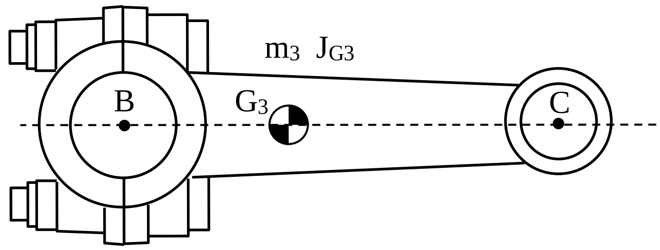
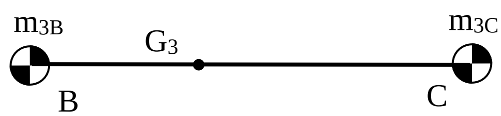
Two-mass Equivalent Link





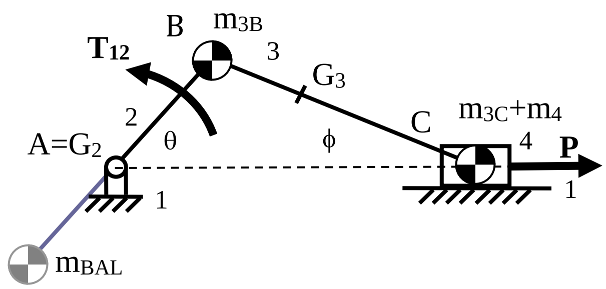
total mass $m_{3B} + m_{3C} = m_3$

centroid location $m_{3B} = m_3 \dfrac{CG_3}{BC}$ $\quad m_{3C} = m_3 \dfrac{BG_3}{BC}$

check approximate mass moment $J_{G3} \approx J_{G3\_APP} = m_{3B}\left(BG_3\right)^2 + m_{3C}\left(CG_3\right)^2$

(for slender rod $J_{G3\_APP} = 3\,J_{G3\_ACTUAL}$)

Shaking Force for Slider Crank



assume crank is statically balanced $G_2 = A$ constant crank speed $\ddot{\theta} = 0$

split link 3 into m_{3B} and m_{3C} assume $\ddot{\phi}$ is small

Case I - static force analysis for only d'Alembert inertial force P with no friction

$$P = -(m_{3C} + m_4)\ddot{s} \qquad \ddot{s} = -R\dot{\theta}^2\left(\cos\theta + \tfrac{R}{L}\cos 2\theta\right) \qquad \mu = 0 \qquad L\sin\phi = R\sin\theta$$

$$T_{1on2} = \frac{P\,R\,\sin(\theta+\phi)}{\cos\phi} \qquad F^X_{1on2} = -P \qquad F^Y_{1on2} = P\tan\phi \qquad F^X_{1on4} = 0 \qquad F^Y_{1on4} = -P\tan\phi$$

Case II - static force analysis for only inertial force $m_{3B}R\omega^2$

$$T_{1on2} = 0 \qquad F^X_{1on2} = -m_{3B}R\omega^2\cos\theta \qquad F^Y_{1on2} = -m_{3B}R\omega^2\sin\theta$$

Case III - place additional counterbalance m_{BAL} on the crank at radius R and 180º from pin B

Superposition = Case I + Case II + Case III

$$T_{1on2} = -\tfrac{1}{2}(m_{3C} + m_4)R^2\dot{\theta}^2\left(\tfrac{R}{2L}\sin\theta - \sin 2\theta - \tfrac{3R}{2L}\sin 3\theta\right)$$

$$F^x_{2on1} = \left(m_{3B} - m_{BAL}\right)R\dot{\theta}^2\cos\theta - \left(m_{3C} + m_4\right)\ddot{s}$$

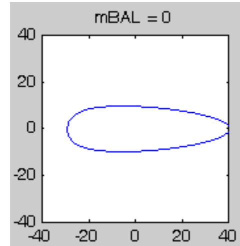
$$F^y_{2on1} = \left(m_{3B} - m_{BAL}\right)R\dot{\theta}^2\sin\theta + \left(m_{3C} + m_4\right)\ddot{s}\tan\phi$$

$$F^y_{4on1} = -\left(m_{3C} + m_4\right)\ddot{s}\tan\phi$$

$$\{F_{SHAKING}\} = \{F_{2on1}\} + \{F_{4on1}\} = R\dot{\theta}^2 \begin{Bmatrix} (m_{3B} - m_{BAL})\cos\theta + (m_{3C} + m_4)(\cos\theta + \frac{R}{L}\cos 2\theta) \\ (m_{3B} - m_{BAL})\sin\theta \end{Bmatrix}$$

Model 1 – no additional balancing mass

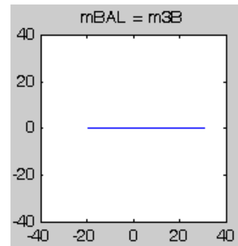
$m_{BAL} = 0$



Model 2 – unidirectional in-line shaking force

$$\{F_{SHAKING}\} = R\dot{\theta}^2 \begin{Bmatrix} (m_{3B} - m_{BAL})\cos\theta + (m_{3C} + m_4)(\cos\theta + \tfrac{R}{L}\cos 2\theta) \\ (m_{3B} - m_{BAL})\sin\theta \end{Bmatrix}$$

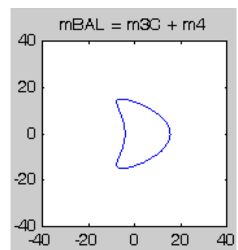
$$m_{BAL} = m_{3B} \qquad \{F_{SHAKING}\} = R\dot{\theta}^2 \begin{Bmatrix} (m_{3C} + m_4)(\cos\theta + \tfrac{R}{L}\cos 2\theta) \\ 0 \end{Bmatrix}$$



Model 3 – minimize the maximum magnitude of shaking force (approximate)

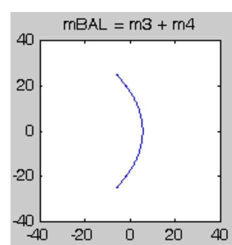
$$\{F_{SHAKING}\} = R\dot{\theta}^2 \begin{Bmatrix} (m_{3B} - m_{BAL})\cos\theta + (m_{3C} + m_4)(\cos\theta + \tfrac{R}{L}\cos 2\theta) \\ (m_{3B} - m_{BAL})\sin\theta \end{Bmatrix}$$

$$m_{BAL} = m_{3C} + m_4 \qquad \{F_{SHAKING}\} = R\dot{\theta}^2 \begin{Bmatrix} m_{3B}\cos\theta + (m_{3C} + m_4)\tfrac{R}{L}\cos 2\theta \\ (m_{3B} - m_{3C} - m_4)\sin\theta \end{Bmatrix}$$



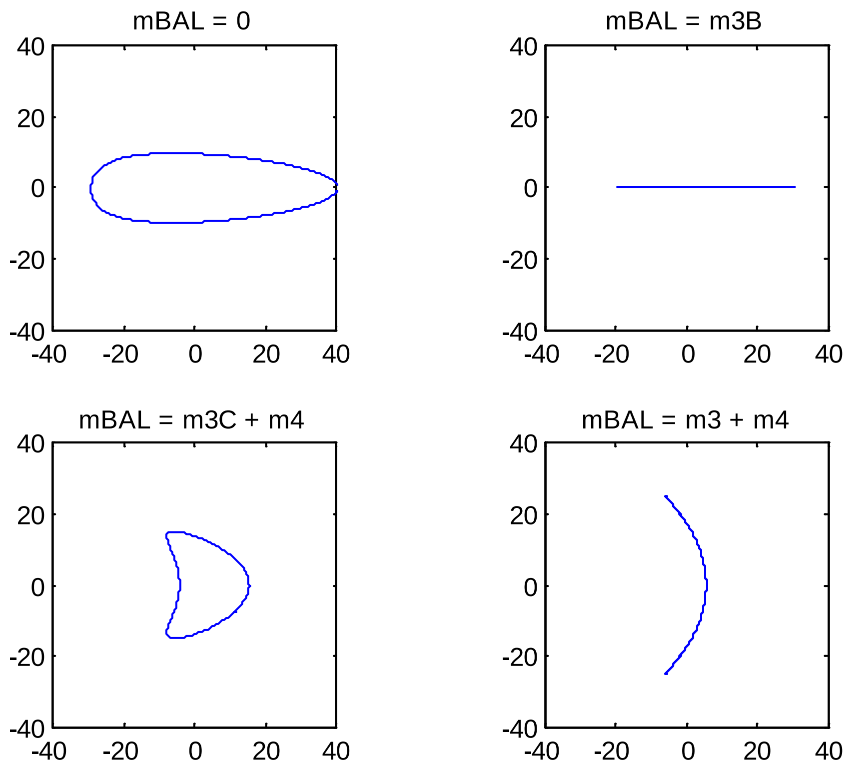
Model 4 – minimize in-line shaking force (approximate)

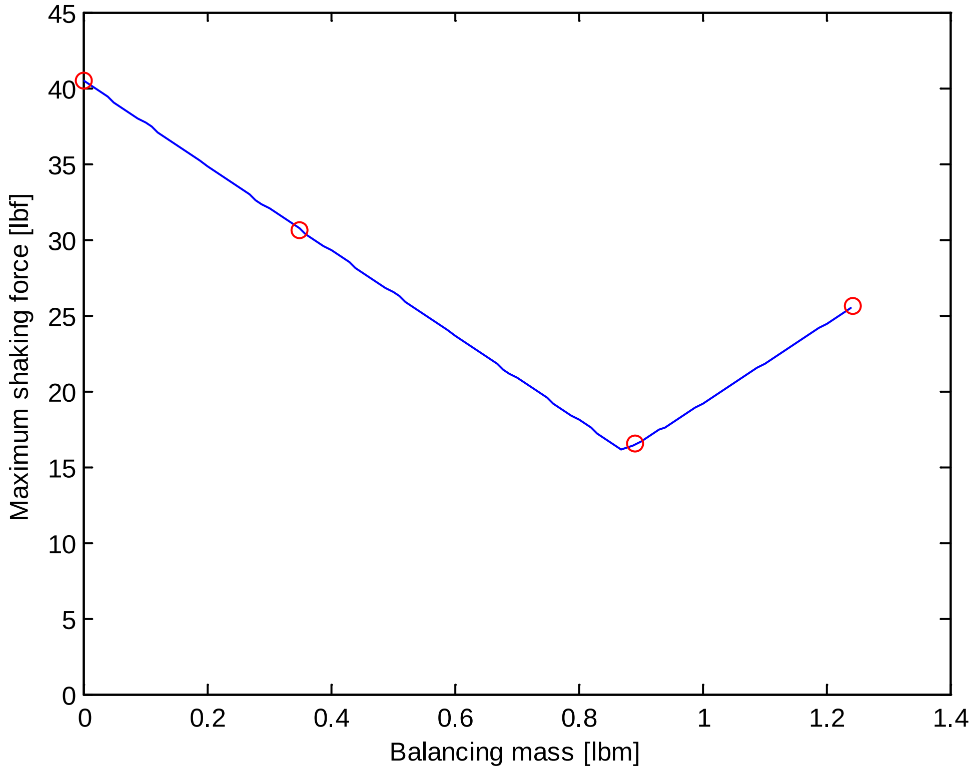
$$\{F_{SHAKING}\} = R\dot{\theta}^2 \begin{Bmatrix} (m_{3B} - m_{BAL})\cos\theta + (m_{3C} + m_4)(\cos\theta + \tfrac{R}{L}\cos 2\theta) \\ (m_{3B} - m_{BAL})\sin\theta \end{Bmatrix}$$



$$m_{BAL} = m_3 + m_4 = m_{3B} + m_{3C} + m_4$$

$$\{F_{SHAKING}\} = R\dot{\theta}^2 (m_{3C} + m_4) \begin{Bmatrix} \tfrac{R}{L}\cos 2\theta \\ -\sin\theta \end{Bmatrix}$$


mBAL = 0
mBAL = m3B
mBAL = m3C + m4
mBAL = m3 + m4
40
20
0
-20
-40
-40
-20
0
20
40


Maximum shaking force [lbf]
Balancing mass [lbm]
45
40
35
30
25
20
15
10
5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4

```
% shake.m - single cylinder shaking force Notes_08_04
% HJSIII, 13.03.14

clear

% constants
d2r = pi / 180;

% geometry [inches]
R = 0.985;
L = 4.33;

% crank speed [rad/sec]
w = 104.719;

% masses [lbm]
m3B = 0.351;
m3C = 0.111;
m4 = 0.781;

% crank angle
th_deg = (0:360)';
th = th_deg * d2r;

% piston acceleration [inch/s/s]
sdd = -R*w*w*(cos(th) +R*cos(2*th)/L);

% plot four models
figure( 1 )
  clf
  res = [];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Model 1 - mBAL = 0
  subplot( 2,2,1 )
  mBAL = 0;

% shaking force [lbm.in/s/s] * [lbf*s*s / 32.174 lbm.ft] * [ ft / 12 in ]
  Fx = ( (m3B-mBAL)*R*w*w*cos(th) - (m3C+m4)*sdd ) / 386;  % [lbf]
  Fy = ( (m3B-mBAL)*R*w*w*sin(th) ) / 386;
  Fs = sqrt( Fx.*Fx + Fy.*Fy );
  Fs_max = max( Fs );
  res = [ res ; mBAL  Fs_max ];

% plot shaking force curve
  plot( Fx, Fy )
  axis square
  axis( [ -40 40  -40 40 ] )
  title( 'mBAL = 0' )

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Model 2 - mBAL = m3B
  subplot( 2,2,2 )
  mBAL = m3B;

% shaking force [lbm.in/s/s] * [lbf*s*s / 32.174 lbm.ft] * [ ft / 12 in ]
  Fx = ( (m3B-mBAL)*R*w*w*cos(th) - (m3C+m4)*sdd ) / 386;  % [lbf]
  Fy = ( (m3B-mBAL)*R*w*w*sin(th) ) / 386;
  Fs = sqrt( Fx.*Fx + Fy.*Fy );
  Fs_max = max( Fs );
  res = [ res ; mBAL  Fs_max ];

% plot shaking force curve
  plot( Fx, Fy )
  axis square
  axis( [ -40 40  -40 40 ] )
  title( 'mBAL = m3B' )
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Model 3 - mBAL = m3C + m4
  subplot( 2,2,3 )
  mBAL = m3C + m4;

% shaking force [lbm.in/s/s] * [lbf*s*s / 32.174 lbm.ft] * [ ft / 12 in ]
  Fx = ( (m3B-mBAL)*R*w*w*cos(th) - (m3C+m4)*sdd ) / 386;  % [lbf]
  Fy = ( (m3B-mBAL)*R*w*w*sin(th) ) / 386;
  Fs = sqrt( Fx.*Fx + Fy.*Fy );
  Fs_max = max( Fs );
  res = [ res ; mBAL  Fs_max ];

% plot shaking force curve
  plot( Fx, Fy )
  axis square
  axis( [ -40 40  -40 40 ] )
  title( 'mBAL = m3C + m4' )

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Model 4 - mBAL = m3 + m4
  subplot( 2,2,4 )
  mBAL = m3B + m3C + m4;

% shaking force [lbm.in/s/s] * [lbf*s*s / 32.174 lbm.ft] * [ ft / 12 in ]
  Fx = ( (m3B-mBAL)*R*w*w*cos(th) - (m3C+m4)*sdd ) / 386;  % [lbf]
  Fy = ( (m3B-mBAL)*R*w*w*sin(th) ) / 386;
  Fs = sqrt( Fx.*Fx + Fy.*Fy );
  Fs_max = max( Fs );
  res = [ res ; mBAL  Fs_max ];

% plot shaking force curve
  plot( Fx, Fy )
  axis square
  axis( [ -40 40  -40 40 ] )
  title( 'mBAL = m3 + m4' )

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% plot maximum shaking force versus balancing mass
keep = [];
for mBAL = 0 : 0.01 : (m3B+m3C+m4),

% shaking force [lbm.in/s/s] * [lbf*s*s / 32.174 lbm.ft] * [ ft / 12 in ]
  Fx = ( (m3B-mBAL)*R*w*w*cos(th) - (m3C+m4)*sdd ) / 386;  % [lbf]
  Fy = ( (m3B-mBAL)*R*w*w*sin(th) ) / 386;
  Fs = sqrt( Fx.*Fx + Fy.*Fy );
  Fs_max = max( Fs );
  keep = [ keep ; mBAL  Fs_max ];
end

% plot results
figure( 2 )
  clf
  plot( keep(:,1),keep(:,2),'b', res(:,1),res(:,2),'ro' )
  axis( [ 0 1.4  0 45 ] )
  xlabel( 'Balancing mass [lbm]' )
  ylabel( 'Maximum shaking force [lbf]' )

% bottom of shake
```

Model 5 – minimize the maximum magnitude of shaking force (exact) does not work ?????

$$F_{SHAKING}^2 = R^2\dot{\theta}^4\left((m_{3B} - m_{BAL})^2 + 2(m_{3B} - m_{BAL})(m_{3C} + m_4)\cos\theta\left(\cos\theta + \tfrac{R}{L}\cos 2\theta\right) + (m_{3C} + m_4)^2\left(\cos\theta + \tfrac{R}{L}\cos 2\theta\right)^2 \right)$$

$$\frac{\partial\left(F_{SHAKING}^2\right)}{\partial m_{BAL}} = R^2\dot{\theta}^4\left(-2(m_{3B} - m_{BAL}) - 2(m_{3C} + m_4)\cos\theta\left(\cos\theta + \tfrac{R}{L}\cos 2\theta\right) \right) = 0$$

$$m_{BAL} = m_{3B} + (m_{3C} + m_4)\left(\cos^2\theta + \tfrac{R}{L}\cos\theta\cos 2\theta\right)$$

$$m_{BAL} = m_{3B} + (m_{3C} + m_4)\left(\cos^2\theta + \tfrac{R}{L}\cos^3\theta - \tfrac{R}{L}\cos\theta\sin^2\theta\right)$$

$$m_{BAL} = m_{3B} + (m_{3C} + m_4)\left(2\tfrac{R}{L}\cos^3\theta + \cos^2\theta - \tfrac{R}{L}\cos\theta\right)$$

$$\partial\left(2\tfrac{R}{L}\cos^3\theta + \cos^2\theta - \tfrac{R}{L}\cos\theta\right)/\partial\theta = -6\tfrac{R}{L}\cos^2\theta\sin\theta - 2\cos\theta\sin\theta + \tfrac{R}{L}\sin\theta = 0$$

$$\sin\theta = 0,\ \theta = 0^\circ \qquad m_{BAL} = m_{3B} + (m_{3C} + m_4)\left(1 + \tfrac{R}{L}\right)$$

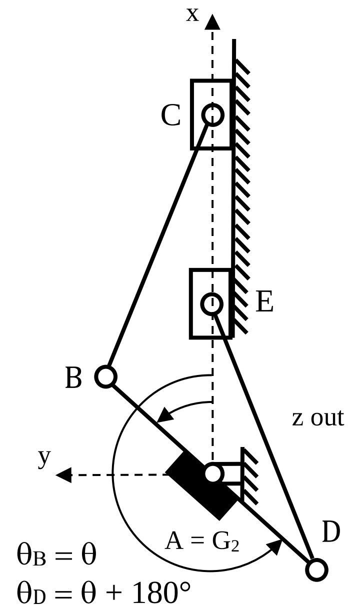
$$\sin\theta = 0,\ \theta = 180^\circ \qquad m_{BAL} = m_{3B} + (m_{3C} + m_4)\left(1 - \tfrac{R}{L}\right)$$

$$\cos\theta = \frac{-1 \pm \sqrt{1 + 6\left(\tfrac{R}{L}\right)^2}}{6\tfrac{R}{L}} \qquad m_{BAL} = m_{3B} + (m_{3C} + m_4)\left(2\tfrac{R}{L}\cos^3\theta + \cos^2\theta - \tfrac{R}{L}\cos\theta\right)$$

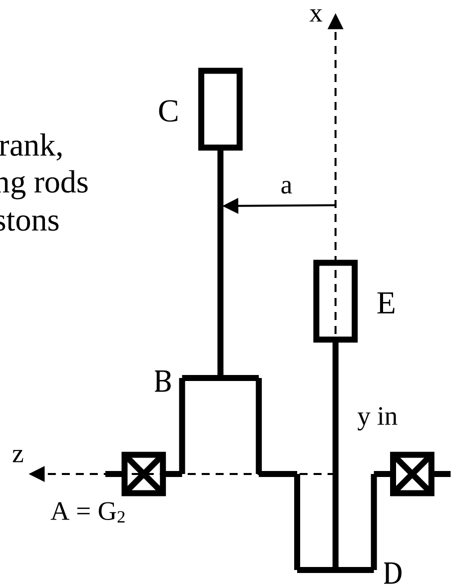
```
% model5.m - min/max shaking force for slider crank
% does not work ???????????????
% HJSIII, 13.03.14

clear

% geometry [inches]
R = 0.985;
L = 4.33;

% masses [lbm]
m3B = 0.351;
m3C = 0.111;
m4 = 0.781;

% cosine solution
rho = R / L;
c1 = (-1 + sqrt( 1 + 6*rho*rho ) ) /6 /rho;
c2 = (-1 - sqrt( 1 + 6*rho*rho ) ) /6 /rho;

% values
v1 = 2*rho*c1*c1*c1 + c1*c1 - rho*c1;
v2 = 2*rho*c2*c2*c2 + c2*c2 - rho*c2;
```

```
% balancing masses
mBAL1 = m3B + (m3C+m4)*v1
mBAL2 = m3B + (m3C+m4)*v2
```

Shaking Force for In-line Two Cylinder Air Compressor



same crank, connecting rods and pistons



only ABC $$\{F_{SHAKING}\} = R\dot{\theta}^2 \begin{Bmatrix} (m_{3B} - m_{BAL})\cos\theta \\ +(m_{3C} + m_4)\cos\theta \\ +(m_{3C} + m_4)\frac{R}{L}\cos 2\theta \\ \\ (m_{3B} - m_{BAL})\sin\theta \end{Bmatrix}$$

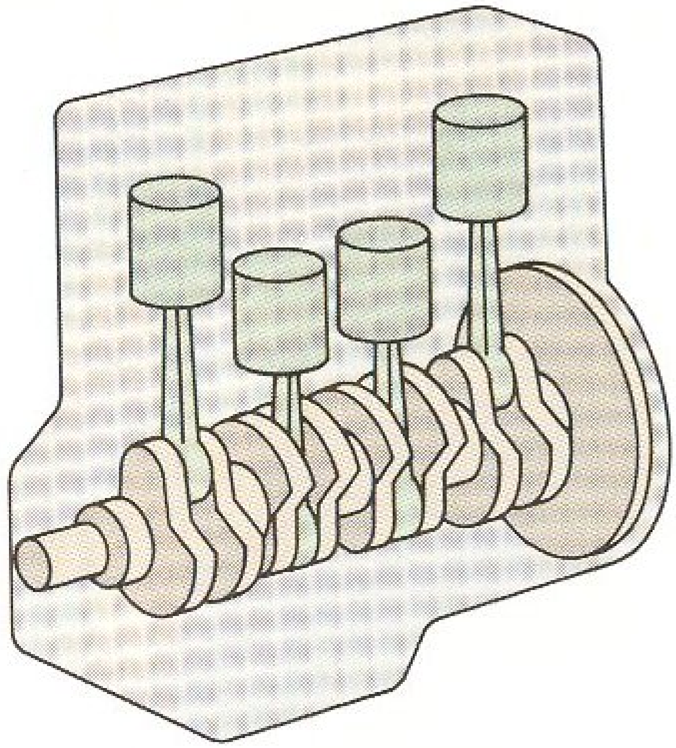
both $$\{F_{SHAKING}\} = R\dot{\theta}^2 \begin{Bmatrix} (m_{3B} - m_{BAL})(\cos\theta + \cos(\theta + 180°)) \\ +(m_{3C} + m_4)(\cos\theta + \cos(\theta + 180°)) \\ +(m_{3C} + m_4)\frac{R}{L}(\cos 2\theta + \cos(2\theta + 360°)) \\ \\ (m_{3B} - m_{BAL})(\sin\theta + \sin(\theta + 180°)) \end{Bmatrix}$$

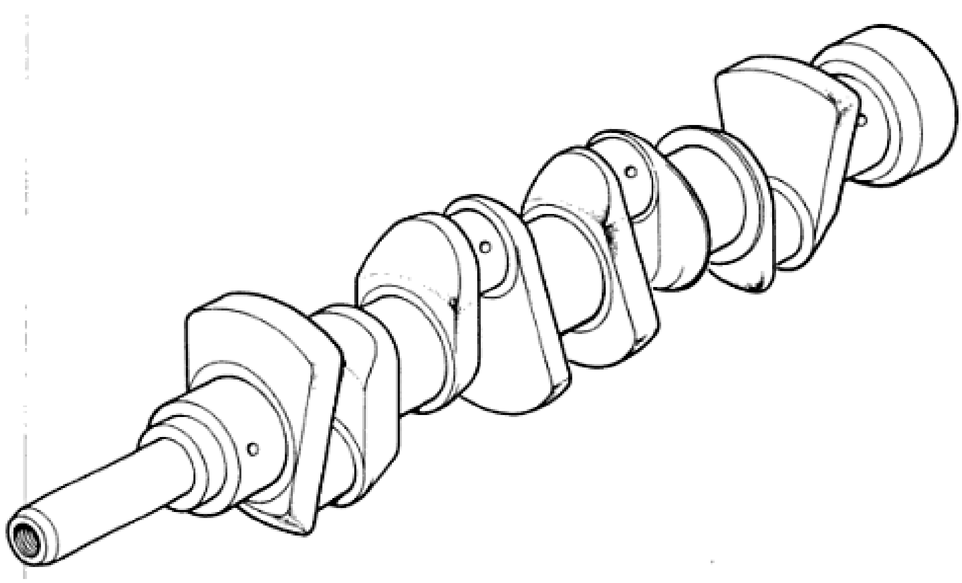
$$\cos\theta + \cos(\theta + 180°) = \cos\theta + \cos\theta\cos 180° - \sin\theta\sin 180° = 0$$
$$\cos 2\theta + \cos(2\theta + 360°) = 2\cos 2\theta$$
$$\sin\theta + \sin(\theta + 180°) = \sin\theta + \sin\theta\cos 180° + \cos\theta\sin 180° = 0$$

both $$\{F_{SHAKING}\} = R\dot{\theta}^2 \begin{Bmatrix} 2\frac{R}{L}(m_{3C} + m_4)\cos 2\theta \\ 0 \end{Bmatrix}$$

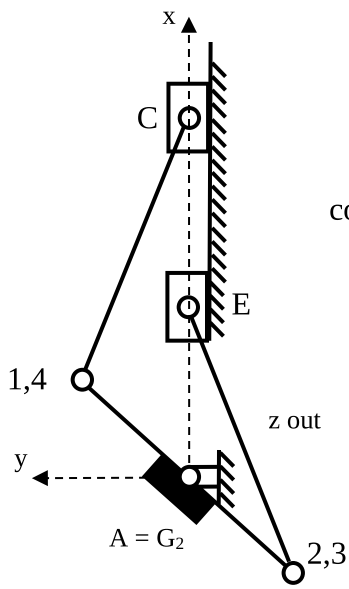
shaking moments $M^X_{SHAKING} = -a\left(F^Y_{2on1} + F^Y_{4on1}\right)$ $M^Y_{SHAKING} = +a\ F^X_{2on1}$

In-line Four Cylinder Engine

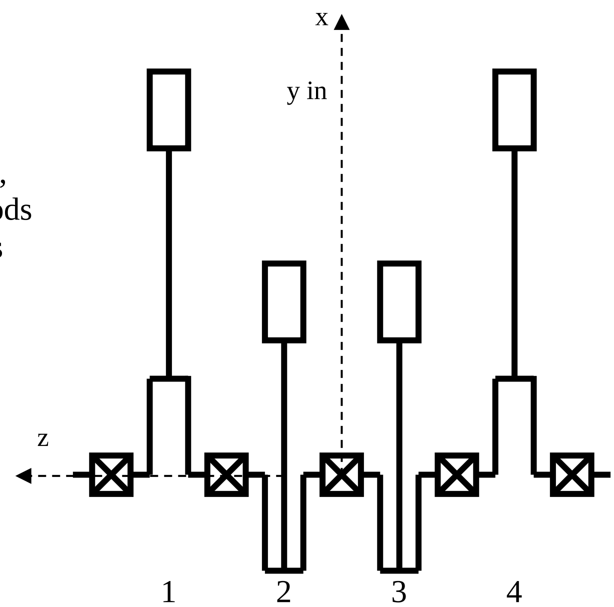




Shaking Force for In-line Four Cylinder Engine



same crank, connecting rods and pistons



$\theta_1 = \theta_4 = \theta$

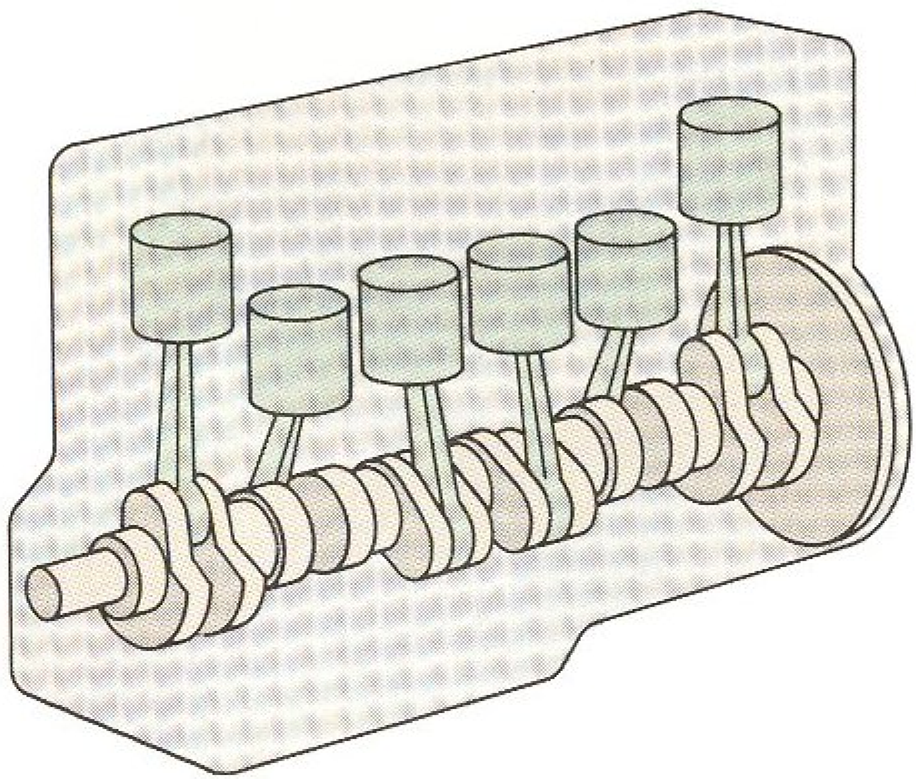
$\theta_2 = \theta_3 = \theta + 180°$

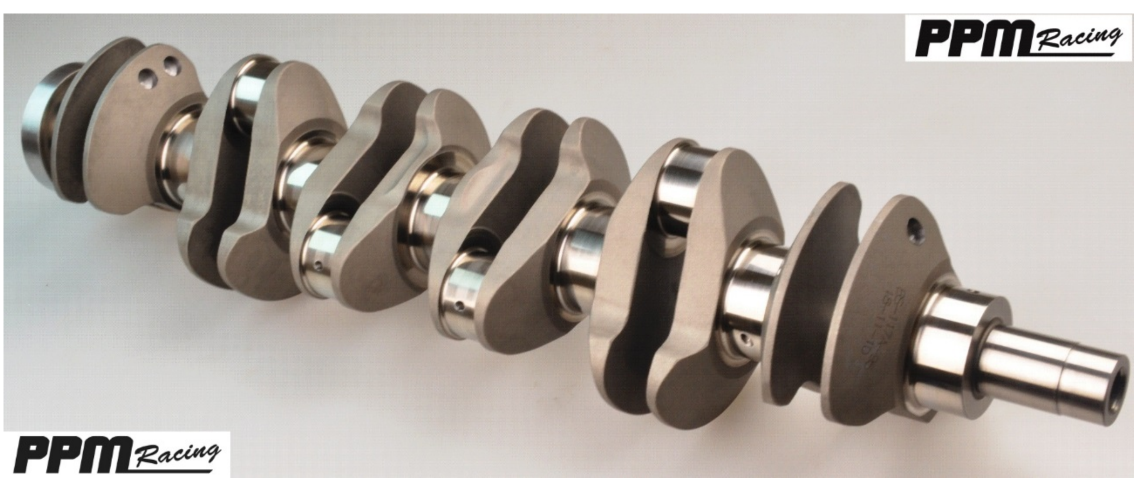
same as two two-cylinder compressors mirrored end to end

$$\{F_{SHAKING}\} = R\dot{\theta}^2 \begin{Bmatrix} 4\frac{R}{L}(m_{3C} + m_4)\cos 2\theta \\ 0 \end{Bmatrix}$$

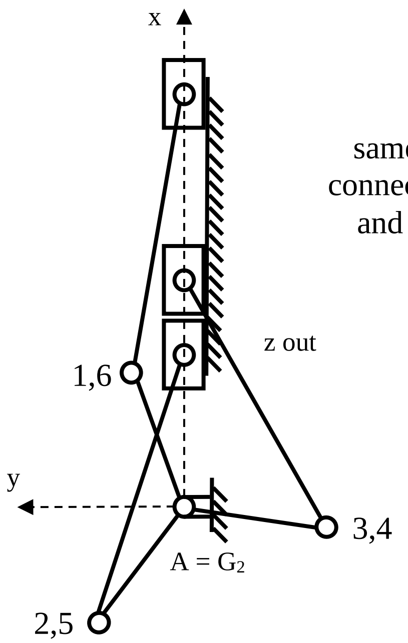
shaking moments $\quad M^X_{SHAKING} = 0 \quad M^Y_{SHAKING} = 0$

In-line Six Cylinder Engine

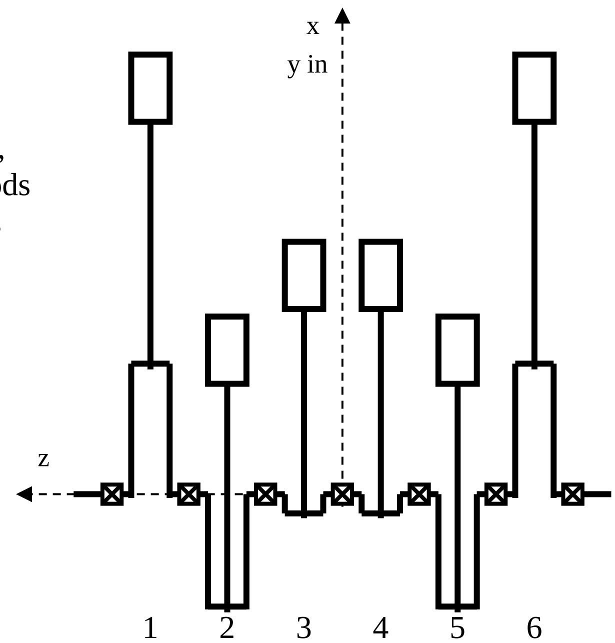




Shaking Force for In-line Six Cylinder Engine



same crank,
connecting rods
and pistons



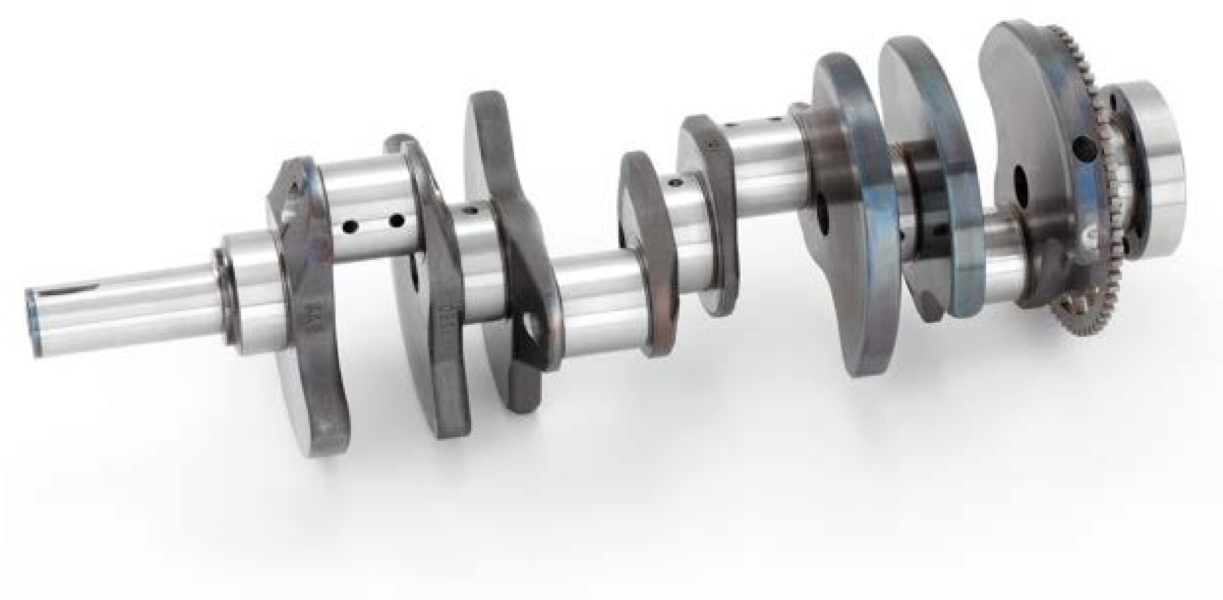
$$\theta_1 = \theta_6 = \theta$$
$$\theta_2 = \theta_5 = \theta + 120°$$
$$\theta_3 = \theta_4 = \theta + 240°$$

$$\cos\theta + \cos(\theta + 120°) + \cos(\theta + 240°) = 0$$
$$\cos 2\theta + \cos(2\theta + 240°) + \cos(2\theta + 480°) = 0$$
$$\sin\theta + \sin(\theta + 120°) + \sin(\theta + 240°) = 0$$

$$\{F_{SHAKING}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad M^{X}_{SHAKING} = 0 \qquad M^{Y}_{SHAKING} = 0$$

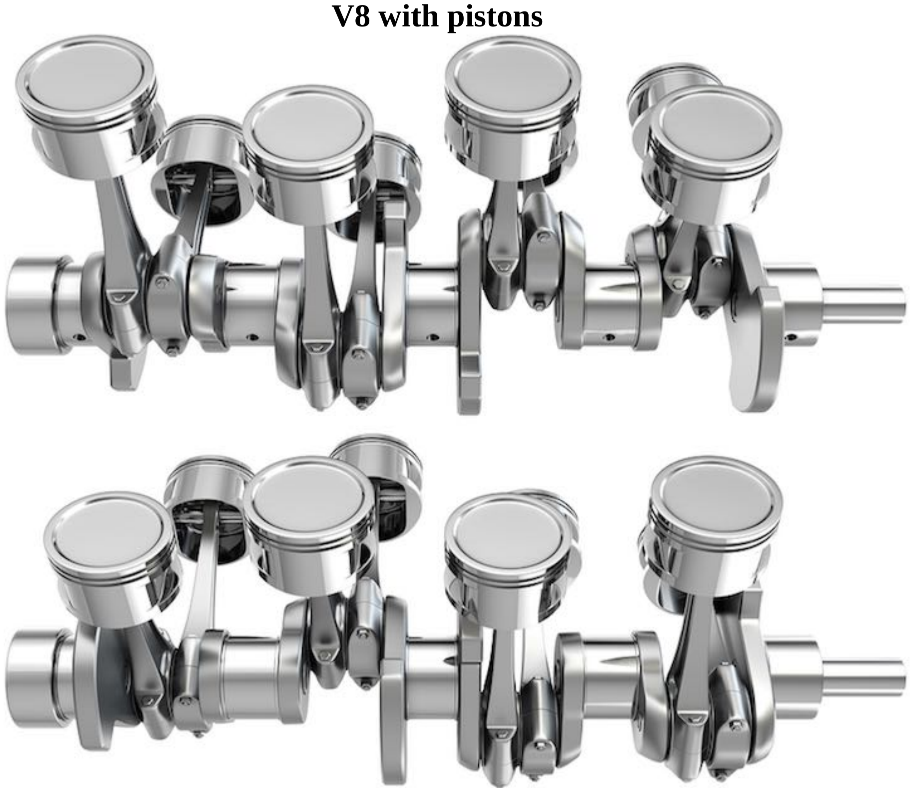
V8 separate cranks

V8 dual cranks

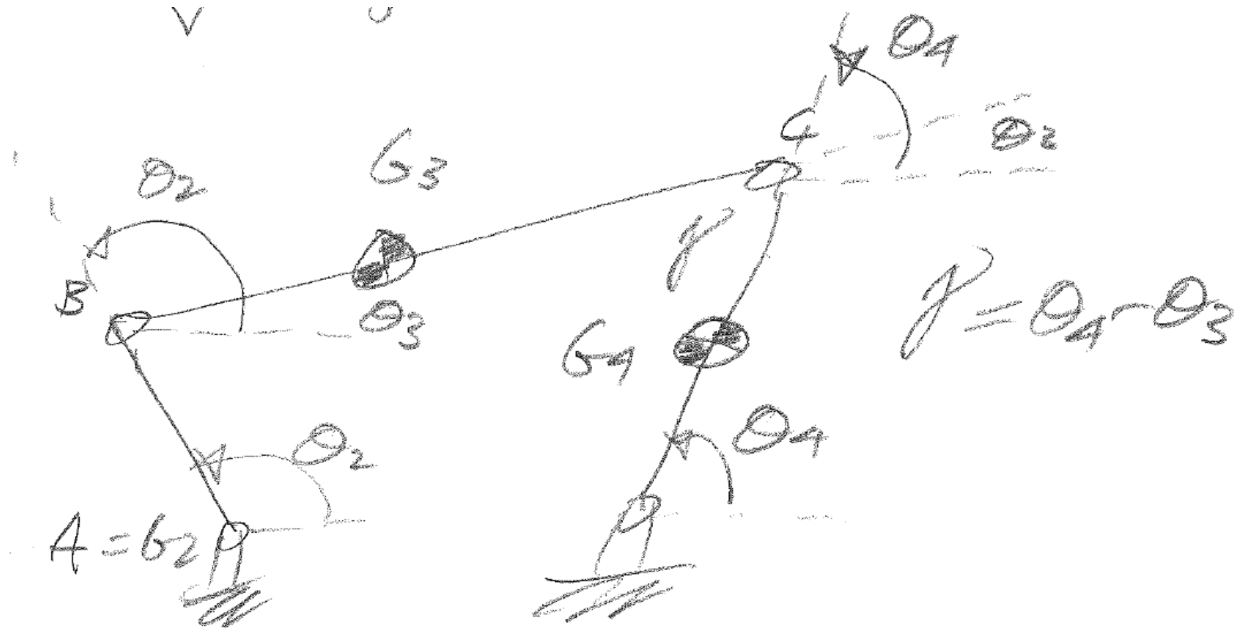


V8 split cranks

V8 with pistons



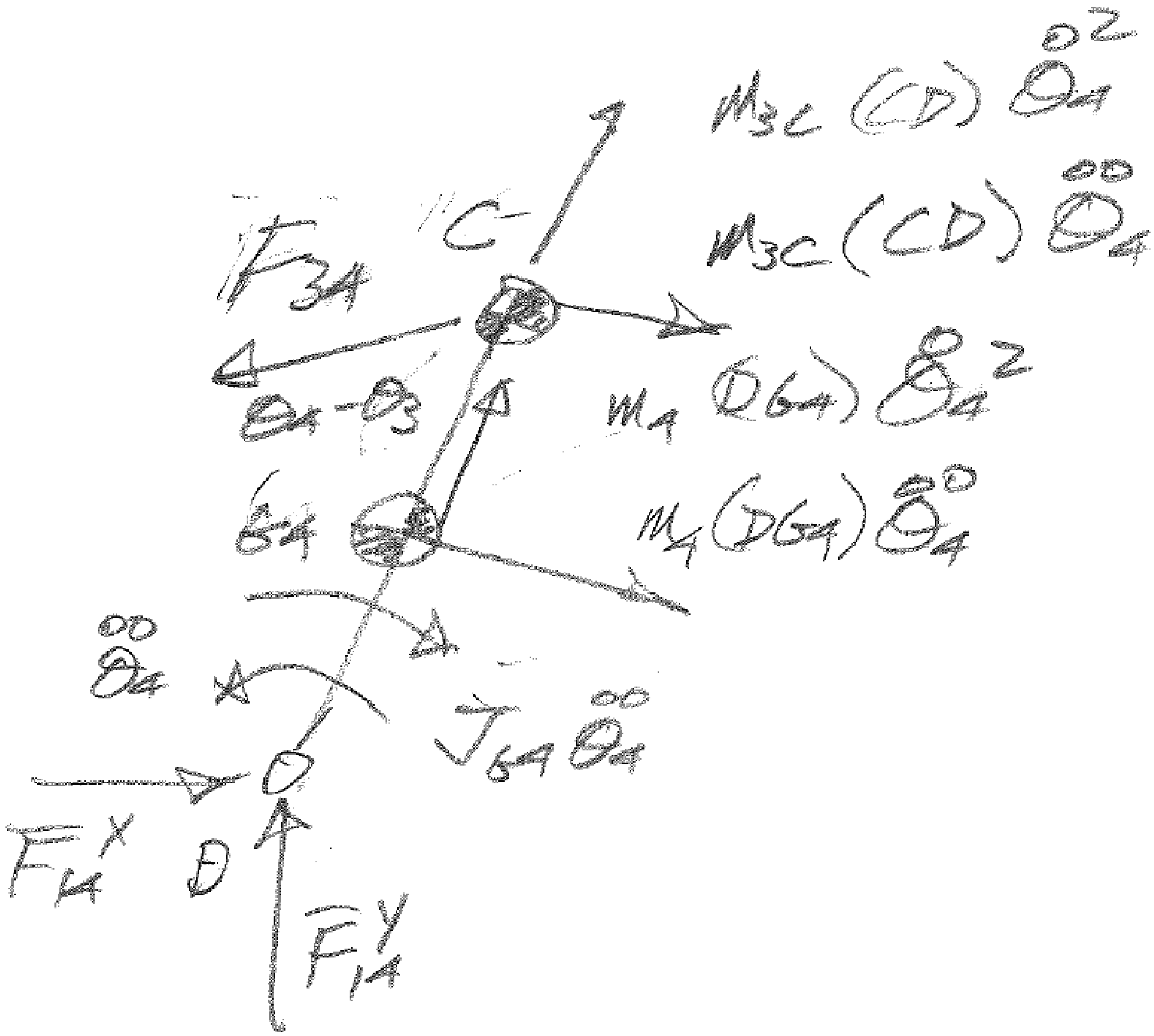
Shaking Force for Four Bar



assume crank is statically balanced $G_2 = A$

split link 3 into m_{3B} and m_{3C} assume $\ddot{\theta}_3$ is small link 3 becomes two-force member

static force analysis with d'Alembert inertial forces

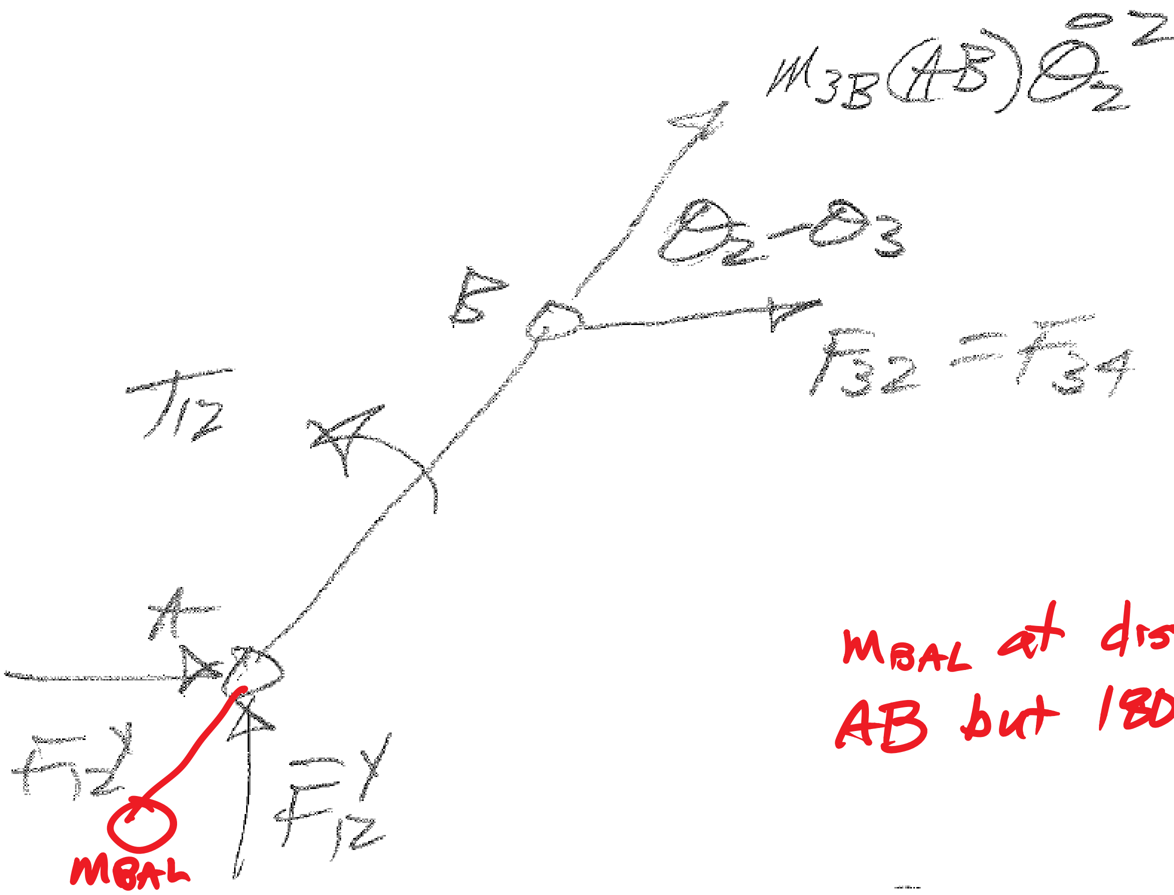


ΣM on 4 about D CCW + $\quad \left(F_{3on4}\sin\gamma\right)CD = \ddot{\theta}_4\left(J_{G4} + m_4\left(DG_4\right)^2 + m_{3C}\left(CD\right)^2\right)$

$$F_{3on4} = \ddot{\theta}_4\left(J_{G4} + m_4\left(DG_4\right)^2 + m_{3C}\left(CD\right)^2\right)/\,CD\sin\left(\theta_4 - \theta_3\right)$$

$$F^x_{4on1} = \left(m_4\left(DG_4\right) + m_{3C}\left(CD\right)\right)\dot{\theta}_4^2\cos\theta_4 + \left(m_4\left(DG_4\right) + m_{3C}\left(CD\right)\right)\ddot{\theta}_4\sin\theta_4 - F_{3on4}\cos\theta_3$$

$$F^y_{4on1} = \left(m_4\left(DG_4\right) + m_{3C}\left(CD\right)\right)\dot{\theta}_4^2\sin\theta_4 - \left(m_4\left(DG_4\right) + m_{3C}\left(CD\right)\right)\ddot{\theta}_4\cos\theta_4 - F_{3on4}\sin\theta_3$$



MBAL at distance AB but 180° from B

ΣM on 2 about A CCW + $T_{1\_on\_2}-(F_{3on2}\sin(\theta_2-\theta_3))AB=0$

$$T_{1on2}=\ddot{\theta}_4\left(J_{G4}+m_4(DG_4)^2+m_{3C}(CD)^2\right)\frac{AB\sin(\theta_2-\theta_3)}{CD\sin(\theta_4-\theta_3)}$$

$$T_{1on2}=\ddot{\theta}_4\left(J_{G4}+m_4(DG_4)^2+m_{3C}(CD)^2\right)\frac{\dot{\theta}_4}{\dot{\theta}_2}$$ from Notes_03_02

$\dot{\theta}_4$ and $\ddot{\theta}_4$ from Notes_03_02

$$F_{2on1}^x=(m_{3B}-m_{BAL})(AB)\dot{\theta}_2^2\cos\theta_2+F_{3on2}\cos\theta_3$$

$$F_{2on1}^y=(m_{3B}-m_{BAL})(AB)\dot{\theta}_2^2\sin\theta_2+F_{3on2}\sin\theta_3$$

$$\left\{F_{SHAKING}\right\} = \begin{Bmatrix} F_{2on1}^{x} + F_{4on1}^{x} \\ F_{2on1}^{y} + F_{4on1}^{y} \end{Bmatrix}$$

$$\left\{F_{SHAKING}\right\} = \begin{bmatrix} \cos\theta_2 & \sin\theta_4 & \cos\theta_4 \\ \sin\theta_2 & -\cos\theta_4 & \sin\theta_4 \end{bmatrix} \begin{Bmatrix} \left(m_{3B} - m_{BAL}\right)\left(AB\right)\dot{\theta}_2^2 \\ \left(m_4\left(DG_4\right) + m_{3C}\left(CD\right)\right)\ddot{\theta}_4 \\ \left(m_4\left(DG_4\right) + m_{3C}\left(CD\right)\right)\dot{\theta}_4^2 \end{Bmatrix}$$