Area, Centroid and Area Moments for Polygonal Objects

Summations shown below are for closed CCW boundary sequences. CW boundary sequences will produce negative values for area and moments. Closed boundaries require $x_{n+1} = x_1$.

The term a_i is twice the signed area of the elementary triangle formed by (x_i,y_i) and (x_{i+1},y_{i+1}) and the origin.

For improved accuracy, a temporary local origin at the mean of the boundary points should be used.

$$a_i = x_i y_{i+1} - x_{i+1} y_i$$

$$A = \frac{1}{2}\sum_1^n a_i$$

$$x_c = \frac{1}{6A}\sum_1^n a_i\left(x_i + x_{i+1}\right)$$

$$y_c = \frac{1}{6A}\sum_1^n a_i\left(y_i + y_{i+1}\right)$$

$$I_{xx} = \frac{1}{12}\sum_1^n a_i\left(y_i^{\,2} + y_i y_{i+1} + y_{i+1}^{\,2}\right)$$

$$I_{yy} = \frac{1}{12}\sum_1^n a_i\left(x_i^{\,2} + x_i x_{i+1} + x_{i+1}^{\,2}\right)$$

$$I_{xy} = \frac{1}{24}\sum_1^n a_i\left(x_i y_{i+1} + 2x_i y_i + 2x_{i+1} y_{i+1} + x_{i+1} y_i\right)$$

$$\text{perimeter} = \sum_1^n \sqrt{\left(x_{i+1} - x_i\right)^2 + \left(y_{i+1} - y_i\right)^2}$$

centroidal moments

$$I_{uu} = I_{xx} - Ay_c^{\,2} \qquad I_{vv} = I_{yy} - Ax_c^{\,2} \qquad I_{uv} = I_{xy} - Ax_c y_c \qquad J = I_{uu} + I_{vv} = I_1 + I_2$$

principal moments

$$I_1, I_2 = \left(I_{uu} + I_{vv}\right)/2 \pm \sqrt{\left(I_{uu} + I_{vv}\right)^2/4 + I_{uv}^{\,2}} \qquad \tan 2\theta = 2I_{uv}/\left(I_{vv} - I_{uu}\right)$$

$$I_1, I_2 = \text{eig}\begin{bmatrix} I_{uu} & -I_{uv} \\ -I_{uv} & I_{vv} \end{bmatrix}$$

Object with Holes

1) Digitize the outline of the object in the CCW direction. Be certain to close the outline (i.e. the first and last points must be the same).

2) Digitize outlines of holes in the CW direction. Be certain to close the outlines (i.e. the first and last points must be the same).

3) Append the data strings. Remember to addend the first point in the outline to the end of each string.

4) Repeat steps 2) and 3) for multiple holes. For example, an object with three holes will contain a closed CCW outline for the object, followed by three closed CW outlines, one for each hole. Outlines for holes must be separated by the first point for the outline.

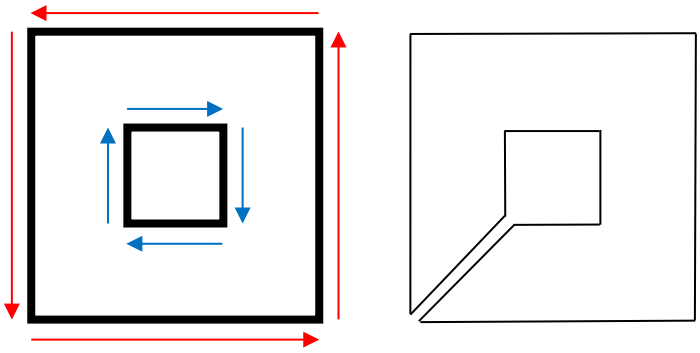
5) Area, centroid and moment computations will be correct. Perimeter will NOT be correct.

Sample data for figure at right

```
x_outline  =  [  0  3  3  0  0  ];
y_outline  =  [  0  0  3  3  0  ];

x_hole  =  [  1  1  2  2  1  ];
y_hole  =  [  1  2  2  1  1  ];

x  =  [ x_outline x_hole x_outline(1) ];
y  =  [ y_outline y_hole y_outline(1) ];
```



Sample data for three holes

```
x  =  [ x_outline  x_holeA  x_outline(1)  x_holeB  x_outline(1)  x_holeC  x_outline(1) ];
y  =  [ y_outline  y_holeA  y_outline(1)  y_holeB  y_outline(1)  y_holeC  y_outline(1) ];
```

```
% test_polygeom.m - test polygeom
%   area, centroid, perimeter and area moments of polygonal outline
% H.J. Sommer III - 16.12.09 - tested under MATLAB v9.0

clear

% constants
d2r = pi / 180;

% 3x5 test rectangle with long axis at 30 degrees
% area=15, x_cen=3.415, y_cen=6.549, perimeter=16
% I1=11.249, I2=31.247, J=42.496
x = [ 2.000  0.500  4.830  6.330 ]';
y = [ 4.000  6.598  9.098  6.500 ]';

% get geometry
[ geom, iner, cpmo ] = polygeom( x, y );

% show results
area = geom(1);
x_cen = geom(2);
y_cen = geom(3);
perimeter = geom(4);
disp( [ ' ' ] )
disp( [ '3x5 test rectangle with long axis at 30 degrees' ] )
disp( [ ' ' ] )
disp( [ '      area     x_cen     y_cen     perim' ] )
disp( [ area  x_cen  y_cen  perimeter ] )

I1 = cpmo(1);
angle1 = cpmo(2);
I2 = cpmo(3);
angle2 = cpmo(4);
disp( [ ' ' ] )
disp( [ '        I1        I2' ] )
disp( [ I1 I2 ] )
disp( [ '    angle1    angle2' ] )
disp( [ angle1/d2r angle2/d2r ] )

% plot outline
xplot = x( [ 1:end 1] );
yplot = y( [ 1:end 1] );
rad = 10;
x1 = [ x_cen-rad*cos(angle1)  x_cen+rad*cos(angle1) ];
y1 = [ y_cen-rad*sin(angle1)  y_cen+rad*sin(angle1) ];
x2 = [ x_cen-rad*cos(angle2)  x_cen+rad*cos(angle2) ];
y2 = [ y_cen-rad*sin(angle2)  y_cen+rad*sin(angle2) ];
plot( xplot,yplot,'b', x_cen,y_cen,'ro', ...
      x1,y1,'g:', x2,y2,'g:'  )
axis( [ 0  rad  0  rad ] )
axis square

% bottom of test_polygeom.m
```

```
function [ geom, iner, cpmo ] = polygeom( x, y )
%POLYGEOM Geometry of a planar polygon
%
%   POLYGEOM( X, Y ) returns area, X centroid,
%   Y centroid and perimeter for the planar polygon
%   specified by vertices in vectors X and Y.
%
%   [ GEOM, INER, CPMO ] = POLYGEOM( X, Y ) returns
%   area, centroid, perimeter and area moments of
%   inertia for the polygon.
%   GEOM = [ area   X_cen  Y_cen  perimeter ]
%   INER = [ Ixx    Iyy    Ixy    Iuu    Ivv    Iuv ]
%     u,v are centroidal axes parallel to x,y axes.
%   CPMO = [ I1     ang1   I2     ang2   J ]
%     I1,I2 are centroidal principal moments about axes
%         at angles ang1,ang2.
%     ang1 and ang2 are in radians.
%     J is centroidal polar moment.  J = I1 + I2 = Iuu + Ivv

% H.J. Sommer III - 16.12.09 - tested under MATLAB v9.0
%
% sample data
% x = [ 2.000  0.500  4.830  6.330 ]';
% y = [ 4.000  6.598  9.098  6.500 ]';
% 3x5 test rectangle with long axis at 30 degrees
% area=15, x_cen=3.415, y_cen=6.549, perimeter=16
% Ixx=659.561, Iyy=201.173, Ixy=344.117
% Iuu=16.249, Ivv=26.247, Iuv=8.660
% I1=11.249, ang1=30deg, I2=31.247, ang2=120deg, J=42.496
%
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% begin function POLYGEOM

% check if inputs are same size
if ~isequal( size(x), size(y) ),
  error( 'X and Y must be the same size');
end

% temporarily shift data to mean of vertices for improved accuracy
xm = mean(x);
ym = mean(y);
x = x - xm;
y = y - ym;

% summations for CCW boundary
xp = x( [2:end 1] );
yp = y( [2:end 1] );
a = x.*yp - xp.*y;

A = sum( a ) /2;
xc = sum( (x+xp).*a  ) /6/A;
yc = sum( (y+yp).*a  ) /6/A;
Ixx = sum( (y.*y +y.*yp + yp.*yp).*a  ) /12;
Iyy = sum( (x.*x +x.*xp + xp.*xp).*a  ) /12;
Ixy = sum( (x.*yp +2*x.*y +2*xp.*yp + xp.*y).*a  ) /24;

dx = xp - x;
dy = yp - y;
P = sum( sqrt( dx.*dx +dy.*dy ) );

% check for CCW versus CW boundary
if A < 0,
  A = -A;
  Ixx = -Ixx;
  Iyy = -Iyy;
  Ixy = -Ixy;
end
```

```
% centroidal moments
Iuu = Ixx - A*yc*yc;
Ivv = Iyy - A*xc*xc;
Iuv = Ixy - A*xc*yc;
J = Iuu + Ivv;

% replace mean of vertices
x_cen = xc + xm;
y_cen = yc + ym;
Ixx = Iuu + A*y_cen*y_cen;
Iyy = Ivv + A*x_cen*x_cen;
Ixy = Iuv + A*x_cen*y_cen;

% principal moments and orientation
I = [ Iuu  -Iuv ;
     -Iuv   Ivv ];
[ eig_vec, eig_val ] = eig(I);
I1 = eig_val(1,1);
I2 = eig_val(2,2);
ang1 = atan2( eig_vec(2,1), eig_vec(1,1) );
ang2 = atan2( eig_vec(2,2), eig_vec(1,2) );

% return values
geom = [ A  x_cen  y_cen  P ];
iner = [ Ixx  Iyy  Ixy  Iuu  Ivv  Iuv ];
cpmo = [ I1  ang1  I2  ang2  J ];

% bottom of polygeom
```