Two-Dimensional Constraints

General

$$\{\Phi\} = \{0\}$$

$$\{\dot{\Phi}\} = \{0\}$$
$$\{\dot{\Phi}\} = [\Phi_q]\{\dot{q}\} + \{\Phi_t\} = \{0\}$$

$$[\Phi_q]\{\dot{q}\} = \{\nu\} \qquad\qquad \{\nu\} = -\{\Phi_t\}$$

$$\{\ddot{\Phi}\} = \{0\}$$
$$\{\ddot{\Phi}\} \equiv [\Phi_q]\{\ddot{q}\} + ([\Phi_q]\{\dot{q}\})_q\{\dot{q}\} + 2[\Phi_{qt}]\{\dot{q}\} + \{\Phi_{tt}\} = \{0\}$$

$$[\Phi_q]\{\ddot{q}\} = \{\gamma\} \qquad\qquad \{\gamma\} \equiv -([\Phi_q]\{\dot{q}\})_q\{\dot{q}\} - 2[\Phi_{qt}]\{\dot{q}\} - \{\Phi_{tt}\}$$

$$\{\dddot{\Phi}\} = \{0\}$$

$$[\Phi_q]\{\dddot{q}\} = \{\eta\}$$
$$\{\eta\} \equiv -3([\Phi_q]\{\dot{q}\})_q\{\ddot{q}\} - (([\Phi_q]\{\dot{q}\})_q\{\dot{q}\})_q\{\dot{q}\} - 3[\Phi_{qt}]\{\ddot{q}\} - 3([\Phi_{qt}]\{\dot{q}\})_q\{\dot{q}\} - 3[\Phi_{qtt}]\{\dot{q}\} - \{\Phi_{ttt}\}$$

$$\{\ddddot{\Phi}\} = \{0\}$$

$$[\Phi_q]\{\ddddot{q}\} = \{\sigma\}$$
$$\begin{aligned}\{\sigma\} \equiv &-4([\Phi_q]\{\dddot{q}\})_q\{\dot{q}\} - 3([\Phi_q]\{\ddot{q}\})_q\{\ddot{q}\} - 6(([\Phi_q]\{\dot{q}\})_q\{\dot{q}\})_q\{\ddot{q}\} - \Big((([\Phi_q]\{\dot{q}\})_q\{\dot{q}\})_q\{\dot{q}\}\Big)_q\{\dot{q}\} \\ &- 4[\Phi_{qt}]\{\dddot{q}\} - 12([\Phi_{qt}]\{\dot{q}\})_q\{\ddot{q}\} - 4(([\Phi_{qt}]\{\dot{q}\})_q\{\dot{q}\})_q\{\dot{q}\} - 6[\Phi_{qtt}]\{\ddot{q}\} - 6([\Phi_{qtt}]\{\dot{q}\})_q\{\dot{q}\} \\ &- 4[\Phi_{qttt}]\{\dot{q}\} - \{\Phi_{tttt}\}\end{aligned}$$

Scleronomic constraints

independent of time such as mechanical joints

$$\{\gamma\} \equiv -\left(\left[\Phi_q\right]\{\dot{q}\}\right)_q \{\dot{q}\}$$

$$\{\eta\} \equiv \{\dot{\gamma}\} - \left(\left[\Phi_q\right]\{\ddot{q}\}\right)_q \{\dot{q}\}$$

$$\{\sigma\} \equiv \{\dot{\eta}\} - \left(\left[\Phi_q\right]\{\dddot{q}\}\right)_q \{\dot{q}\}$$

Revolute

$$\{\Phi\}_{REV} = \{r_j\}^P - \{r_i\}^P = \{0_{2x1}\}$$

Note: Haug uses $\{r_i\}^P - \{r_j\}^P$

$$\left[\Phi_{qi}\right]_{REV} = -\left[\ [I_2] \qquad [B_i]\{s_i\}'^P\ \right]$$
$$\left[\Phi_{qj}\right]_{REV} = \left[\ [I_2] \qquad [B_j]\{s_j\}'^P\ \right]$$

$$\{\nu\}_{REV} = \{0_{2x1}\}$$

$$\{\gamma\}_{REV} = \dot{\phi}_j^2 [A_j]\{s_j\}'^P - \dot{\phi}_i^2 [A_i]\{s_i\}'^P$$

$$\{\eta\}_{REV} = \dot{\phi}_j^3 [B_j]\{s_j\}'^P + 3\,\dot{\phi}_j\ddot{\phi}_j [A_j]\{s_j\}'^P - \dot{\phi}_i^3 [B_i]\{s_i\}'^P - 3\,\dot{\phi}_i\ddot{\phi}_i [A_i]\{s_i\}'^P$$

$$\begin{aligned}\{\sigma\}_{REV} = {} & 6\,\dot{\phi}_j^2\ddot{\phi}_j [B_j]\{s_j\}'^P + \left(4\,\dot{\phi}_j\dddot{\phi}_j + 3\ddot{\phi}_j^2 - \dot{\phi}_j^4\right)[A_j]\{s_j\}'^P \\ & - 6\,\dot{\phi}_i^2\ddot{\phi}_i [B_i]\{s_i\}'^P - \left(4\,\dot{\phi}_i\dddot{\phi}_i + 3\ddot{\phi}_i^2 - \dot{\phi}_i^4\right)[A_i]\{s_i\}'^P\end{aligned}$$

$$\{\kappa_i\}_{REV} = \begin{Bmatrix} [0_{2x1}] \\ \{s_i\}'^{PT} [A_i]^T \{\lambda\}_{REV}\, \dot{\phi}_i \end{Bmatrix}$$

$$\{\kappa_j\}_{REV} = \begin{Bmatrix} [0_{2x1}] \\ -\{s_j\}'^{PT} [A_j]^T \{\lambda\}_{REV}\, \dot{\phi}_j \end{Bmatrix}$$

Double revolute

$$\Phi_{REV\_REV} = \{d_{ij}\}^T\{d_{ij}\} - L^2 = 0 \qquad L = \text{constant length}$$

for $\{d_{ij}\} = \{r_j\}^P - \{r_i\}^P$

and $\{\dot{d}_{ij}\} = \{\dot{r}_j\}^P - \{\dot{r}_i\}^P$

and $\{\ddot{d}_{ij}\} = \{\ddot{r}_j\}^P - \{\ddot{r}_i\}^P$

and $\{\dddot{d}_{ij}\} = \{\dddot{r}_j\}^P - \{\dddot{r}_i\}^P$

$$[\Phi_{qi}]_{REV\_REV} = 2\{d_{ij}\}^T[\Phi_{qi}]_{REV}$$
$$[\Phi_{qj}]_{REV\_REV} = 2\{d_{ij}\}^T[\Phi_{qj}]_{REV}$$

$$\nu_{REV\_REV} = 0$$

$$\gamma_{REV\_REV} = 2\{d_{ij}\}^T\{\gamma\}_{REV} - 2\{\dot{d}_{ij}\}^T\{\dot{d}_{ij}\}$$

$$\eta_{REV\_REV} = 2\{d_{ij}\}^T\{\eta\}_{REV} - 6\{\dot{d}_{ij}\}^T\{\ddot{d}_{ij}\}$$

$$\sigma_{REV\_REV} = 2\{d_{ij}\}^T\{\sigma\}_{REV} - 8\{\dot{d}_{ij}\}^T\{\dddot{d}_{ij}\} - 6\{\ddot{d}_{ij}\}^T\{\ddot{d}_{ij}\}$$

$$\{\kappa_i\}_{REV\_REV} = -2\lambda_{REV\_REV}\begin{Bmatrix}\{\dot{d}_{ij}\} \\ \{s_i\}'^{PT}\left([B_i]^T\{\dot{d}_{ij}\} - [A_i]^T\{d_{ij}\}\right)\dot{\phi}_i\end{Bmatrix}$$

$$\{\kappa_j\}_{REV\_REV} = 2\lambda_{REV\_REV}\begin{Bmatrix}\{\dot{d}_{ij}\} \\ \{s_j\}'^{PT}\left([B_j]^T\{\dot{d}_{ij}\} - [A_j]^T\{d_{ij}\}\dot{\phi}_j\right)\end{Bmatrix}$$

<u>Parallel vectors (planar parallel-1)</u>

$\{a_i\}$ parallel to $\{a_j\}$

$$\Phi_{PARALLEL} = \{a_i\}^T[R]^T\{a_j\} = 0$$

for $\{a_i\} = \{r_i\}^Q - \{r_i\}^P$ and $\{a_j\} = \{r_j\}^Q - \{r_j\}^P$

$$[\Phi_{qi}]_{PARALLEL} = \left[\{0_{1x2}\} \qquad -\{a_i\}^T\{a_j\}\right] \qquad \{a_i\}^T\{a_j\} = \pm\text{norm}\{a_i\} \times \text{norm}\{a_j\} = \text{constant}$$

$$\left[\Phi_{qj}\right]_{PARALLEL} = \left[\{0_{1x2}\} \qquad + \{a_i\}^T \{a_j\}\right]$$

$$\nu_{PARALLEL} = 0$$

$$\gamma_{PARALLEL} = 0$$

$$\eta_{PARALLEL} = 0$$

$$\sigma_{PARALLEL} = 0$$

$$\{\kappa_i\}_{PARALLEL} = \{0_{3x1}\}$$

$$\{\kappa_j\}_{PARALLEL} = \{0_{3x1}\}$$

<u>Pin-in-slot (planar parallel-2)</u>

$\{a_i\}$ parallel to $\{d_{ij}\}$

$$\Phi_{PIN\_SLOT} = \{a_i\}^T [R]^T \{d_{ij}\} = 0$$

for $\{d_{ij}\} = \{r_j\}^P - \{r_i\}^P$ and $\{a_i\} = \{r_i\}^Q - \{r_i\}^P$

and $\{\dot{d}_{ij}\}$ $\{\ddot{d}_{ij}\}$ $\{\dddot{d}_{ij}\}$ from above

$$\left[\Phi_{qi}\right]_{PIN\_SLOT} = \{a_i\}^T [R]^T \left[\Phi_{qi}\right]_{REV} - \left[\{0_{1x2}\} \qquad \{a_i\}^T \{d_{ij}\}\right]$$

$$\left[\Phi_{qj}\right]_{PIN\_SLOT} = \{a_i\}^T [R]^T \left[\Phi_{qj}\right]_{REV}$$

$$\nu_{PIN\_SLOT} = 0$$

$$\gamma_{PIN\_SLOT} = \{a_i\}^T \left(2\,\dot{\phi}_i \{\dot{d}_{ij}\} + [R]^T \left(\dot{\phi}_i^{\,2} \{d_{ij}\} + \{\gamma\}_{REV}\right)\right)$$

$$\eta_{PIN\_SLOT} = \{a_i\}^T \left(3\,\dot{\phi}_i \{\ddot{d}_{ij}\} + 3\,\ddot{\phi}_i \{\dot{d}_{ij}\} - \dot{\phi}_i^{\,3} \{d_{ij}\} + [R]^T \left(3\,\dot{\phi}_i^{\,2} \{\dot{d}_{ij}\} + 3\,\dot{\phi}_i \ddot{\phi}_i \{d_{ij}\} + \{\eta\}_{REV}\right)\right)$$

$$\sigma_{PIN\_SLOT} = \{a_i\}^T \begin{pmatrix} 4\,\dot{\phi}_i \{\dddot{d}_{ij}\} + 6\,\ddot{\phi}_i \{\ddot{d}_{ij}\} + 4\left(\dddot{\phi}_i - \dot{\phi}_i^{\,3}\right)\{\dot{d}_{ij}\} - 6\,\dot{\phi}_i^{\,2} \ddot{\phi}_i \{d_{ij}\} \\ + [R]^T \left(6\,\dot{\phi}_i^{\,2} \{\ddot{d}_{ij}\} + 12\,\dot{\phi}_i \ddot{\phi}_i \{\dot{d}_{ij}\} + \left(4\dot{\phi}_i \dddot{\phi}_i + 3\ddot{\phi}_i^{\,2} - \dot{\phi}_i^{\,4}\right)\{d_{ij}\} + \{\sigma\}_{REV}\right) \end{pmatrix}$$

$$\{\kappa_i\}_{PIN\_SLOT} = \lambda_{PIN\_SLOT} \begin{Bmatrix} \dot{\phi}_i \{a_i\} \\ -\{\dot{d}_{ij}\}^T \{a_i\} \end{Bmatrix}$$

$$\{\kappa_j\}_{PIN} = \lambda_{PIN} \begin{Bmatrix} -\{a_i\}\dot{\phi}_j \\ \{s_j\}'^{PT}[B_j]^T\{a_i\}\left(\dot{\phi}_j - \dot{\phi}_i\right) \end{Bmatrix}$$

Relative angle driver

$$\Phi_{ANGLE} = \phi_j - \phi_i - C - f(t) = 0 \qquad C = \text{constant}$$

$$\lfloor \Phi_{qi} \rfloor_{ANGLE} = [0 \quad 0 \quad -1]$$
$$\lfloor \Phi_{ji} \rfloor_{ANGLE} = [0 \quad 0 \quad 1]$$

$$\nu_{ANGLE} = f_t$$

$$\gamma_{ANGLE} = f_{tt}$$

$$\eta_{ANGLE} = f_{ttt}$$

$$\sigma_{ANGLE} = f_{tttt}$$

$$\{\kappa_i\}_{ANGLE} = \{0_{3x1}\}$$

$$\{\kappa_j\}_{ANGLE} = \{0_{3x1}\}$$

Gear pair driver (chain/sprockets, belt/pulleys)

$$\Phi_{GEAR} = \phi_j - K\phi_i - C = 0 \qquad K = \text{constant}, C = \text{constant}$$
$$\text{external gears } K = -\rho_i / \rho_j, \text{ internal gears } K = +\rho_i / \rho_j$$

$$\lfloor \Phi_{qi} \rfloor_{GEAR} = [0 \quad 0 \quad -K]$$
$$\lfloor \Phi_{qj} \rfloor_{GEAR} = [0 \quad 0 \quad 1]$$

$$\nu_{GEAR} = 0$$

$$\gamma_{GEAR} = 0$$

$$\eta_{GEAR} = 0$$

$$\sigma_{GEAR} = 0$$

Gear pair on rotating link k

$$\Phi_{GEAR\_ON\_K} = \left(\phi_j - \phi_k\right) - K\left(\phi_i - \phi_k\right) - C = 0 \qquad K = \text{constant}, C = \text{constant} \quad \text{from above}$$

$$\lfloor \Phi_{qi} \rfloor_{GEAR\_ON\_K} = \begin{bmatrix} 0 & 0 & -K \end{bmatrix}$$
$$\lfloor \Phi_{qj} \rfloor_{GEAR\_ON\_K} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$
$$\lfloor \Phi_{qk} \rfloor_{GEAR\_ON\_K} = \begin{bmatrix} 0 & 0 & (K-1) \end{bmatrix}$$

$$\nu_{GEAR\_ON\_K} = 0$$

$$\gamma_{GEAR\_ON\_K} = 0$$

$$\eta_{GEAR\_ON\_K} = 0$$

$$\sigma_{GEAR\_ON\_K} = 0$$

Relative coordinate driver (translation, rotation, gears, pure rolling)

$$\Phi_{RCD} = q_j - Kq_i - C - f(t) = 0 \qquad K = \text{constant}. C = \text{constant}$$

$$\lfloor \Phi_{qi} \rfloor_{RCD} = \begin{bmatrix} -K & 0 & 0 \end{bmatrix} \qquad q_i = x_i$$
$$\lfloor \Phi_{qi} \rfloor_{RCD} = \begin{bmatrix} 0 & -K & 0 \end{bmatrix} \qquad q_i = y_i$$
$$\lfloor \Phi_{qi} \rfloor_{RCD} = \begin{bmatrix} 0 & 0 & -K \end{bmatrix} \qquad q_i = \phi_i$$

$$\lfloor \Phi_{qj} \rfloor_{RCD} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad q_j = x_j$$
$$\lfloor \Phi_{qj} \rfloor_{RCD} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad q_j = y_j$$
$$\lfloor \Phi_{qj} \rfloor_{RCD} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad q_j = \phi_j$$

$$\nu_{RCD} = f_t$$

$$\gamma_{RCD} = f_{tt}$$

$$\eta_{RCD} = f_{ttt}$$

$$\sigma_{RCD} = f_{tttt}$$

Planar parallel-2 distance driver (see pin-in-slot)

$$\Phi_{PP2DD} = \{a_i\}^T\{d_{ij}\}/L - f(t) = 0 \qquad L = |\{a_i\}| = \text{constant length}$$

$$[\Phi_{qi}]_{PP2DD} = \left(\{a_i\}^T[\Phi_{qi}]_{REV} + \begin{bmatrix}\{0_{1x2}\} & \{a_i\}^T[R]^T\{d_{ij}\}\end{bmatrix}\right)/L$$
$$[\Phi_{qj}]_{PP2DD} = \{a_i\}^T[\Phi_{qj}]_{REV}/L$$

$$\nu_{PP2DD} = f_t$$

$$\gamma_{PP2DD} = \{a_i\}^T\left(-2\,\dot{\phi}_i[R]^T\{\dot{d}_{ij}\} + \dot{\phi}_i^{\,2}\{d_{ij}\} + \{\gamma\}_{REV}\right)/L + f_{tt}$$

$$\eta_{PP2DD} = \{a_i\}^T\left(-[R]^T\left(3\,\dot{\phi}_i\{\ddot{d}_{ij}\} + 3\,\ddot{\phi}_i\{\dot{d}_{ij}\} - \dot{\phi}_i^{\,3}\{d_{ij}\}\right) + 3\,\dot{\phi}_i^{\,2}\{\dot{d}_{ij}\} + 3\,\dot{\phi}_i\ddot{\phi}_i\{d_{ij}\} + \{\eta\}_{REV}\right)/L + f_{ttt}$$

$$\sigma_{PP2DD} = \{a_i\}^T\begin{pmatrix} -[R]^T\left(4\,\dot{\phi}_i\{\dddot{d}_{ij}\} + 6\,\ddot{\phi}_i\{\ddot{d}_{ij}\} + 4\left(\dddot{\phi}_i - \dot{\phi}_i^{\,3}\right)\{\dot{d}_{ij}\} - 6\,\dot{\phi}_i^{\,2}\ddot{\phi}_i\{d_{ij}\}\right) \\ +6\,\dot{\phi}_i^{\,2}\{\ddot{d}_{ij}\} + 12\,\dot{\phi}_i\ddot{\phi}_i\{\dot{d}_{ij}\} + \left(4\dot{\phi}_i\dddot{\phi}_i + 3\ddot{\phi}_i^{\,2} - \dot{\phi}_i^{\,4}\right)\{d_{ij}\} + \{\sigma\}_{REV} \end{pmatrix}/L + f_{tttt}$$

Pure rolling along planar parallel-2 distance

$$\Phi_{ROLL} = \{a_i\}^T\{d_{ij}\}/L - \rho(\phi_j - \phi_i) - C = 0$$
$$L = |\{a_i\}| = \text{constant length},\ \rho = \text{rolling radius},\ C = \text{constant}$$

$$[\Phi_{qi}]_{ROLL} = \left(\{a_i\}^T[\Phi_{qi}]_{REV} + \begin{bmatrix}\{0_{1x2}\} & \{a_i\}^T[R]^T\{d_{ij}\} + \rho L\end{bmatrix}\right)/L$$
$$[\Phi_{qj}]_{ROLL} = \left(\{a_i\}^T[\Phi_{qj}]_{REV} - \begin{bmatrix}\{0_{1x2}\} & \rho L\end{bmatrix}\right)/L$$

$$\nu_{ROLL} = 0$$

$$\gamma_{ROLL} = \gamma_{PP2DD} \qquad \text{for} \quad f_{tt} = 0$$

$$\eta_{ROLL} = \eta_{PP2DD} \qquad \text{for} \quad f_{ttt} = 0$$

$$\sigma_{ROLL} = \sigma_{PP2DD} \qquad \text{for} \quad f_{tttt} = 0$$

Planar relative distance driver (see double revolute)

$$\Phi_{PRDD} = \{d_{ij}\}^T\{d_{ij}\} - (f(t))^2 = 0 \qquad f(t) > 0$$

$$\lfloor \Phi_{qi} \rfloor_{PRDD} = \lfloor \Phi_{qi} \rfloor_{REV\_REV}$$
$$\lfloor \Phi_{qj} \rfloor_{PRDD} = \lfloor \Phi_{qj} \rfloor_{REV\_REV}$$

$$\nu_{PRDD} = 2\ f\ f_t$$

$$\gamma_{PRDD} = \gamma_{REV\_REV} + 2\ f_t^{\ 2} + 2\ f\ f_{tt}$$

$$\eta_{PRDD} = \eta_{REV\_REV} + 6\ f_t\ f_{tt} + 2\ f\ f_{ttt}$$

$$\sigma_{PRDD} = \sigma_{REV\_REV} + 6\ f_{tt}^{\ 2} + 8\ f_t\ f_{ttt} + 2\ f\ f_{tttt}$$

Acceleration Right-hand Side for Revolute

$$\{\gamma\} \equiv -\left([\Phi_q]\{\dot{q}\}\right)_q\{\dot{q}\} - 2[\Phi_q]_t\{\dot{q}\} - \{\Phi_{tt}\}$$

$$\{\Phi\}_{REV} = \{r_j\}^P - \{r_i\}^P = \{0_{2x1}\}$$

$$\{q_i\} = \begin{Bmatrix} \{r_i\} \\ \phi_i \end{Bmatrix} \qquad \{\dot{q}_i\} = \begin{Bmatrix} \{\dot{r}_i\} \\ \dot{\phi}_i \end{Bmatrix}$$

$$[\Phi_{qi}]_{REV} = -\begin{bmatrix} [I_2] & [B_i]\{s_i\}'^P \end{bmatrix}$$

$$[\Phi_{qi}]\{\dot{q}_i\} = -\begin{bmatrix} [I_2] & [B_i]\{s_i\}'^P \end{bmatrix}\begin{Bmatrix} \{\dot{r}_i\} \\ \dot{\phi}_i \end{Bmatrix} = -\{\dot{r}_i\} - \dot{\phi}_i[B_i]\{s_i\}'^P$$

$$\left([\Phi_{qi}]\{\dot{q}_i\}\right)_{qi} = \begin{bmatrix} [0_{2x2}] & \dot{\phi}_i[A_i]\{s_i\}'^P \end{bmatrix} \qquad [B_i]_{\phi i} = -[A_i]$$

$$\left([\Phi_{qi}]\{\dot{q}_i\}\right)_{qi}\{\dot{q}_i\} = \begin{bmatrix} [0_{2x2}] & \dot{\phi}_i[A_i]\{s_i\}'^P \end{bmatrix}\begin{Bmatrix} \{\dot{r}_i\} \\ \dot{\phi}_i \end{Bmatrix} = \dot{\phi}_i^2[A_i]\{s_i\}'^P$$

$$[\Phi_{qi}]_{REV} = \begin{bmatrix} [I_2] & [B_i]\{s_i\}'^P \end{bmatrix}$$

$$[\Phi_{qi}]_t = [0_{2x3}] \qquad [\Phi_{qi}]_t\{\dot{q}_i\} = \{0_{2x1}\}$$

$$\{\Phi_t\} = \{0_{2x1}\} \qquad \{\Phi_{tt}\} = \{0_{2x1}\}$$

$$\{\gamma\} \equiv -\left([\Phi_q]\{\dot{q}\}\right)_q\{\dot{q}\} - 2[\Phi_q]_t\{\dot{q}\} - \{\Phi_{tt}\}$$

$\{\gamma\}_{REV} = -\dot{\phi}_i^2[A_i]\{s_i\}'^P$ for body i

$\{\gamma\}_{REV} = \dot{\phi}_j^2[A_j]\{s_j\}'^P$ for body j

$$\{\gamma\}_{REV} = \dot{\phi}_j^2[A_j]\{s_j\}'^P - \dot{\phi}_i^2[A_i]\{s_i\}'^P$$