Two-Dimensional Experimental Kinematics

Digitize locations of landmarks $\{r_i\}^{Pk}$ on body i for points k=1 to n at given time t

All points must be attached to body i

Use landmark weighting factor $f^{Pk} = 1$ if point k is available at time t. Use $f^{Pk} = 0$ if point k not available at given time t.

Determine $\{\dot{r}_i\}^{Pk}$ $\{\ddot{r}_i\}^{Pk}$ $\{\dddot{r}_i\}^{Pk}$ $\{\ddddot{r}_i\}^{Pk}$ at given time t.

Mean values

$$\{r_i\}^{mean} = \left(\sum_{k=1}^{n} f^{Pk} \{r_i\}^{Pk}\right) / \sum_{k=1}^{n} f^{Pk}$$

$$\{\dot{r}_i\}^{mean} = \left(\sum_{k=1}^{n} f^{Pk} \{\dot{r}_i\}^{Pk}\right) / \sum_{k=1}^{n} f^{Pk}$$

$$\{\ddot{r}_i\}^{mean} = \left(\sum_{k=1}^{n} f^{Pk} \{\ddot{r}_i\}^{Pk}\right) / \sum_{k=1}^{n} f^{Pk}$$

$$\{\dddot{r}_i\}^{mean} = \left(\sum_{k=1}^{n} f^{Pk} \{\dddot{r}_i\}^{Pk}\right) / \sum_{k=1}^{n} f^{Pk}$$

$$\{\ddddot{r}_i\}^{mean} = \left(\sum_{k=1}^{n} f^{Pk} \{\ddddot{r}_i\}^{Pk}\right) / \sum_{k=1}^{n} f^{Pk}$$

$$S = \sum_{k=1}^{n} \left(f^{Pk} \left(\{r_i\}^{Pk} - \{r_i\}^{mean}\right)^T \left(\{r_i\}^{Pk} - \{r_i\}^{mean}\right)\right)$$

Velocity

$$\omega_i = \left(\sum_{k=1}^{n} f^{Pk} \left(\{\dot{r}_i\}^{Pk}\right)^T [R] \left(\{r_i\}^{Pk} - \{r_i\}^{mean}\right)\right) / S \qquad \text{for} \qquad [R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$[\tilde{\omega}_i] = \omega_i [R]$$

Point ICR is the instantaneous center of rotation for body i with respect to the ground. Note that the ICR is not attached to the body. Point P on the body coincident with ICR has zero velocity.

$$\{\dot{r}_i\}^{P} = [\tilde{\omega}_i]\left(\{r_i\}^{P} - \{r\}^{ICR}\right) \qquad \text{for any point P attached to body i}$$

$$\{r\}^{ICR} = \{r_i\}^{mean} - [\tilde{\omega}_i]^{-1} \{\dot{r}_i\}^{mean} \qquad \text{for} \qquad \{\dot{r}_i\}^{P\_at\_ICR} = 0$$

Acceleration

$$\dot{\omega}_i = \left(\sum_{k=1}^{n} f^{Pk} \left(\{\ddot{r}_i\}^{Pk}\right)^T [R]\left(\{r_i\}^{Pk} - \{r_i\}^{mean}\right)\right) / S$$

$$[\tilde{\dot{\omega}}_i] = \dot{\omega}_i [R] \qquad [\beta] = [\tilde{\dot{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i] = \begin{bmatrix} -\omega_i^2 & -\dot{\omega}_i \\ \dot{\omega}_i & -\omega_i^2 \end{bmatrix}$$

Point IAP is the instantaneous acceleration pole for body i. Note that the IAP is not attached to the body. Point P on the body coincident with IAP has zero acceleration.

$$\{\ddot{r}_i\}^P = [\beta]\left(\{r_i\}^P - \{r\}^{IAP}\right) \qquad \text{for any point P attached to body i}$$

$$\{r\}^{IAP} = \{r_i\}^{mean} - [\beta]^{-1}\{\ddot{r}_i\}^{mean} \qquad \text{for} \qquad \{\ddot{r}_i\}^{P\_at\_IAP} = 0 \qquad \text{OLD}$$

Jerk

$$\ddot{\omega}_i = \omega_i^3 + \left(\sum_{k=1}^{n} f^{Pk} \left(\{\dddot{r}_i\}^{Pk}\right)^T [R]\left(\{r_i\}^{Pk} - \{r_i\}^{mean}\right)\right) / S$$

$$[\tilde{\ddot{\omega}}_i] = \ddot{\omega}_i [R] \qquad [\eta] = [\tilde{\ddot{\omega}}_i] + 3[\tilde{\dot{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i] = \begin{bmatrix} -3\omega_i\dot{\omega}_i & \omega_i^3 - \ddot{\omega}_i \\ \ddot{\omega}_i - \omega_i^3 & -3\omega_i\dot{\omega}_i \end{bmatrix}$$

Point IJP is the instantaneous jerk pole for the body. Note that the IJP is not attached to the body. Point P on the body coincident with IJP has zero jerk.

$$\{\dddot{r}_i\}^P = [\eta]\left(\{r_i\}^P - \{r\}^{IJP}\right) \qquad \text{for any point P attached to the body}$$

$$\{r\}^{IJP} = \{r_i\}^{mean} - [\eta]^{-1}\{\dddot{r}_i\}^{mean} \qquad \text{for} \qquad \{\dddot{r}_i\}^{P\_at\_IJP} = 0$$

Snap

$$\dddot{\omega}_i = 6\omega_i^2\dot{\omega}_i + \left(\sum_{k=1}^{n} f^{Pk} \left(\{\ddddot{r}_i\}^{Pk}\right)^T [R]\left(\{r_i\}^{Pk} - \{r_i\}^{mean}\right)\right) / S$$

$$[\tilde{\dddot{\omega}}_i] = \dddot{\omega}_i [R] \qquad [\sigma] = [\tilde{\dddot{\omega}}_i] + 6[\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\dot{\omega}}_i] + 4[\tilde{\omega}_i][\tilde{\ddot{\omega}}_i] + 3[\tilde{\dot{\omega}}_i][\tilde{\dot{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i]$$

$$[\sigma] = \begin{bmatrix} -4\omega_i\ddot{\omega}_i - 3\dot{\omega}_i^2 + \omega_i^4 & \omega_i^2\dot{\omega}_i - \dddot{\omega}_i \\ \dddot{\omega}_i - \omega_i^2\dot{\omega}_i & -4\omega_i\ddot{\omega}_i - 3\dot{\omega}_i^2 + \omega_i^4 \end{bmatrix}$$

Point ISP is the instantaneous snap pole for the body. Note that the ISP is not attached to the body. Point P on the body coincident with ISP has zero snap.

$$\{\ddddot{r}_i\}^{P} = [\sigma]\left(\{r_i\}^{P} - \{r\}^{ISP}\right) \quad \text{for any point P attached to the body}$$

$$\{r\}^{ISP} = \{r_i\}^{mean} - [\sigma]^{-1}\{\ddddot{r}_i\}^{mean} \quad \text{for} \quad \{\ddddot{r}_i\}^{P\_at\_ISP} = 0$$

Centrode

Location of the ICR measured relative to ground changes as body i moves and sweeps a locus called the fixed centrode for body i. The time derivative of the locus describes how the ICR moves. Tracking the location of the ICR relative to a coordinate frame fixed to body i provides a locus called the moving centrode. Motion of body i may be characterized as pure rolling of the moving centrode on the fixed centrode because body i instantaneously has zero velocity at each location of the ICR.

$$\{\dot{r}\}^{ICR} = \left([\tilde{\omega}]\{\ddot{r}_i\}^{mean} - [\beta]\{\dot{r}_i\}^{mean}\right) / \omega_i^{\,2}$$

The second time derivative of the location of the ICR also changes as body i moves. First and second time derivatives of position along a locus may be combined to determine curvature κ of the fixed centrode. If body i is part of a mechanism with mobility of one, curvature of the centrode at each location will be invariant to speed of the mechanism.

$$\{\ddot{r}\}^{ICR} = \left(\begin{bmatrix} 0 & \omega_i\ddot{\omega}_i - 2\dot{\omega}_i^{\,2} \\ 2\dot{\omega}_i^{\,2} - \omega_i\ddot{\omega}_i & 0 \end{bmatrix}\{\dot{r}_i\}^{mean} + \begin{bmatrix} \omega_i^{\,3} & 2\omega_i\dot{\omega}_i \\ -2\omega_i\dot{\omega}_i & \omega_i^{\,3} \end{bmatrix}\{\ddot{r}_i\}^{mean} + \begin{bmatrix} 0 & -\omega_i^{\,2} \\ \omega_i^{\,2} & 0 \end{bmatrix}\{\dddot{r}_i\}^{mean}\right) / \omega_i^{\,3}$$

$$\kappa^{ICR} = \left(\left(\{\ddot{r}\}^{ICR}\right)^T [R]\{\dot{r}\}^{ICR}\right) / \left(\left(\{\dot{r}\}^{ICR}\right)^T \{\dot{r}\}^{ICR}\right)^{3/2}$$

Relative velocity

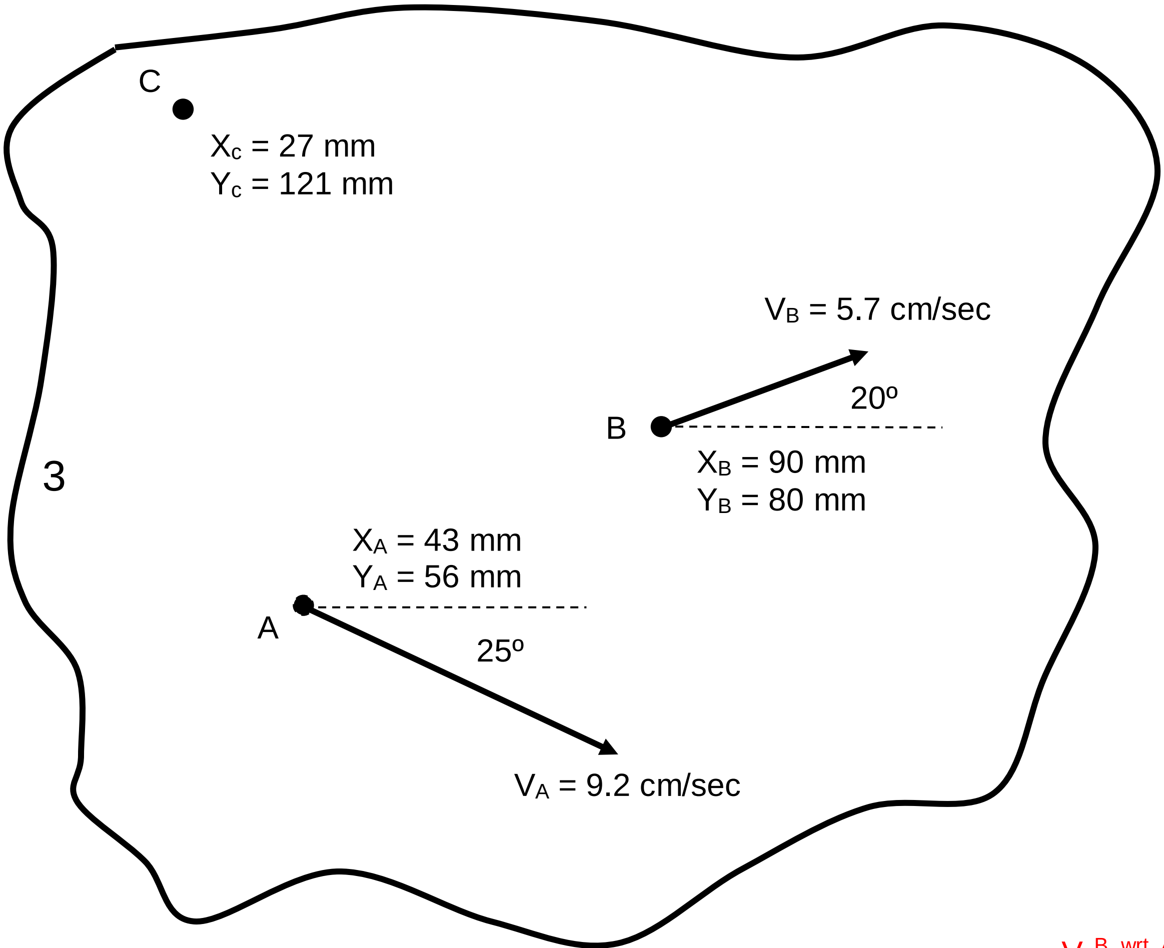
For planar motion, the relative angular velocity of body j with respect to body i is the difference between the two angular velocities. The relative instantaneous center of rotation (RICR) for body j about body i describes a unique point that has zero relative velocity between the two bodies. Note that location of the RICR is measured with respect to the ground.

$$\omega_{j\_wrt\_i} = \omega_j - \omega_i$$

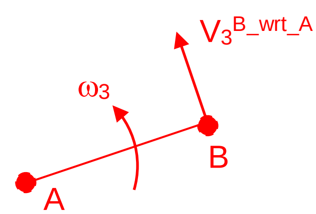
$$\{r_{j\_wrt\_i}\}^{ICR} = \left([\tilde{\omega}_j]\{r_j\}^{ICR} - [\tilde{\omega}_i]\{r_i\}^{ICR}\right) / \omega_{j\_wrt\_i}$$

Rigid Body

Determine the velocity of point C on rigid body link 3. The rigid body and the velocity vectors are drawn to scale. Link 3 is NOT pinned to the ground. Show your work.



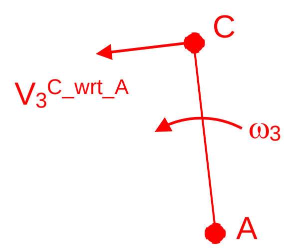
$$\{\dot{r}_3\}^B = \begin{Bmatrix} 53.56 \text{ mmps} \\ 19.50 \text{ mmps} \end{Bmatrix} \qquad \{\dot{r}_3\}^A = \begin{Bmatrix} 83.38 \text{ mmps} \\ -38.88 \text{ mmps} \end{Bmatrix}$$



$$\{\dot{r}_3\}^{B\_wrt\_A} = \{\dot{r}_3\}^B - \{\dot{r}_3\}^A = \begin{Bmatrix} -29.82 \text{ mmps} \\ 58.38 \text{ mmps} \end{Bmatrix} = 65.55 \text{ mmps} \angle 117.1°$$

$$\text{norm}\{\dot{r}_3\}^{B\_wrt\_A} = \omega_3(AB) \qquad AB = 52.77 \text{ mm} \qquad \omega_3 = 1.242 \text{ rad/sec CCW}$$

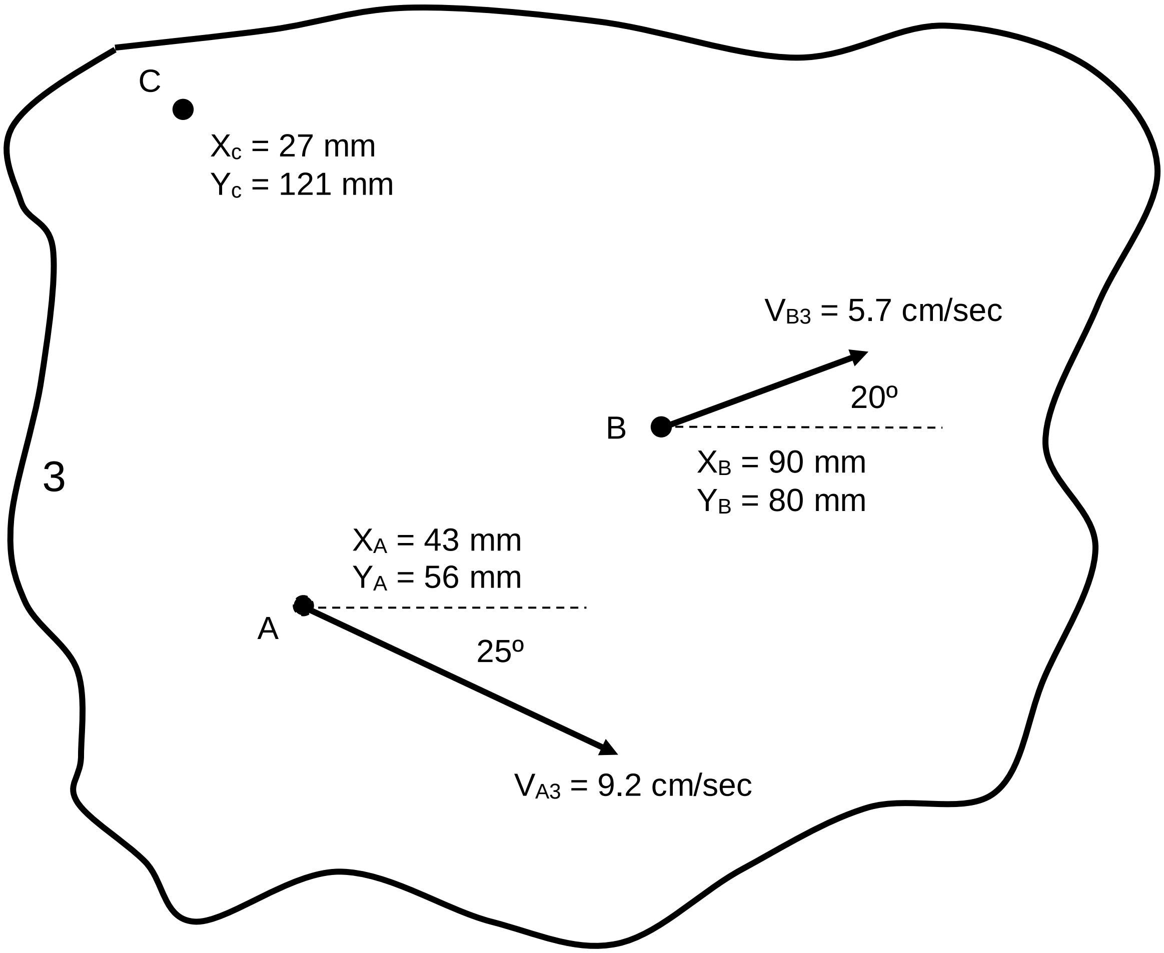
$$\{r_3\}^{C\_wrt\_A} = \begin{Bmatrix} -16 \text{ mm} \\ 65 \text{ mm} \end{Bmatrix} \qquad \{\dot{r}_3\}^{C\_wrt\_A} = \omega_3[R]\{r_3\}^{C\_wrt\_A} = \begin{Bmatrix} -80.73 \text{ mmps} \\ -19.87 \text{ mmps} \end{Bmatrix}$$



$$\{\dot{r}_3\}^C = \{\dot{r}_3\}^A + \{\dot{r}_3\}^{C\_wrt\_A} = \begin{Bmatrix} 2.65 \text{ mmps} \\ -58.75 \text{ mmps} \end{Bmatrix} = 58.81 \text{ mmps} \angle -87.4°$$

Rigid Body

Determine the velocity of point C on rigid body link 3. The rigid body and the velocity vectors are drawn to scale. Link 3 is NOT pinned to the ground. Show your work.



point A = P1 $\quad f^{P1} = 1 \quad \{r_3\}^{P1} = \begin{Bmatrix} 43 \text{ mm} \\ 56 \text{ mm} \end{Bmatrix} \quad \{\dot{r}_3\}^{P1} = \begin{Bmatrix} 83.38 \text{ mmps} \\ -38.88 \text{ mmps} \end{Bmatrix}$ from above

point B = P2 $\quad f^{P2} = 1 \quad \{r_3\}^{P2} = \begin{Bmatrix} 90 \text{ mm} \\ 80 \text{ mm} \end{Bmatrix} \quad \{\dot{r}_3\}^{P2} = \begin{Bmatrix} 53.56 \text{ mmps} \\ 19.50 \text{ mmps} \end{Bmatrix}$

use lm2kin2d per attached code

$\omega_3 = +1.2422$ rad/sec $\qquad \{r\}^{ICR} = \begin{Bmatrix} 74.3006 \\ 123.1197 \end{Bmatrix}$ mm

$$\{r_3\}^C = \begin{Bmatrix} 27 \text{ mm} \\ 121 \text{ mm} \end{Bmatrix} \qquad \{\dot{r}_3\}^C = \begin{bmatrix} 0 & -\omega_3 \\ \omega_3 & 0 \end{bmatrix} \left(\{r_3\}^C - \{r\}^{ICR} \right) = \begin{Bmatrix} 2.6332 \text{ mmps} \\ -58.7571 \text{ mmps} \end{Bmatrix} = 58.82 \text{ mmps} \angle -87.43°$$

```
% rbk.m - rigid body kinematics
% HJSIII, 14.01.13

clear

% constants
Rmat = [ 0  -1 ; 1  0 ];

% inputs
f    = [  1     1  ];

r    = [ 43     90 ;
         56     80 ];

rd   = [ 83.38  53.56 ;
        -38.88  19.5 ];

rdd  = zeros(2,2);

rddd = zeros(2,2);

% call function
[ w, rICR, wd, rIAP, wdd, rIJP, rdICR, kappa ] = lm2kin2d( f, r, rd, rdd, rddd );
w
rICR

% find velocity of C
r3C = [ 27  121 ]';
r3Cd = w * Rmat * ( r3C - rICR )

% bottom of rbk
```

```
% t_lm2kin2d.m - test 2D kinematics from landmark motion
% HJSIII, 14.01.13

clear

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% example inputs - web cutter four bar
% Haug page 197 - not ME 581 web cutter
%             B3        C3
f =    [     1         1      ];

r =    [     3.7588    3.9407 ;
             1.3681   29.3675 ];

rd =   [    -5.4874   22.5296 ;
            15.0764   14.8943 ];

rdd =  [   -60.4716  -42.8075 ;
           -22.0098  -50.1604 ];

rddd = [    88.2815 -673.2083 ;
          -242.5514 -291.1768 ];

% expected outputs
w_test =        -1.0006;
rICR_test =  [  18.8257 ;  6.8520 ];

wd_test =       -0.6374;
rIAP_test =  [ -49.1784 ; 13.0846 ];

wdd_test =      26.1823;
rIJP_test =  [  12.8647 ;  3.9747 ];

rdICR_test = [ -37.0807 ; 72.0169 ];
kappa_test =    -0.0151 ;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% test function
[ w, rICR, wd, rIAP, wdd, rIJP, rdICR, kappa ] = lm2kin2d( f, r, rd, rdd, rddd );
w
rICR
wd
rIAP
wdd
rIJP
rdICR
kappa

% bottom of t_lm2kin2d
```

```
function [ w, rICR, wd, rIAP, wdd, rIJP, rdICR, kappa ] = lm2kin2d( f, r, rd, rdd, rddd )
% 2D instantaneous kinematics of a rigid body from landmark motion
% HJSIII, 14.01.13
%
% USAGE
% function [ w, rICR, wd, rIAP, wdd, rIJP, rdICR, kappa ] = lm2kin2d( f, r, rd, rdd, rddd )

% INPUTS
% f    - 1xn vector of weights - f(j)=1 means data valid, f(j)=0 means data not available
% r    - 2xn matrix of x,y landmark location
% rd   - 2xn matrix of x,y landmark velocity
% rdd  - 2xn matrix of x,y landmark acceleration
% rddd - 2xn matrix of x,y landmark jerk
%
% OUTPUTS
% rICR  - 2x1 location of instantaneous center of rotation
% w     - angular velocity
% rIAP  - 2x1 location of instantaneous acceleration pole
% wd    - angular acceleration
% rIJP  - 2x1 location of instantaneous jerk pole
% wdd   - angular jerk
% rdICR - 2x1 time derivative of location of instantaneous center
% kappa - curvature of centrode

% constants
Rmat = [ 0  -1 ; 1  0 ];

% mean values
[ nr, n ] = size( r );
fmat = diag( f );
sf = sum( f' );
rm    = sum( fmat*r' )'    /sf;
rdm   = sum( fmat*rd' )'   /sf;
rddm  = sum( fmat*rdd' )'  /sf;
rdddm = sum( fmat*rddd' )' /sf;

% centered location
rc = r - rm*ones(1,n);
S = trace( rc * fmat * rc' );

% velocity
vmat = rd * fmat * rc';
w = ( vmat(2,1) - vmat(1,2) ) /S;
wsk = w * Rmat;
rICR = rm - inv(wsk) * rdm;

% acceleration
amat = rdd * fmat * rc';
wd = ( amat(2,1) - amat(1,2) ) /S;
wdsk = wd * Rmat;
beta = wdsk + wsk*wsk;
rIAP = rm - inv(beta) * rddm;

% jerk
jmat = rddd * fmat * rc';
wdd = w*w*w + ( jmat(2,1) - jmat(1,2) ) /S;
wddsk = wdd * Rmat;
eta = wddsk + 3*wsk*wdsk + wsk*wsk*wsk;
rIJP = rm - inv(eta) * rdddm;

% snap
%rddddm = sum( fmat*rdddd' )' /sf;
%smat = rdddd * fmat * rc';
%wddd = 6*w*w*wd + ( smat(2,1) - smat(1,2) ) /S;
%wdddsk = wddd * Rmat;
%sigma = wdddsk + 6*wsk*wsk*wdsk + 4*wsk*wddsk + 3*wdsk*wdsk + wsk*wsk*wsk*wsk;
%rISP = rm - inv(sigma) * rddddm;

% centrode
rdICR = ( wsk*rddm - beta*rdm ) /w/w;
nrdICR = norm( rdICR );

sk1 = [ 0  (w*wdd-2*wd*wd) ; -(w*wdd-2*wd*wd)  0 ];
```

```
sk2 = [ w*w*w  2*w*wd ; -2*w*wd  w*w*w ];
sk3 = [ 0  -w*w ; w*w  0 ];
rddICR = ( sk1*rdm + sk2*rddm + sk3*rdddm ) /w/w/w;

kappa = ( rdICR(1)*rddICR(2) - rdICR(2)*rddICR(1) ) /nrdICR/nrdICR/nrdICR;

% bottom of lm2kin2d
```