta_3 \sin\theta_3 + r_4\dot\theta_4 \sin\theta_4 = 0$

Imaginary Components: $j\, r_2\, \dot\theta_2 \cos\theta_2 + j\, r_3\dot\theta_3 \cos\theta_3 - j\, r_4\dot\theta_4 \cos\theta_4 = 0$

Matrix Notation:
$$\begin{bmatrix} -r_3 \sin\theta_3 & r_4 \sin\theta_4 \\ r_3 \cos\theta_3 & -r_4 \cos\theta_4 \end{bmatrix} \begin{Bmatrix} \dot\theta_3 \\ \dot\theta_4 \end{Bmatrix} = \begin{Bmatrix} r_2\dot\theta_2 \sin\theta_2 \\ -r_2\dot\theta_2 \cos\theta_2 \end{Bmatrix}$$

$$\det[J] = r_3 r_4 (\sin\theta_3 \cos\theta_4 - \cos\theta_3 \sin\theta_4) = r_3 r_4 \sin(\theta_3 - \theta_4) = -r_3 r_4 \sin\gamma$$

$$\dot\theta_3 = -r_2\dot\theta_2 \sin(\theta_2 - \theta_4) / r_3 \sin(\theta_3 - \theta_4) \qquad \dot\theta_4 = -r_2\dot\theta_2 \sin(\theta_2 - \theta_3) / r_4 \sin(\theta_3 - \theta_4)$$

Given: Position and velocity solutions and $\ddot\theta_2$

Find: $\ddot\theta_3$ and $\ddot\theta_4$

Acceleration:
$$j\, r_2\ddot\theta_2 e^{j\theta_2} - r_2\dot\theta_2^2 e^{j\theta_2} + j\, r_3\ddot\theta_3 e^{j\theta_3} - r_3\dot\theta_3^2 e^{j\theta_3} - j\, r_4\ddot\theta_4 e^{j\theta_4} + r_4\dot\theta_4^2 e^{j\theta_4} = 0$$

Real: $-r_2\ddot\theta_2 \sin\theta_2 - r_2\dot\theta_2^2 \cos\theta_2 - r_3\ddot\theta_3 \sin\theta_3 - r_3\dot\theta_3^2 \cos\theta_3 + r_4\ddot\theta_4 \sin\theta_4 + r_4\dot\theta_4^2 \cos\theta_4 = 0$

Imaginary: $j\, r_2\ddot\theta_2 \cos\theta_2 - j\, r_2\dot\theta_2^2 \sin\theta_2 + j\, r_3\ddot\theta_3 \cos\theta_3 - j\, r_3\dot\theta_3^2 \sin\theta_3 - j\, r_4\ddot\theta_4 \cos\theta_4 + j\, r_4\dot\theta_4^2 \sin\theta_4 = 0$

Matrix:
$$\begin{bmatrix} -r_3 \sin\theta_3 & r_4 \sin\theta_4 \\ r_3 \cos\theta_3 & -r_4 \cos\theta_4 \end{bmatrix} \begin{Bmatrix} \ddot\theta_3 \\ \ddot\theta_4 \end{Bmatrix} = \begin{Bmatrix} r_2\ddot\theta_2 \sin\theta_2 + r_2\dot\theta_2^2 \cos\theta_2 + r_3\dot\theta_3^2 \cos\theta_3 - r_4\dot\theta_4^2 \cos\theta_4 \\ -r_2\ddot\theta_2 \cos\theta_2 + r_2\dot\theta_2^2 \sin\theta_2 + r_3\dot\theta_3^2 \sin\theta_3 - r_4\dot\theta_4^2 \sin\theta_4 \end{Bmatrix}$$

$$\ddot\theta_3 = \left(-r_2\ddot\theta_2 \sin(\theta_2 - \theta_4) - r_2\dot\theta_2^2 \cos(\theta_2 - \theta_4) - r_3\dot\theta_3^2 \cos(\theta_3 - \theta_4) + r_4\dot\theta_4^2\right) / r_3 \sin(\theta_3 - \theta_4)$$

$$\ddot\theta_4 = \left(-r_2\ddot\theta_2 \sin(\theta_2 - \theta_3) - r_2\dot\theta_2^2 \cos(\theta_2 - \theta_3) - r_3\dot\theta_3^2 + r_4\dot\theta_4^2 \cos(\theta_3 - \theta_4)\right)) / r_4 \sin(\theta_3 - \theta_4)$$

Given: Position, velocity and acceleration solutions and $\dddot{\theta}_2$

Find: $\dddot{\theta}_3$ and $\dddot{\theta}_4$

Jerk:

$$
\begin{aligned}
& j\,(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)e^{j\theta_2} - 3r_2\dot{\theta}_2\ddot{\theta}_2 e^{j\theta_2} + j\,(r_3\dddot{\theta}_3 - r_3\dot{\theta}_3^3)e^{j\theta_3} - 3r_3\dot{\theta}_3\ddot{\theta}_3 e^{j\theta_3} \\
& \qquad - j\,(r_4\dddot{\theta}_4 - r_4\dot{\theta}_4^3)e^{j\theta_4} + 3r_4\dot{\theta}_4\ddot{\theta}_4 e^{j\theta_4} = 0
\end{aligned}
$$

Real Components:

$$
\begin{aligned}
& -(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\sin\theta_2 - 3r_2\dot{\theta}_2\ddot{\theta}_2\cos\theta_2 - (r_3\dddot{\theta}_3 - r_3\dot{\theta}_3^3)\sin\theta_3 - 3r_3\dot{\theta}_3\ddot{\theta}_3\cos\theta_3 \\
& \qquad + (r_4\dddot{\theta}_4 - r_4\dot{\theta}_4^3)\sin\theta_4 + 3r_4\dot{\theta}_4\ddot{\theta}_4\cos\theta_4 = 0
\end{aligned}
$$

Imaginary Components:

$$
\begin{aligned}
& j\,(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\cos\theta_2 - j\,3r_2\dot{\theta}_2\ddot{\theta}_2\sin\theta_2 + j\,(r_3\dddot{\theta}_3 - r_3\dot{\theta}_3^3)\cos\theta_3 - j\,3r_3\dot{\theta}_3\ddot{\theta}_3\sin\theta_3 \\
& \qquad - j\,(r_4\dddot{\theta}_4 - r_4\dot{\theta}_4^3)\cos\theta_4 + j\,3r_4\dot{\theta}_4\ddot{\theta}_4\sin\theta_4 = 0
\end{aligned}
$$

Matrix Notation:

$$
\begin{bmatrix} -r_3\sin\theta_3 & r_4\sin\theta_4 \\ r_3\cos\theta_3 & -r_4\cos\theta_4 \end{bmatrix}
\begin{Bmatrix} \dddot{\theta}_3 \\ \dddot{\theta}_4 \end{Bmatrix} =
$$

$$
\begin{Bmatrix}
(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\sin\theta_2 + 3r_2\dot{\theta}_2\ddot{\theta}_2\cos\theta_2 - r_3\dot{\theta}_3^3\sin\theta_3 + 3r_3\dot{\theta}_3\ddot{\theta}_3\cos\theta_3 + r_4\dot{\theta}_4^3\sin\theta_4 - 3r_4\dot{\theta}_4\ddot{\theta}_4\cos\theta_4 \\
-(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\cos\theta_2 + 3r_2\dot{\theta}_2\ddot{\theta}_2\sin\theta_2 + r_3\dot{\theta}_3^3\cos\theta_3 + 3r_3\dot{\theta}_3\ddot{\theta}_3\sin\theta_3 - r_4\dot{\theta}_4^3\cos\theta_4 - 3r_4\dot{\theta}_4\ddot{\theta}_4\sin\theta_4
\end{Bmatrix}
$$

Given: Position, velocity, acceleration and jerk solutions and $\ddddot{\theta}_2$

Find: $\ddddot{\theta}_3$ and $\ddddot{\theta}_4$

Snap:

$$
\begin{aligned}
& + j\,r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\,\ddot{\theta}_2)e^{j\theta_2} - r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)e^{j\theta_2} \\
& + j\,r_3(\ddddot{\theta}_3 - 6\dot{\theta}_3^2\ddot{\theta}_3)e^{j\theta_3} - r_3(4\dot{\theta}_3\dddot{\theta}_3 + 3\ddot{\theta}_3^2 - \dot{\theta}_3^4)e^{j\theta_3} \\
& - j\,r_4(\ddddot{\theta}_4 - 6\dot{\theta}_4^2\ddot{\theta}_4)e^{j\theta_4} + r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)e^{j\theta_4} = 0
\end{aligned}
$$

Real Components:

$$
\begin{aligned}
& - r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\,\ddot{\theta}_2)\sin\theta_2 - r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)\cos\theta_2 \\
& - r_3(\ddddot{\theta}_3 - 6\dot{\theta}_3^2\ddot{\theta}_3)\sin\theta_3 - r_3(4\dot{\theta}_3\dddot{\theta}_3 + 3\ddot{\theta}_3^2 - \dot{\theta}_3^4)\cos\theta_3 \\
& + r_4(\ddddot{\theta}_4 - 6\dot{\theta}_4^2\ddot{\theta}_4)\sin\theta_4 + r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)\cos\theta_4 = 0
\end{aligned}
$$

Imaginary Components:

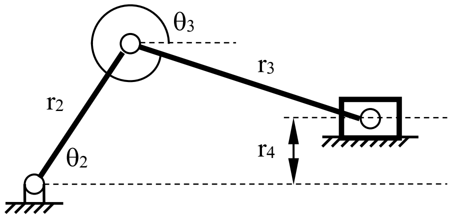
$$
\begin{aligned}
& + j\,r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\,\ddot{\theta}_2)\cos\theta_2 - j\,r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)\sin\theta_2 \\
& + j\,r_3(\ddddot{\theta}_3 - 6\dot{\theta}_3^2\ddot{\theta}_3)\cos\theta_3 - j\,r_3(4\dot{\theta}_3\dddot{\theta}_3 + 3\ddot{\theta}_3^2 - \dot{\theta}_3^4)\sin\theta_3 \\
& - j\,r_4(\ddddot{\theta}_4 - 6\dot{\theta}_4^2\ddot{\theta}_4)\cos\theta_4 + j\,r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)\sin\theta_4 = 0
\end{aligned}
$$

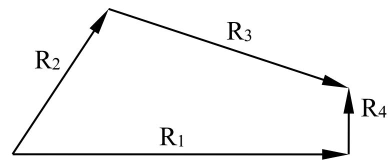
Matrix Notation:

$$
\begin{bmatrix} -r_3 \sin\theta_3 & r_4 \sin\theta_4 \\ r_3 \cos\theta_3 & -r_4 \cos\theta_4 \end{bmatrix}
\begin{Bmatrix} \ddddot{\theta}_3 \\ \ddddot{\theta}_4 \end{Bmatrix}
=
\begin{Bmatrix}
\begin{array}{l}
+r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\,\ddot{\theta}_2)\sin\theta_2 + r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)\cos\theta_2 \\
-6r_3\dot{\theta}_3^2\,\ddot{\theta}_3 \sin\theta_3 + r_3(4\dot{\theta}_3\dddot{\theta}_3 + 3\ddot{\theta}_3^2 - \dot{\theta}_3^4)\cos\theta_3 \\
+6r_4\dot{\theta}_4^2\,\ddot{\theta}_4 \sin\theta_4 - r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)\cos\theta_4 \\
\\
-r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\,\ddot{\theta}_2)\cos\theta_2 + r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)\sin\theta_2 \\
+6r_3\dot{\theta}_3^2\ddot{\theta}_3 \cos\theta_3 + r_3(4\dot{\theta}_3\dddot{\theta}_3 + 3\ddot{\theta}_3^2 - \dot{\theta}_3^4)\sin\theta_3 \\
-6r_4\dot{\theta}_4^2\ddot{\theta}_4 \cos\theta_4 - r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)\sin\theta_4
\end{array}
\end{Bmatrix}
$$

Complex Number Analysis of In-Line and Offset Slider Crank

Given: constants r_2 r_3 r_4 θ_1 ($\theta_4 = \theta_1 + 90°$) and variable θ_2
Find: θ_3 and r_1





Loop Equation: $\overline{R}_2 + \overline{R}_3 - \overline{R}_4 - \overline{R}_1 = 0$

Position: $r_2 e^{j\theta_2} + r_3 e^{j\theta_3} - r_4 e^{j\theta_4} - r_1 e^{j\theta_1} = 0$

Real Components: $r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_4 \cos\theta_4 - r_1 \cos\theta_1 = 0$

Imaginary Components: $j\, r_2 \sin\theta_2 + j\, r_3 \sin\theta_3 - j\, r_4 \sin\theta_4 - j\, r_1 \sin\theta_1 = 0$

<u>Use Newton-Raphson iterative solution for position</u>.

Given: Position solution and $\dot\theta_2$ and $\dot r_2 = \dot r_3 = \dot r_4 = \dot\theta_1 = \dot\theta_4 = 0$

Find: $\dot\theta_3$ and $\dot r_1$

Velocity: $j\, r_2\, \dot\theta_2\, e^{j\theta_2} + j\, r_3\, \dot\theta_3\, e^{j\theta_3} - \dot r_1\, e^{j\theta_1} = 0$

Real Components: $-r_2\dot\theta_2 \sin\theta_2 - r_3\dot\theta_3 \sin\theta_3 - \dot r_1 \cos\theta_1 = 0$

Imaginary Components: $j\, r_2\dot\theta_2 \cos\theta_2 + j\, r_3\dot\theta_3 \cos\theta_3 - j\, \dot r_1 \sin\theta_1 = 0$

Matrix Notation:
$$\begin{bmatrix} -r_3 \sin\theta_3 & -\cos\theta_1 \\ r_3 \cos\theta_3 & -\sin\theta_1 \end{bmatrix} \begin{Bmatrix} \dot\theta_3 \\ \dot r_1 \end{Bmatrix} = \begin{Bmatrix} r_2\dot\theta_2 \sin\theta_2 \\ -r_2\dot\theta_2 \cos\theta_2 \end{Bmatrix}$$

$$\det[J] = r_3(\sin\theta_3 \sin\theta_i + \cos\theta_3 \cos\theta_1) = r_3 \cos(\theta_3 - \theta_1)$$

$$\dot\theta_3 = -r_2\dot\theta_2 \cos(\theta_2 - \theta_1) / r_3 \cos(\theta_3 - \theta_1) \qquad \dot r_1 = -r_2\dot\theta_2 \sin(\theta_2 - \theta_3) / \cos(\theta_3 - \theta_1)$$

Given: Position and velocity solutions and $\ddot\theta_2$

Find: $\ddot\theta_3$ and $\ddot r_1$

Acceleration:
$$j\, r_2\, \ddot\theta_2\, e^{j\theta_2} - r_2\, \dot\theta_2^2\, e^{j\theta_2} + j\, r_3\, \ddot\theta_3\, e^{j\theta_3} - r_3\, \dot\theta_3^2\, e^{j\theta_3} - \ddot r_1\, e^{j\theta_1} = 0$$

Real: $-r_2\ddot\theta_2 \sin\theta_2 - r_2\dot\theta_2^2 \cos\theta_2 - r_3\ddot\theta_3 \sin\theta_3 - r_3\dot\theta_3^2 \cos\theta_3 - \ddot r_1 \cos\theta_1 = 0$

Imaginary: $j\, r_2\ddot\theta_2 \cos\theta_2 - j\, r_2\dot\theta_2^2 \sin\theta_2 + j\, r_3\ddot\theta_3 \cos\theta_3 - j\, r_3\dot\theta_3^2 \sin\theta_3 - j\, \ddot r_1 \sin\theta_1 = 0$

Matrix Notation:
$$\begin{bmatrix} -r_3 \sin\theta_3 & -\cos\theta_1 \\ r_3 \cos\theta_3 & -\sin\theta_1 \end{bmatrix} \begin{Bmatrix} \ddot\theta_3 \\ \ddot r_1 \end{Bmatrix} = \begin{Bmatrix} r_2\ddot\theta_2 \sin\theta_2 + r_2\dot\theta_2^2 \cos\theta_2 + r_3\dot\theta_3^2 \cos\theta_3 \\ -r_2\ddot\theta_2 \cos\theta_2 + r_2\dot\theta_2^2 \sin\theta_2 + r_3\dot\theta_3^2 \sin\theta_3 \end{Bmatrix}$$

$$\ddot\theta_3 = \left(-r_2\ddot\theta_2 \cos(\theta_2 - \theta_1) + r_2\dot\theta_2^2 \sin(\theta_2 - \theta_1) + r_3\dot\theta_3^2 \sin(\theta_3 - \theta_1)\right) / r_3 \cos(\theta_3 - \theta_1)$$

$$\ddot r_1 = \left(r_2\ddot\theta_2 \sin(\theta_3 - \theta_2) - r_2\dot\theta_2^2 \cos(\theta_3 - \theta_2) - r_3\dot\theta_3^2\right) / \cos(\theta_3 - \theta_1)$$

Given: Position and velocity solutions and $\dddot{\theta}_2$

Find: $\dddot{\theta}_3$ and $\dddot{r}_1$

Jerk:

$$j\,(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)e^{j\theta_2} - 3r_2\dot{\theta}_2\ddot{\theta}_2 e^{j\theta_2} + j\,(r_3\dddot{\theta}_3 - r_3\dot{\theta}_3^3)e^{j\theta_3} - 3r_3\dot{\theta}_3\ddot{\theta}_3 e^{j\theta_3} - \dddot{r}_1\, e^{j\theta_1} = 0$$

Real Components:

$$-(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\sin\theta_2 - 3r_2\dot{\theta}_2\ddot{\theta}_2\cos\theta_2 - (r_3\dddot{\theta}_3 - r_3\dot{\theta}_3^3)\sin\theta_3 - 3r_3\dot{\theta}_3\ddot{\theta}_3\cos\theta_3 - \dddot{r}_1\cos\theta_1 = 0$$

Imaginary Components:

$$j\,(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\cos\theta_2 - j\,3r_2\dot{\theta}_2\ddot{\theta}_2\sin\theta_2 + j\,(r_3\dddot{\theta}_3 - r_3\dot{\theta}_3^3)\cos\theta_3 - j\,3r_3\dot{\theta}_3\ddot{\theta}_3\sin\theta_3 - j\,\dddot{r}_1\sin\theta_1 = 0$$

Matrix Notation:

$$\begin{bmatrix} -r_3\sin\theta_3 & -\cos\theta_1 \\ r_3\cos\theta_3 & -\sin\theta_1 \end{bmatrix}\begin{Bmatrix} \dddot{\theta}_3 \\ \dddot{r}_1 \end{Bmatrix} = \begin{Bmatrix} (r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\sin\theta_2 + 3r_2\dot{\theta}_2\ddot{\theta}_2\cos\theta_2 - r_3\dot{\theta}_3^3\sin\theta_3 + 3r_3\dot{\theta}_3\ddot{\theta}_3\cos\theta_3 \\ -(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\cos\theta_2 + 3r_2\dot{\theta}_2\ddot{\theta}_2\sin\theta_2 + r_3\dot{\theta}_3^3\cos\theta_3 + 3r_3\dot{\theta}_3\ddot{\theta}_3\sin\theta_3 \end{Bmatrix}$$

Given: Position and velocity solutions and $\ddddot{\theta}_2$

Find: $\ddddot{\theta}_3$ and $\ddddot{r}_1$

Snap:

$$+\,j\,(r_2\ddddot{\theta}_2 - 6r_2\dot{\theta}_2^2\ddot{\theta}_2)e^{j\theta_2} - (4r_2\dddot{\theta}_2\dot{\theta}_2 + 3r_2\ddot{\theta}_2^2 - r_2\dot{\theta}_2^4)e^{j\theta_2}$$

$$+\,j\,(r_3\ddddot{\theta}_3 - 6r_3\dot{\theta}_3^2\ddot{\theta}_3)e^{j\theta_3} - (4r_3\dddot{\theta}_3\dot{\theta}_3 + 3r_3\ddot{\theta}_3^2 - r_3\dot{\theta}_3^4)e^{j\theta_3} - \ddddot{r}_1\, e^{j\theta_1} = 0$$

Real Components:

$$-(r_2\ddddot{\theta}_2 - 6r_2\dot{\theta}_2^2\ddot{\theta}_2)\sin\theta_2 - (4r_2\dddot{\theta}_2\dot{\theta}_2 + 3r_2\ddot{\theta}_2^2 - r_2\dot{\theta}_2^4)\cos\theta_2$$

$$-(r_3\ddddot{\theta}_3 - 6r_3\dot{\theta}_3^2\ddot{\theta}_3)\sin\theta_3 - (4r_3\dddot{\theta}_3\dot{\theta}_3 + 3r_3\ddot{\theta}_3^2 - r_3\dot{\theta}_3^4)\cos\theta_3 - \ddddot{r}_1\, e^{j\theta_1}\cos\theta_1 = 0$$

Imaginary Components:

$$+\,j\,(r_2\ddddot{\theta}_2 - 6r_2\dot{\theta}_2^2\ddot{\theta}_2)\cos\theta_2 - j\,(4r_2\dddot{\theta}_2\dot{\theta}_2 + 3r_2\ddot{\theta}_2^2 - r_2\dot{\theta}_2^4)\sin\theta_2$$

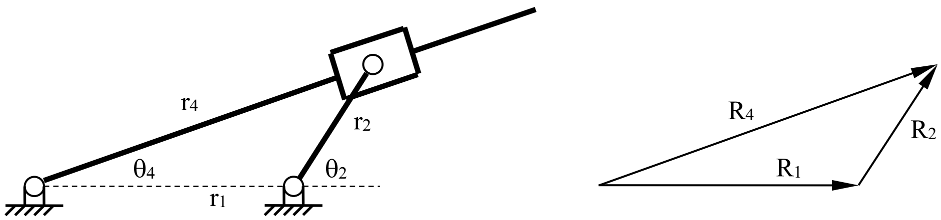
$$+\,j\,(r_3\ddddot{\theta}_3 - 6r_3\dot{\theta}_3^2\ddot{\theta}_3)\cos\theta_3 - j\,(4r_3\dddot{\theta}_3\dot{\theta}_3 + 3r_3\ddot{\theta}_3^2 - r_3\dot{\theta}_3^4)\sin\theta_3 - j\,\ddddot{r}_1\, e^{j\theta_1}\sin\theta_1 = 0$$

Matrix Notation:

$$\begin{bmatrix} -r_3\sin\theta_3 & -\cos\theta_1 \\ r_3\cos\theta_3 & -\sin\theta_1 \end{bmatrix}\begin{Bmatrix} \ddddot{\theta}_3 \\ \ddddot{r}_1 \end{Bmatrix} = \begin{Bmatrix} +(r_2\ddddot{\theta}_2 - 6r_2\dot{\theta}_2^2\ddot{\theta}_2)\sin\theta_2 + (4r_2\dddot{\theta}_2\dot{\theta}_2 + 3r_2\ddot{\theta}_2^2 - r_2\dot{\theta}_2^4)\cos\theta_2 \\ -6r_3\dot{\theta}_3^2\ddot{\theta}_3\sin\theta_3 + (4r_3\dddot{\theta}_3\dot{\theta}_3 + 3r_3\ddot{\theta}_3^2 - r_3\dot{\theta}_3^4)\cos\theta_3 \\ \\ -(r_2\ddddot{\theta}_2 - 6r_2\dot{\theta}_2^2\ddot{\theta}_2)\cos\theta_2 + (4r_2\dddot{\theta}_2\dot{\theta}_2 + 3r_2\ddot{\theta}_2^2 - r_2\dot{\theta}_2^4)\sin\theta_2 \\ +6r_3\dot{\theta}_3^2\ddot{\theta}_3\cos\theta_3 + (4r_3\dddot{\theta}_3\dot{\theta}_3 + 3r_3\ddot{\theta}_3^2 - r_3\dot{\theta}_3^4)\sin\theta_3 \end{Bmatrix}$$

Complex Number Analysis of Inverted Slider Crank

Given: constants r_1 r_2 θ_1 and variable θ_2
Find: r_4 and θ_4



Loop Equation: $\overline{R}_2 - \overline{R}_4 + \overline{R}_1 = 0$

Position: $r_2 e^{j\theta_2} - r_4 e^{j\theta_4} + r_1 e^{j\theta_1} = 0$

Real Components: $r_2 \cos\theta_2 - r_4 \cos\theta_4 + r_1 \cos\theta_1 = 0$

Imaginary Components: $j\, r_2 \sin\theta_2 - j\, r_4 \sin\theta_4 + j\, r_1 \sin\theta_1 = 0$

Use Newton-Raphson iterative solution for position.

Given: Position solution and $\dot{\theta}_2$ and $\dot{r}_1 = \dot{r}_2 = \dot{\theta}_1 = 0$
Find: $\dot{r}_4$ and $\dot{\theta}_4$

Velocity: $j\, r_2\, \dot{\theta}_2\, e^{j\theta_2} - \dot{r}_4\, e^{j\theta_4} - j\, r_4\, \dot{\theta}_4\, e^{j\theta_4} = 0$

Real Components: $-r_2\dot{\theta}_2 \sin\theta_2 - \dot{r}_4 \cos\theta_4 + r_4\dot{\theta}_4 \sin\theta_4 = 0$

Imaginary Components: $j\, r_2\dot{\theta}_2 \cos\theta_2 - j\, \dot{r}_4 \sin\theta_4 - j\, r_4\dot{\theta}_4 \cos\theta_4 = 0$

Matrix Notation:
$$\begin{bmatrix} -\cos\theta_4 & +r_4 \sin\theta_4 \\ -\sin\theta_4 & -r_4 \cos\theta_4 \end{bmatrix} \begin{Bmatrix} \dot{r}_4 \\ \dot{\theta}_4 \end{Bmatrix} = \begin{Bmatrix} r_2\dot{\theta}_2 \sin\theta_2 \\ -r_2\dot{\theta}_2 \cos\theta_2 \end{Bmatrix}$$

$\det[J] = r_4\left(\cos^2\theta_4 + \sin^2\theta_4\right) = r_4$

$\dot{r}_4 = -r_2\dot{\theta}_2 \sin(\theta_2 - \theta_4)$ $\qquad\qquad$ $\dot{\theta}_4 = r_2\dot{\theta}_2 \cos(\theta_2 - \theta_4)/r_4$

Given: Position and velocity solutions and $\ddot{\theta}_2$
Find: $\ddot{r}_4$ and $\ddot{\theta}_4$

Acceleration:
$$j\, r_2\, \ddot{\theta}_2\, e^{j\theta_2} - r_2\, \dot{\theta}_2^2\, e^{j\theta_2} - \ddot{r}_4\, e^{j\theta_4} - j\, r_4\, \ddot{\theta}_4\, e^{j\theta_4} + r_4\, \dot{\theta}_4^2\, e^{j\theta_4} - j\, 2\, \dot{r}_4\, \dot{\theta}_4\, e^{j\theta_4} = 0$$

Real Components:
$$-r_2\ddot{\theta}_2 \sin\theta_2 - r_2\dot{\theta}_2^2 \cos\theta_2 - \ddot{r}_4 \cos\theta_4 + r_4\ddot{\theta}_4 \sin\theta_4 + r_4\dot{\theta}_4^2 \cos\theta_4 + 2\dot{r}_4\dot{\theta}_4 \sin\theta_4 = 0$$

Imaginary Components:
$$j\, r_2\ddot{\theta}_2 \cos\theta_2 - j\, r_2\dot{\theta}_2^2 \sin\theta_2 - j\, \ddot{r}_4 \sin\theta_4 - j\, r_4\ddot{\theta}_4 \cos\theta_4 + j\, r_4\dot{\theta}_4^2 \sin\theta_4 - j\, 2\dot{r}_4\dot{\theta}_4 \cos\theta_4 = 0$$

Matrix Notation:
$$\begin{bmatrix} -\cos\theta_4 & +r_4 \sin\theta_4 \\ -\sin\theta_4 & -r_4 \cos\theta_4 \end{bmatrix} \begin{Bmatrix} \ddot{r}_4 \\ \ddot{\theta}_4 \end{Bmatrix} = \begin{Bmatrix} r_2\ddot{\theta}_2 \sin\theta_2 + r_2\dot{\theta}_2^2 \cos\theta_2 - r_4\dot{\theta}_4^2 \cos\theta_4 - 2\dot{r}_4\dot{\theta}_4 \sin\theta_4 \\ -r_2\ddot{\theta}_2 \cos\theta_2 + r_2\dot{\theta}_2^2 \sin\theta_2 - r_4\dot{\theta}_4^2 \sin\theta_4 + 2\dot{r}_4\dot{\theta}_4 \cos\theta_4 \end{Bmatrix}$$

$\ddot{r}_4 = -r_2\ddot{\theta}_2 \sin(\theta_2 - \theta_4) - r_2\dot{\theta}_2^2 \cos(\theta_2 - \theta_4) + r_4\dot{\theta}_4^2$

$\ddot{\theta}_4 = \left(r_2\ddot{\theta}_2 \cos(\theta_2 - \theta_4) - r_2\dot{\theta}_2^2 \sin(\theta_2 - \theta_4) - 2\dot{r}_4\dot{\theta}_4\right)/r_4$

Given: Position, velocity and acceleration solutions and $\dddot{\theta}_2$

Find: $\dddot{r}_4$ and $\dddot{\theta}_4$

Jerk:

$$j\,(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)e^{j\theta_2} - 3r_2\dot{\theta}_2\ddot{\theta}_2 e^{j\theta_2} - (\dddot{r}_4 - 3\dot{r}_4\dot{\theta}_4^2)e^{j\theta_4} - j\,(3\ddot{r}_4\dot{\theta}_4 + 3\dot{r}_4\ddot{\theta}_4)e^{j\theta_4}$$
$$- j\,(r_4\dddot{\theta}_4 - r_4\dot{\theta}_4^3)e^{j\theta_4} + 3r_4\dot{\theta}_4\ddot{\theta}_4 e^{j\theta_4} = 0$$

Real Components:

$$-(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\sin\theta_2 - 3r_2\dot{\theta}_2\ddot{\theta}_2\cos\theta_2 - (\dddot{r}_4 - 3\dot{r}_4\dot{\theta}_4^2)\cos\theta_4 + (3\ddot{r}_4\dot{\theta}_4 + 3\dot{r}_4\ddot{\theta}_4)\sin\theta_4$$
$$+(r_4\dddot{\theta}_4 - r_4\dot{\theta}_4^3)\sin\theta_4 + 3r_4\dot{\theta}_4\ddot{\theta}_4\cos\theta_4 = 0$$

Imaginary Components:

$$j\,(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\cos\theta_2 - j\,3r_2\dot{\theta}_2\ddot{\theta}_2\sin\theta_2 - j\,(\dddot{r}_4 - 3\dot{r}_4\dot{\theta}_4^2)\sin\theta_4 - j\,(3\ddot{r}_4\dot{\theta}_4 + 3\dot{r}_4\ddot{\theta}_4)\cos\theta_4$$
$$- j\,(r_4\dddot{\theta}_4 - r_4\dot{\theta}_4^3)\cos\theta_4 + j\,3r_4\dot{\theta}_4\ddot{\theta}_4\sin\theta_4 = 0$$

Matrix Notation:

$$\begin{bmatrix} -\cos\theta_4 & +r_4\sin\theta_4 \\ -\sin\theta_4 & -r_4\cos\theta_4 \end{bmatrix}\begin{Bmatrix} \dddot{r}_4 \\ \dddot{\theta}_4 \end{Bmatrix} =$$
$$\begin{Bmatrix} (r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\sin\theta_2 + 3r_2\dot{\theta}_2\ddot{\theta}_2\cos\theta_2 - (3\ddot{r}_4\dot{\theta}_4 + 3\dot{r}_4\ddot{\theta}_4 - r_4\dot{\theta}_4^3)\sin\theta_4 - \left(3r_4\dot{\theta}_4\ddot{\theta}_4 + 3\dot{r}_4\dot{\theta}_4^2\right)\cos\theta_4 \\ -(r_2\dddot{\theta}_2 - r_2\dot{\theta}_2^3)\cos\theta_2 + 3r_2\dot{\theta}_2\ddot{\theta}_2\sin\theta_2 + (3\ddot{r}_4\dot{\theta}_4 + 3\dot{r}_4\ddot{\theta}_4 - r_4\dot{\theta}_4^3)\cos\theta_4 - \left(3r_4\dot{\theta}_4\ddot{\theta}_4 + 3\dot{r}_4\dot{\theta}_4^2\right)\sin\theta_4 \end{Bmatrix}$$

Given: Position, velocity, acceleration and jerk solutions and $\ddddot{\theta}_2$

Find: $\ddddot{\theta}_3$ and $\ddddot{\theta}_4$

Snap:

$$-r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)e^{j\theta_2} + j\,r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\ddot{\theta}_2)e^{j\theta_2}$$
$$-(\ddddot{r}_4 - 6\ddot{r}_4\dot{\theta}_4^2 - 12\dot{r}_4\dot{\theta}_4\ddot{\theta}_4)e^{j\theta_4} - j\,(4\dddot{r}_4\dot{\theta}_4 + 6\ddot{r}_4\ddot{\theta}_4 + 4\dot{r}_4\dddot{\theta}_4 - 4\dot{r}_4\dot{\theta}_4^3)e^{j\theta_4}$$
$$+ r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)e^{j\theta_4} - j\,r_4(\ddddot{\theta}_4 - 6\dot{\theta}_4^2\ddot{\theta}_4)e^{j\theta_4} = 0$$

Real Components:

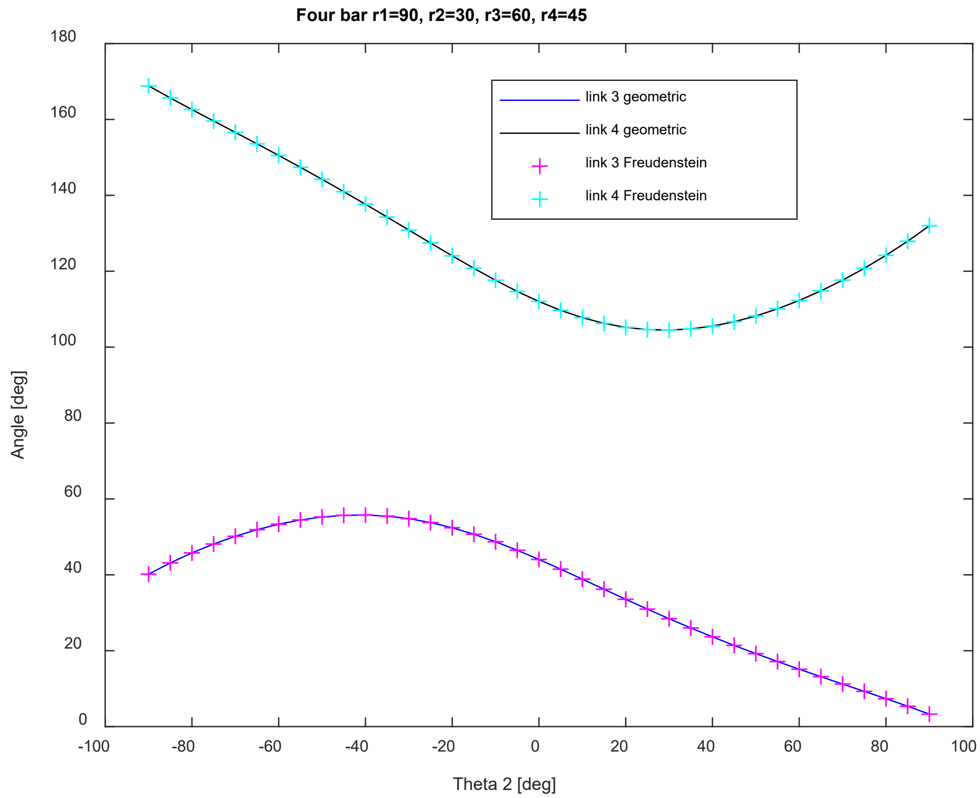
$$-r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)\cos\theta_2 - r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\ddot{\theta}_2)\sin\theta_2$$
$$-(\ddddot{r}_4 - 6\ddot{r}_4\dot{\theta}_4^2 - 12\dot{r}_4\dot{\theta}_4\ddot{\theta}_4)\cos\theta_4 + (4\dddot{r}_4\dot{\theta}_4 + 6\ddot{r}_4\ddot{\theta}_4 + 4\dot{r}_4\dddot{\theta}_4 - 4\dot{r}_4\dot{\theta}_4^3)\sin\theta_4$$
$$+ r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)\cos\theta_4 + r_4(\ddddot{\theta}_4 - 6\dot{\theta}_4^2\ddot{\theta}_4)\sin\theta_4 = 0$$

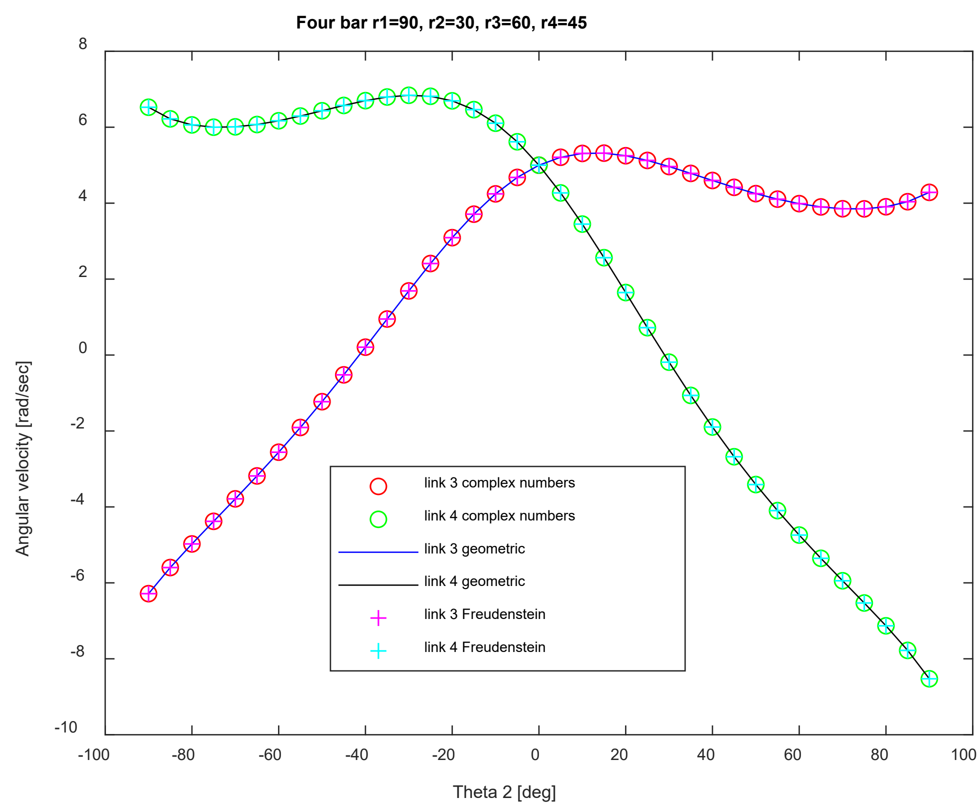
Imaginary Components:

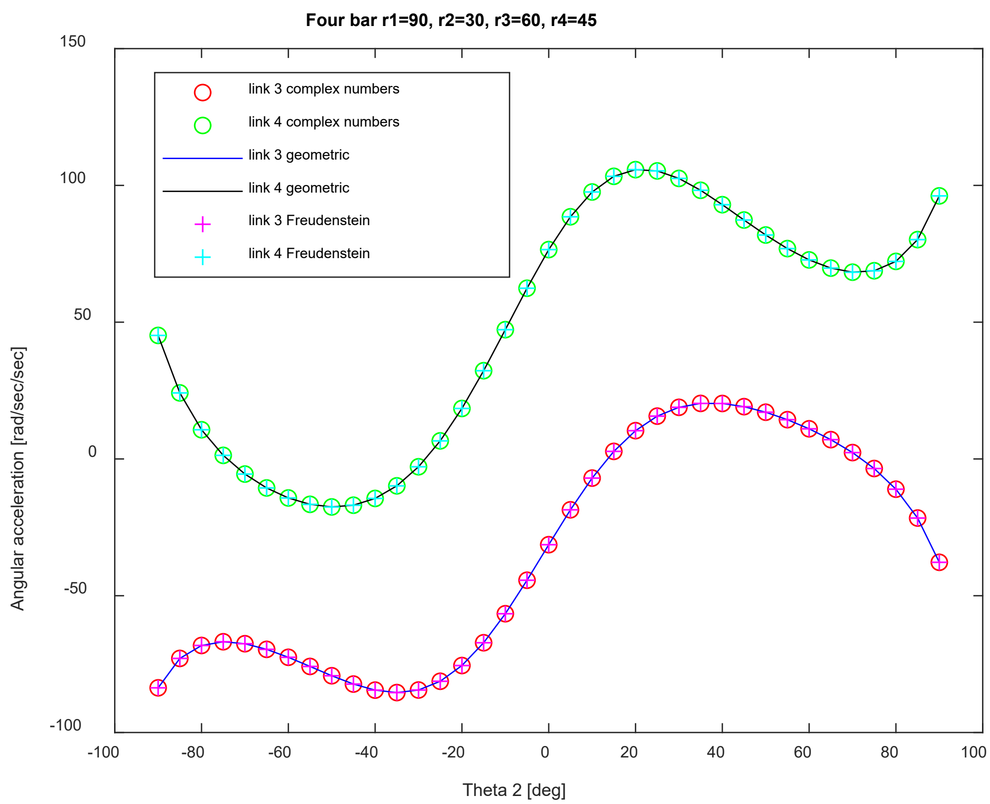
$$-j\,r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)\sin\theta_2 + j\,r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\ddot{\theta}_2)\cos\theta_2$$
$$-j\,(\ddddot{r}_4 - 6\ddot{r}_4\dot{\theta}_4^2 - 12\dot{r}_4\dot{\theta}_4\ddot{\theta}_4)\sin\theta_4 - j\,(4\dddot{r}_4\dot{\theta}_4 + 6\ddot{r}_4\ddot{\theta}_4 + 4\dot{r}_4\dddot{\theta}_4 - 4\dot{r}_4\dot{\theta}_4^3)\cos\theta_4$$
$$+ j\,r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)\sin\theta_4 - j\,r_4(\ddddot{\theta}_4 - 6\dot{\theta}_4^2\ddot{\theta}_4)\cos\theta_4 = 0$$

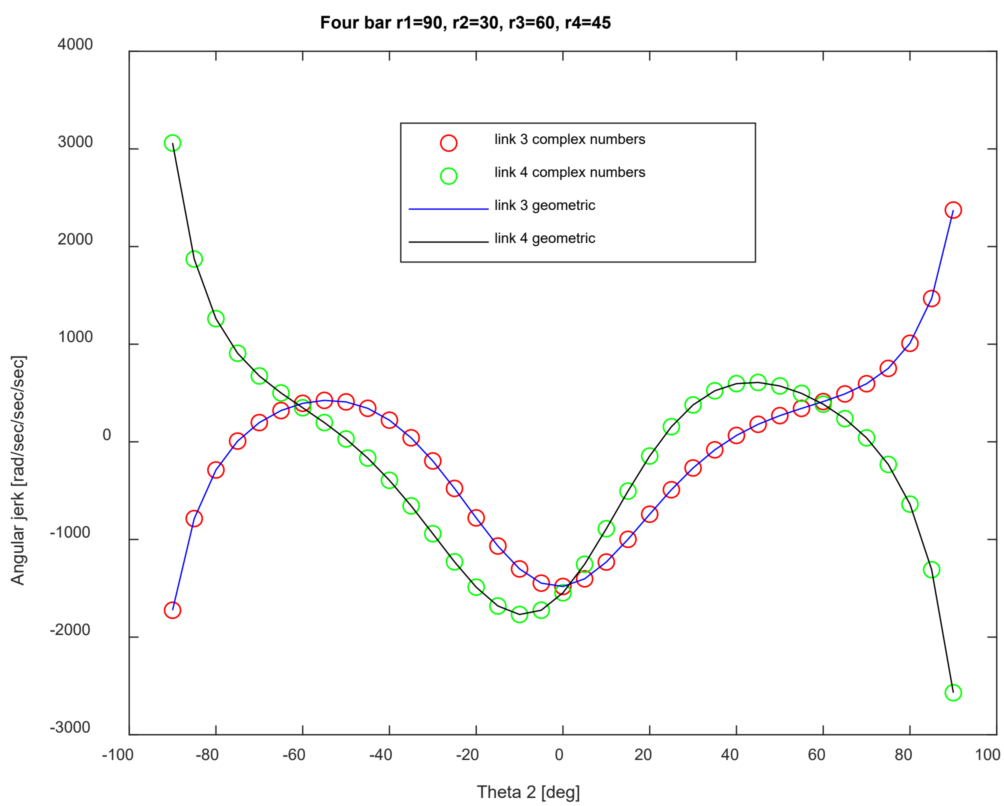
Matrix Notation:

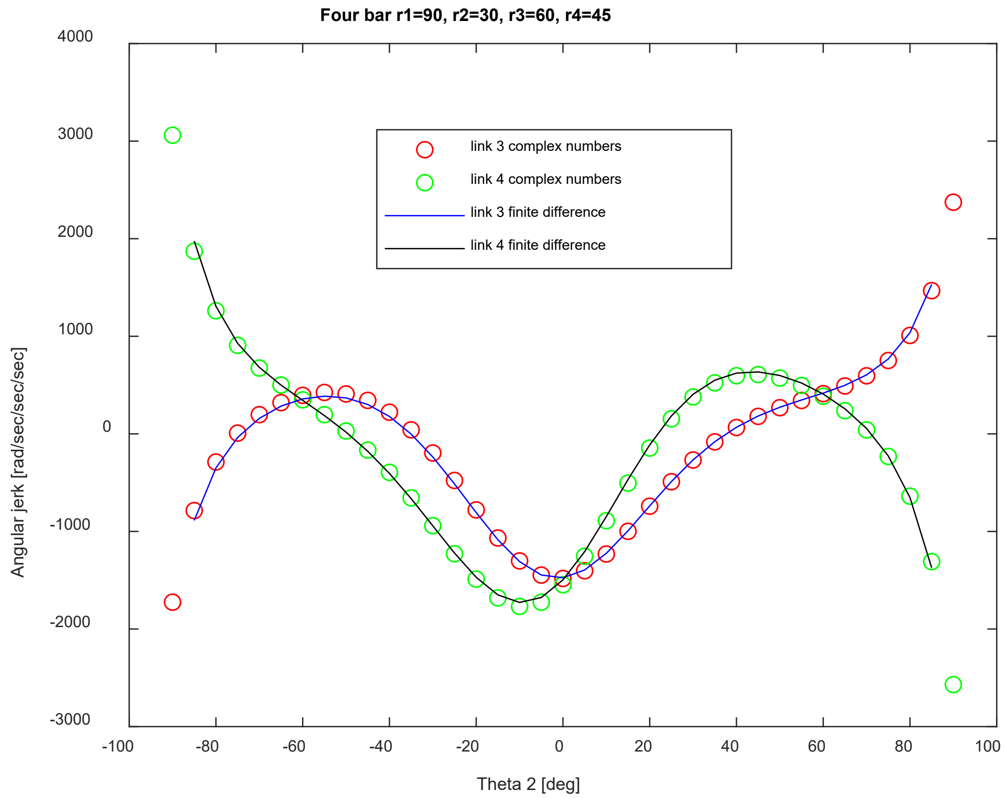
$$\begin{bmatrix} -\cos\theta_4 & +r_4\sin\theta_4 \\ -\sin\theta_4 & -r_4\cos\theta_4 \end{bmatrix} \begin{Bmatrix} \ddddot{r}_4 \\ \ddddot{\theta}_4 \end{Bmatrix} = \begin{Bmatrix} \begin{array}{l} +r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\ddot{\theta}_2)\sin\theta_2 + r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)\cos\theta_2 \\ -(4\dddot{r}_4\dot{\theta}_4 + 6\ddot{r}_4\ddot{\theta}_4 + 4\dot{r}_4\dddot{\theta}_4 - 4\dot{r}_4\dot{\theta}_4^3)\sin\theta_4 - (6\ddot{r}_4\dot{\theta}_4^2 + 12\dot{r}_4\dot{\theta}_4\ddot{\theta}_4)\cos\theta_4 \\ +6r_4\dot{\theta}_4^2\ddot{\theta}_4\sin\theta_4 - r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)\cos\theta_4 \\ \\ -r_2(\ddddot{\theta}_2 - 6\dot{\theta}_2^2\ddot{\theta}_2)\cos\theta_2 + r_2(4\dot{\theta}_2\dddot{\theta}_2 + 3\ddot{\theta}_2^2 - \dot{\theta}_2^4)\sin\theta_2 \\ +(4\dddot{r}_4\dot{\theta}_4 + 6\ddot{r}_4\ddot{\theta}_4 + 4\dot{r}_4\dddot{\theta}_4 - 4\dot{r}_4\dot{\theta}_4^3)\cos\theta_4 - (6\ddot{r}_4\dot{\theta}_4^2 + 12\dot{r}_4\dot{\theta}_4\ddot{\theta}_4)\sin\theta_4 \\ -6r_4\dot{\theta}_4^2\ddot{\theta}_4\cos\theta_4 - r_4(4\dot{\theta}_4\dddot{\theta}_4 + 3\ddot{\theta}_4^2 - \dot{\theta}_4^4)\sin\theta_4 \end{array} \end{Bmatrix}$$


Four bar r1=90, r2=30, r3=60, r4=45
link 3 geometric
link 4 geometric
link 3 Freudenstein
link 4 Freudenstein
Angle [deg]
Theta 2 [deg]


Four bar r1=90, r2=30, r3=60, r4=45
link 3 complex numbers
link 4 complex numbers
link 3 geometric
link 4 geometric
link 3 Freudenstein
link 4 Freudenstein
Angular velocity [rad/sec]
Theta 2 [deg]


Four bar r1=90, r2=30, r3=60, r4=45
link 3 complex numbers
link 4 complex numbers
link 3 geometric
link 4 geometric
link 3 Freudenstein
link 4 Freudenstein
Angular acceleration [rad/sec/sec]
Theta 2 [deg]
150
100
50
0
-50
-100
-100
-80
-60
-40
-20
0
20
40
60
80
100


Four bar r1=90, r2=30, r3=60, r4=45
link 3 complex numbers
link 4 complex numbers
link 3 geometric
link 4 geometric
Angular jerk [rad/sec/sec/sec]
Theta 2 [deg]
4000
3000
2000
1000
0
-1000
-2000
-3000
-100
-80
-60
-40
-20
0
20
40
60
80
100


Four bar r1=90, r2=30, r3=60, r4=45
link 3 complex numbers
link 4 complex numbers
link 3 finite difference
link 4 finite difference
Angular jerk [rad/sec/sec/sec]
Theta 2 [deg]
4000
3000
2000
1000
0
-1000
-2000
-3000
-100
-80
-60
-40
-20
0
20
40
60
80
100

```
% fourbar_kin.m - four bar kinematics
%   geometric, Freudenstein and complex number methods
% HJSIII, 20.04.28

% general constants
d2r = pi / 180;

% link lengths and coupler angle
r1 = 90;
r2 = 30;
r3 = 60;
r4 = 45;

% Freudenstein constants
K1 = r1 / r2;
K2 = r1 / r4;
K3 = (r2*r2 -r3*r3 +r4*r4 +r1*r1) /2 /r2 /r4;
K4 = r1 / r3;
K5 = (r4*r4 -r1*r1 -r2*r2 -r3*r3) /2 /r2 /r3;

% crank
w2 = -10;   % [rad/sec]
a2 = 2;     % [rad/sec/sec]
g2 = -0.5;  % [rad/sec/sec/sec]

% theta 2 at toggle positions +/- 109 deg
th2max = acos( (r1*r1 + r2*r2 - (r3+r4)*(r3+r4)) /2 /r1 /r2 );
th2max_deg = fix( th2max / d2r ) - 3;
th2min_deg = -th2max_deg;

% reduced motion
th2min_deg = -90;
th2max_deg = 90;
%th2min_deg = -109;
%th2max_deg = 109;

% allocate empty array to hold values
keep_g = [];
keep_m = [];
keep_f = [];

% crank angle
for th2_deg = th2min_deg : 5 : th2max_deg,
%for th2_deg = 65 : 65,
th2 = th2_deg * d2r;

% geometric position equations
  e = sqrt( r1*r1 + r2*r2 - 2*r1*r2*cos(th2) );
  alpha = asin( r2 * sin(th2) / e );
  gamma = acos( ( r3*r3 + r4*r4 - e*e ) /2 /r3 /r4 );
  beta = asin( r3 * sin(gamma) / e );
  th4 = pi - alpha - beta;
  th3 = th4 - gamma;

% Freudenstein position equations
  af = K1 + (K2-1)*cos(th2) - K3;
  bf = 2 * sin(th2);
  cf = -K1 +(K2+1)*cos(th2) - K3;
  u1 = ( -bf + sqrt( bf*bf - 4*af*cf ) ) /2 /af;
  u2 = ( -bf - sqrt( bf*bf - 4*af*cf ) ) /2 /af;
  th4_f1 = 2 * atan( u1 );
  th4_f2 = 2 * atan( u2 );

  ff = K1 - (K4+1)*cos(th2) - K5;
  gf = 2 * sin(th2);
  hf = -K1 -(K4-1)*cos(th2) - K5;
  v1 = ( -gf + sqrt( gf*gf - 4*ff*hf ) ) /2 /ff;
  v2 = ( -gf - sqrt( gf*gf - 4*ff*hf ) ) /2 /ff;
  th3_f1 = 2 * atan( v1 );
  th3_f2 = 2 * atan( v2 );

% matrix velocity equations
  JAC = [ -r3*sin(th3)   r4*sin(th4) ;
```

```
         r3*cos(th3)  -r4*cos(th4) ];

  vrhs = [  r2*w2*sin(th2) ;
           -r2*w2*cos(th2) ];

  vsol = inv(JAC) * vrhs;
  w3 = vsol(1);
  w4 = vsol(2);

% geometric velocity solutions
  th2d = w2;
  ed = r1*r2*th2d*sin(th2) / e;
  ad = ( r2*th2d*cos(th2) - ed*sin(alpha) ) /e /cos(alpha);
  gd = e*ed /r3 /r4 /sin(gamma);
  bd = ( r3*gd*cos(gamma) - ed*sin(beta) ) /e / cos(beta);
  th4d = -ad - bd;
  th3d = th4d - gd;

% Freudenstein velocity solution
  den3_f = K1*sin(th3) + sin(th2-th3);
  th3d_f = th2d * ( -K4*sin(th2) + sin(th2-th3) ) / den3_f;
  den4_f = K1*sin(th4) + sin(th2-th4);
  th4d_f = th2d * (  K2*sin(th2) + sin(th2-th4) ) / den4_f;

% matrix acceleration equations
  arhs = [  r2*a2*sin(th2)+r2*w2*w2*cos(th2)+r3*w3*w3*cos(th3)-r4*w4*w4*cos(th4) ;
           -r2*a2*cos(th2)+r2*w2*w2*sin(th2)+r3*w3*w3*sin(th3)-r4*w4*w4*sin(th4) ];

  asol = inv(JAC) * arhs;
  a3 = asol(1);
  a4 = asol(2);

% geometric acceleration solutions
  th2dd = a2;
  edd = ( r1*r2*(th2d*th2d*cos(th2) + th2dd*sin(th2)) - ed*ed ) / e;
  add = ( -r2*th2d*th2d*sin(th2) +r2*th2dd*cos(th2) - edd*sin(alpha) ...
          -2*ed*ad*cos(alpha) + e*ad*ad*sin(alpha) ) /e /cos(alpha);
  gdd = ( ed*ed +e*edd - r3*r4*gd*gd*cos(gamma) ) /r3 /r4 /sin(gamma);
  bdd = ( -r3*gd*gd*sin(gamma) +r3*gdd*cos(gamma) -edd*sin(beta) ...
          -2*ed*bd*cos(beta) +e*bd*bd*sin(beta) ) /e / cos(beta);
  th4dd = -add - bdd;
  th3dd = th4dd - gdd;

% Freudenstein accleration solution
  th3dd_f = ( -(K4*sin(th2)-sin(th2-th3))*th2dd -K1*th3d*th3d*cos(th3) ...
              -K4*th2d*th2d*cos(th2) +(th2d-th3d)*(th2d-th3d)*cos(th2-th3) ) / den3_f;
  th4dd_f = (  (K2*sin(th2)+sin(th2-th4))*th2dd -K1*th4d*th4d*cos(th4) ...
              +K2*th2d*th2d*cos(th2) +(th2d-th4d)*(th2d-th4d)*cos(th2-th4) ) / den4_f;

% matrix jerk equations
  jrhs1 =  (r2*g2-r2*w2*w2*w2)*sin(th2) +3*r2*w2*a2*cos(th2) ...
                -r3*w3*w3*w3*sin(th3)  +3*r3*w3*a3*cos(th3) ...
                +r4*w4*w4*w4*sin(th4)  -3*r4*w4*a4*cos(th4) ;

  jrhs2 = -(r2*g2-r2*w2*w2*w2)*cos(th2) +3*r2*w2*a2*sin(th2) ...
                +r3*w3*w3*w3*cos(th3)  +3*r3*w3*a3*sin(th3) ...
                -r4*w4*w4*w4*cos(th4)  -3*r4*w4*a4*sin(th4) ;
  jrhs = [ jrhs1 ; jrhs2 ];

  jsol = inv(JAC) * jrhs;
  g3 = jsol(1);
  g4 = jsol(2);

% geometric jerk solutions
  th2ddd = g2;
  eddd = ( r1*r2*(3*th2d*th2dd*cos(th2) -th2d*th2d*th2d*sin(th2) + th2ddd*sin(th2)) ...
          -3*ed*edd ) / e;
  addd = ( -3*r2*th2d*th2dd*sin(th2) -r2*th2d*th2d*th2d*cos(th2) +r2*th2ddd*cos(th2) ...
         +3*e*ad*add*sin(alpha) +e*ad*ad*ad*cos(alpha) ...
         -3*edd*ad*cos(alpha) -3*ed*add*cos(alpha) +3*ed*ad*ad*sin(alpha) ...
         -eddd*sin(alpha) ) /e /cos(alpha);
  gddd = ( 3*ed*edd +e*eddd - r3*r4*(3*gd*gdd*cos(gamma) -gd*gd*gd*sin(gamma)) ) ...
          /r3 /r4 /sin(gamma);
```

```
  bddd = ( -3*r3*gd*gdd*sin(gamma) -r3*gd*gd*gd*cos(gamma) +r3*gddd*cos(gamma) ...
       -eddd*sin(beta) -3*bd*edd*cos(beta) -3*ed*bdd*cos(beta) ...
       +3*ed*bd*bd*sin(beta) +3*e*bd*bdd*sin(beta) +e*bd*bd*bd*cos(beta) ) ...
       /e / cos(beta);
  th4ddd = -addd - bddd;
  th3ddd = th4ddd - gddd;

% save for plotting
  keep_g = [ keep_g ; th2  th3  th4  th3d  th4d  th3dd  th4dd  th3ddd  th4ddd ];
  keep_m = [ keep_m ; th2  th3  th4  w3    w4    a3     a4     g3      g4     ];
  keep_f = [ keep_f ; th3_f1  th3_f2  th4_f1  th4_f2  th3d_f  th4d_f  th3dd_f  th4dd_f ];
end

% create plotting vectors
th2_deg = keep_m(:,1) / d2r;
th3_deg = keep_m(:,2) / d2r;
th4_deg = keep_m(:,3) / d2r;

w3 = keep_m(:,4);
w4 = keep_m(:,5);
a3 = keep_m(:,6);
a4 = keep_m(:,7);
g3 = keep_m(:,8);
g4 = keep_m(:,9);

th3d = keep_g(:,4);
th4d = keep_g(:,5);
th3dd = keep_g(:,6);
th4dd = keep_g(:,7);
th3ddd = keep_g(:,8);
th4ddd = keep_g(:,9);

th3_deg_f1 = keep_f(:,1) / d2r;
th3_deg_f2 = keep_f(:,2) / d2r;
th4_deg_f1 = keep_f(:,3) / d2r;
th4_deg_f2 = keep_f(:,4) / d2r;
th3d_f     = keep_f(:,5);
th4d_f     = keep_f(:,6);
th3dd_f    = keep_f(:,7);
th4dd_f    = keep_f(:,8);

figure( 1 )
  clf
  plot( th2_deg,th3_deg,'b-', th2_deg,th4_deg,'k-' )
  xlabel( 'Theta 2 [deg]' )
  ylabel( 'Angle [deg]' )
  title( 'Four bar r1=90, r2=30, r3=60, r4=45' )
  hold on
  plot( th2_deg,th3_deg_f1,'m+', th2_deg,th4_deg_f1,'c+' )
  legend( 'link 3 geometric', 'link 4 geometric', 'link 3 Freudenstein', 'link 4 Freudenstein'  )

figure( 2 )
  clf
  plot( th2_deg,w3,'ro', th2_deg,w4,'go', th2_deg,th3d,'b-', th2_deg,th4d,'k-' )
  xlabel( 'Theta 2 [deg]' )
  ylabel( 'Angular velocity [rad/sec]' )
  legend( 'link 3 complex numbers', 'link 4 complex numbers', ...
          'link 3 geometric', 'link 4 geometric' )
  title( 'Four bar r1=90, r2=30, r3=60, r4=45' )
  hold on
  plot( th2_deg,th3d_f,'m+', th2_deg,th4d_f,'c+' )
  legend( 'link 3 complex numbers', 'link 4 complex numbers', ...
          'link 3 geometric', 'link 4 geometric', ...
          'link 3 Freudenstein', 'link 4 Freudenstein' )

figure( 3 )
  clf
  plot( th2_deg,a3,'ro', th2_deg,a4,'go',  th2_deg,th3dd,'b-', th2_deg,th4dd,'k-' )
  xlabel( 'Theta 2 [deg]' )
  ylabel( 'Angular acceleration [rad/sec/sec]' )
  title( 'Four bar r1=90, r2=30, r3=60, r4=45' )
  hold on
  plot( th2_deg,th3dd_f,'m+', th2_deg,th4dd_f,'c+' )
```

```
  legend( 'link 3 complex numbers', 'link 4 complex numbers', ...
          'link 3 geometric', 'link 4 geometric', ...
          'link 3 Freudenstein', 'link 4 Freudenstein' )

figure( 4 )
  clf
  plot( th2_deg,g3,'ro', th2_deg,g4,'go',  th2_deg,th3ddd,'b', th2_deg,th4ddd,'k' )
  xlabel( 'Theta 2 [deg]' )
  ylabel( 'Angular jerk [rad/sec/sec/sec]' )
  legend( 'link 3 complex numbers', 'link 4 complex numbers', ...
          'link 3 geometric', 'link 4 geometric' )
  title( 'Four bar r1=90, r2=30, r3=60, r4=45' )

% check by finite differences
figure( 5 )
  clf
  dt = ( 5 * d2r ) / w2;
  g3_fd = ( [ NaN ; diff(a3) ] + [ diff(a3) ; NaN ] ) /2 /dt;
  g4_fd = ( [ NaN ; diff(a4) ] + [ diff(a4) ; NaN ] ) /2 /dt;
  plot( th2_deg,g3,'ro', th2_deg,g4,'go',  th2_deg,g3_fd,'b', th2_deg,g4_fd,'k' )
  xlabel( 'Theta 2 [deg]' )
  ylabel( 'Angular jerk [rad/sec/sec/sec]' )
  legend( 'link 3 complex numbers', 'link 4 complex numbers', ...
          'link 3 finite difference', 'link 4 finite difference' )
  title( 'Four bar r1=90, r2=30, r3=60, r4=45' )

% bottom - fourbar_kin.m
```