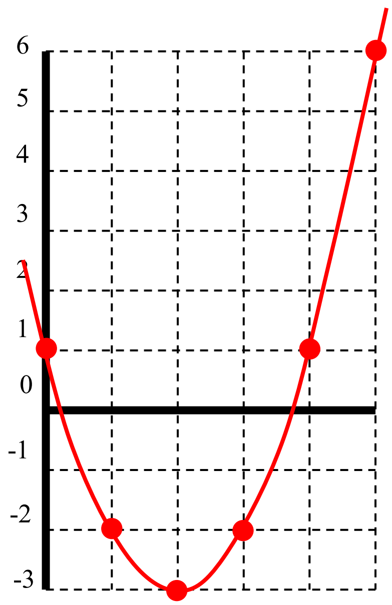
Newton-Raphson Algorithm

$$x_{k+1} = x_k - f_k / \left(\partial f / \partial x\right)_k$$

Example: $x^2 - 4x + 1 = 0$ **Use:** $f(x) = x^2 - 4x + 1$ $\partial f / \partial x = 2x - 4$

x	f(x)
0	1
1	-2
2	-3
3	-2
4	1
5	6



k	x	f(x)	∂f/∂x	f / (∂f/∂x)
1	5	6	6	1
2	4	1	4	0.25
3	3.75	0.0625	3.5	0.01786
4	3.7321	0.0003	3.4643	0.00009
5	3.7320			
1	2.5	-2.75	1	-2.75
2	5.25	7.5625	6.5	1.1635
3	4.0865	1.3536	4.1731	0.3244
4	3.7622	0.1052	3.5243	0.0299
5	3.7323			
1	1	-2	-2	1
2	0	1	-4	-0.25
3	0.25	0.0625	-3.5	-0.01786
4	0.2679	0.00017	-3.4642	-0.00005
5	0.2679			

Newton-Raphson for Four Bar

Given: constants r_1 r_2 r_3 r_4 θ_1 and variable θ_2 **Find:** θ_3 and θ_4

Subject to:
$$f_H = r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_4 \cos\theta_4 - r_1 \cos\theta_1 = 0$$
$$f_V = r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_4 \sin\theta_4 - r_1 \sin\theta_1 = 0$$

Use: $\{q\} = \begin{Bmatrix} \theta_3 \\ \theta_4 \end{Bmatrix}$ generalized coordinates, $\{\Phi\} = \begin{Bmatrix} f_H \\ f_V \end{Bmatrix}$ constraint functions

Taylor series about estimate {q}k:
$$\begin{Bmatrix} f_H \\ f_V \end{Bmatrix}_{k+1} \approx \begin{Bmatrix} f_H \\ f_V \end{Bmatrix}_k + \begin{bmatrix} \dfrac{\partial f_H}{\partial \theta_3} & \dfrac{\partial f_H}{\partial \theta_4} \\ \dfrac{\partial f_V}{\partial \theta_3} & \dfrac{\partial f_V}{\partial \theta_4} \end{bmatrix} \begin{Bmatrix} \Delta\theta_3 \\ \Delta\theta_4 \end{Bmatrix}$$

$$\{\Phi\}_{k+1} \approx \{\Phi\}_k + [\Phi_q]_k \{\Delta q\}_k \qquad [\Phi_q] = [\partial\Phi / \partial q] = \begin{bmatrix} \dfrac{\partial f_H}{\partial \theta_3} & \dfrac{\partial f_H}{\partial \theta_4} \\ \dfrac{\partial f_V}{\partial \theta_3} & \dfrac{\partial f_V}{\partial \theta_4} \end{bmatrix} = \begin{bmatrix} -r_3 \sin\theta_3 & r_4 \sin\theta_4 \\ r_3 \cos\theta_3 & -r_4 \cos\theta_4 \end{bmatrix}$$

Desired constraint functions: $\{\Phi\}_{k+1} = 0 \qquad \{\Delta q\} = -[\Phi_q]_k^{-1} \{\Phi\}_k$

Newton-Raphson equation: $\{q\}_{k+1} = \{q\}_k - [\Phi_q]_k^{-1} \{\Phi\}_k$

Example: r_1 = 90 cm, r_2 = 30 cm, r_3 = 60 cm, r_4 = 45 cm, θ_1 = 0°, θ_2 = 65°
(θ_3 = 13.151°, θ_4 = 114.827° by manual geometric solution)

k	$\{q\} = \begin{Bmatrix} \theta_3 \\ \theta_4 \end{Bmatrix}$ [deg]	$\{\Phi\} = \begin{Bmatrix} f_H \\ f_V \end{Bmatrix}$ [cm]	$[\partial\Phi / \partial q]$ [cm/rad]	$[\partial\Phi / \partial q]^{-1}\{\Phi\}$ [rad]	$[\partial\Phi / \partial q]^{-1}\{\Phi\}$ [deg]
1	0 90	-17.3215 -17.8108	0 45 60 0	-0.2968 -0.3849	-17.0080 -22.0544
2	17.0080 112.0544	-3.0488 3.0323	-17.5503 41.7073 57.3758 16.8969	0.0662 -0.0453	3.7916 -2.5927
3	13.2164 114.6471	-0.1444 0.0068	-13.7178 40.9002 58.4108 18.7663	0.0011 -0.0032	0.0647 -0.1806
4	13.1517 114.8277	-0.00013 0.00019	-13.6518 40.8409 58.4263 18.8951	0.0000039 -0.0000018	0.00023 -0.00011
5	13.1515 114.8278	10^{-14} 10^{-14}	det $[\Phi_q]$ = -2644.1		

$\gamma = \theta_4 - \theta_3 = 101.6763°$ $\quad -r_3 r_4 \sin\gamma = -2644.1$

if θ_4 is the driver $\{q\} = \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$ $[\Phi_q] = \begin{bmatrix} -r_2 \sin\theta_2 & -r_3 \sin\theta_3 \\ r_2 \cos\theta_2 & r_3 \cos\theta_3 \end{bmatrix}$ $\det[\Phi_q] = r_2 r_3 \sin(\theta_3 - \theta_2)$

if θ_3 is the driver $\{q\} = \begin{Bmatrix} \theta_2 \\ \theta_4 \end{Bmatrix}$ $[\Phi_q] = \begin{bmatrix} -r_2 \sin\theta_2 & r_4 \sin\theta_4 \\ r_2 \cos\theta_2 & -r_4 \cos\theta_4 \end{bmatrix}$ $\det[\Phi_q] = r_2 r_4 \sin(\theta_2 - \theta_4)$

Numerical Partial Derivatives

1) use current generalized coordinates $\{q\}$

2) evaluate constraint functions $\{\Phi\}$

3) perturb one (and only one) q_j by a small value ε $\quad q_j^* = q_j + \varepsilon$

4) evaluate perturbed constraint functions $\{\Phi^*\}$

5) compute partial derivatives with respect to that q_j $\quad \dfrac{\partial\{\Phi\}}{\partial q_j} = \dfrac{\{\Phi^*\} - \{\Phi\}}{\varepsilon}$

6) column j of Jacobian $[\partial\Phi / \partial q] = \dfrac{\partial\{\Phi\}}{\partial q_j}$

7) remember to reset and use the original value for q_j

8) repeat steps 3) through 7) for all j

Example: $r_1 = 90$ cm, $r_2 = 30$ cm, $r_3 = 60$ cm, $r_4 = 45$ cm, $\theta_1 = 0°$, $\theta_2 = 65°$

$$\{\Phi\} = \begin{Bmatrix} f_H \\ f_V \end{Bmatrix} = \begin{Bmatrix} r_2\cos\theta_2 + r_3\cos\theta_3 - r_4\cos\theta_4 - r_1\cos\theta_1 \\ r_2\sin\theta_2 + r_3\sin\theta_3 - r_4\sin\theta_4 - r_1\sin\theta_1 \end{Bmatrix}$$

$$\{q\} = \begin{Bmatrix} \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 90 \end{Bmatrix}\text{deg} = \begin{Bmatrix} 0 \\ 1.5708 \end{Bmatrix}\text{rad} \qquad \{\Phi\} = \begin{Bmatrix} -17.3215 \\ -17.8108 \end{Bmatrix}\text{cm} \qquad \varepsilon = 0.001\ \text{rad}$$

$$\{q\} = \begin{Bmatrix} 0 \\ 1.5708 \end{Bmatrix}\text{rad} \qquad \{\Phi\} = \begin{Bmatrix} -17.3215 \\ -17.8108 \end{Bmatrix}\text{cm} \qquad \varepsilon = 0.001\ \text{rad}$$

$$\{q\}^* = \begin{Bmatrix} 0.001 \\ 1.5708 \end{Bmatrix}\text{rad} \qquad \{\Phi\} = \begin{Bmatrix} -17.3215 \\ -17.7508 \end{Bmatrix}\text{cm} \qquad \frac{\partial\{\Phi\}}{\partial q_1} = \begin{Bmatrix} -0.0300 \\ 60.0000 \end{Bmatrix}\text{cm/rad}$$

$$\{q\}^* = \begin{Bmatrix} 0 \\ 1.5718 \end{Bmatrix}\text{rad} \qquad \{\Phi\} = \begin{Bmatrix} -17.2765 \\ -17.7508 \end{Bmatrix}\text{cm} \qquad \frac{\partial\{\Phi\}}{\partial q_1} = \begin{Bmatrix} 45.0000 \\ 0.0225 \end{Bmatrix}\text{cm/rad}$$

$$[\partial\Phi / \partial q] \approx \begin{bmatrix} -0.0300 & 45.0000 \\ 60.0000 & 0.0225 \end{bmatrix}\text{cm/rad}$$

```
% test_jac.m - evaluate Jacobian by numerical partial derivatives
%   used for ME 581 web cutter
% HJSIII, 20.02.19

% hold estimates for generalized coordinates
nq = length(q);
qhold = q;

% evaluate constraints
wc_phi

% hold constraints
phold = PHI;

% perturb one coordinate at a time
for iq = 1:nq,
  q = qhold;
  q(iq) = q(iq) + 0.01;

% change in constraints caused by coordinate perturbation is
% approximately equal to partial derivative
  wc_phi
  jtest(:,iq) = ( PHI - phold ) / 0.01;
end

% reset coordinates and constraints
q = qhold;
wc_phi

% bottom - test_jac
```