Two-Dimensional Inverse Dynamics

Kinematically driven

Must use centroidal coordinate frames !

$$\{q_i\} = \begin{Bmatrix} \{r_i\} \\ \phi_i \end{Bmatrix}$$

$$\{\ddot{q}_i\} = \begin{Bmatrix} \{\ddot{r}_i\} \\ \ddot{\phi}_i \end{Bmatrix}$$

$$\{Q_{on\ i}\} = \begin{Bmatrix} \{F_{on\ i}\} \\ T_{on\ i} \end{Bmatrix}$$

$$[M_i] = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & J_{Gi} \end{bmatrix}$$

Single body

$$[M_i]\{\ddot{q}_i\} = \sum\left(\{Q_{on\ i}\}_{ALL}\right)$$

$\{Q_{on\ i}\}_{ALL}$ $\{Q_{on\ i}\}_{APPLIED}$ $\{Q_{on\ i}\}_{CONSTRAINT}$

System of multiple bodies

$$\{q\} = \begin{Bmatrix} \{q_2\} \\ \{q_3\} \\ \{q_4\} \end{Bmatrix}$$

$$\{\ddot{q}\} = \begin{Bmatrix} \{\ddot{q}_2\} \\ \{\ddot{q}_3\} \\ \{\ddot{q}_4\} \end{Bmatrix}$$

$$[M] = \begin{bmatrix} [M_2] & [0_{3x3}] & [0_{3x3}] \\ [0_{3x3}] & [M_3] & [0_{3x3}] \\ [0_{3x3}] & [0_{3x3}] & [M_4] \end{bmatrix}$$

$$\begin{bmatrix} [M_2] & [0_{3x3}] & [0_{3x3}] \\ [0_{3x3}] & [M_3] & [0_{3x3}] \\ [0_{3x3}] & [0_{3x3}] & [M_4] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}_2\} \\ \{\ddot{q}_3\} \\ \{\ddot{q}_4\} \end{Bmatrix} = \begin{Bmatrix} \sum\left(\{Q_{on\ 2}\}_{ALL}\right) \\ \sum\left(\{Q_{on\ 3}\}_{ALL}\right) \\ \sum\left(\{Q_{on\ 4}\}_{ALL}\right) \end{Bmatrix}$$

$$\{Q\}_{ALL} = \begin{Bmatrix} \sum\left(\{Q_{on\ 2}\}_{ALL}\right) \\ \sum\left(\{Q_{on\ 3}\}_{ALL}\right) \\ \sum\left(\{Q_{on\ 4}\}_{ALL}\right) \end{Bmatrix}$$

$$[M]\{\ddot{q}\} = \{Q\}_{ALL}$$

$$\{Q\}_{ALL} = \{Q\}_{APPLIED} + \{Q\}_{CONSTRAINT}$$

$$\{\Phi\}_{CONSTRAINT} = \begin{Bmatrix} \{\Phi\}_{KINEMATIC} \\ \{\Phi\}_{DRIVER} \end{Bmatrix}$$

$$\{Q\}_{CONSTRAINT} \qquad \{Q\}_{KINEMATIC} \qquad \{Q\}_{DRIVER}$$

Virtual work

$$[M]\{\ddot{q}\}-\{Q\}_{ALL}=\{0\}$$

$$\{\dot{q}\}^T([M]\{\ddot{q}\}-\{Q\}_{ALL})=0$$

$$\{Q\}_{ALL}=\{Q\}_{APPLIED}+\{Q\}_{CONSTRAINT}$$

$\{\dot{q}\}^T\{Q\}_{CONSTRAINT}=0$ for kinematic consistency $[\Phi_q]\{\dot{q}\}=\{0\}$

$\{\dot{q}\}^T([M]\{\ddot{q}\}-\{Q\}_{APPLIED})=0$ subject to $[\Phi_q]\{\dot{q}\}=\{0\}$

Virtual work for one revolute

$$\{\dot{q}\}^T\{Q\}_{CONSTRAINT}=?\quad 0$$

$$\{\dot{q}_i\}=\begin{Bmatrix}\{\dot{r}_i\}\\ \dot{\phi}_i\end{Bmatrix}\qquad \{\dot{q}_j\}=\begin{Bmatrix}\{\dot{r}_j\}\\ \dot{\phi}_j\end{Bmatrix}$$

$$\{F_{on\ i}\}_{REV}^{P}=-\{F_{on\ j}\}_{REV}^{P}$$

$$\{Q_{on\ i}\}_{REV}^{P}=\begin{Bmatrix}\{F_{on\ i}\}_{REV}^{P}\\ ([B_i]\{s_i\}'^{P})^T\{F_{on\ i}\}_{REV}^{P}\end{Bmatrix}\qquad \{Q_{on\ j}\}_{REV}^{P}=\begin{Bmatrix}\{F_{on\ j}\}_{REV}^{P}\\ ([B_j]\{s_j\}'^{P})^T\{F_{on\ j}\}_{REV}^{P}\end{Bmatrix}$$

$$\{\dot{q}_i\}^T\{Q_{on\ i}\}_{REV}^{P}+\{\dot{q}_j\}^T\{Q_{on\ j}\}_{REV}^{P}=?\quad 0$$

$$\begin{Bmatrix}\{\dot{r}_i\}\\ \dot{\phi}_i\end{Bmatrix}^T\begin{Bmatrix}\{F_{on\ i}\}_{REV}^{P}\\ ([B_i]\{s_i\}'^{P})^T\{F_{on\ i}\}_{REV}^{P}\end{Bmatrix}+\begin{Bmatrix}\{\dot{r}_j\}\\ \dot{\phi}_j\end{Bmatrix}^T\begin{Bmatrix}\{F_{on\ j}\}_{REV}^{P}\\ ([B_j]\{s_j\}'^{P})^T\{F_{on\ j}\}_{REV}^{P}\end{Bmatrix}=?\quad 0$$

$$\{\dot{r}_i\}^T\{F_{on\ i}\}_{REV}^{P}+(\dot{\phi}_i[B_i]\{s_i\}'^{P})^T\{F_{on\ i}\}_{REV}^{P}+\{\dot{r}_j\}^T\{F_{on\ j}\}_{REV}^{P}+(\dot{\phi}_j[B_j]\{s_j\}'^{P})^T\{F_{on\ j}\}_{REV}^{P}=?\quad 0$$

$$\{\dot{r}_i\}^P=\{\dot{r}_i\}+\dot{\phi}_i[B_i]\{s_i\}'^{P}\qquad \{\dot{r}_j\}^P=\{\dot{r}_j\}+\dot{\phi}_j[B_j]\{s_j\}'^{P}\qquad \{\dot{r}_i\}^P=\{\dot{r}_j\}^P$$

$(\{\dot{r}_i\}^P)^T\{F_{on\ i}\}_{REV}^{P}+(\{\dot{r}_j\}^P)^T\{F_{on\ j}\}_{REV}^{P}=?\quad 0$ **OK**

Lagrange multiplier theorem

general problem $\{b\}^T\{x\}=0$ subject to $[A]\{x\}=\{0\}$

$\{b\}^T\{x\}+\{\lambda\}^T[A]\{x\}=0$ using Lagrange multipliers $\{\lambda\}$ for any arbitrary $\{x\}$

virtual work $\{\dot{q}\}^T([M]\{\ddot{q}\}-\{Q\}_{APPLIED})=0$ subject to $[\Phi_q]\{\dot{q}\}=\{0\}$

$\{x\}=\{\dot{q}\}$ $\{b\}=[M]\{\ddot{q}\}-\{Q\}_{APPLIED}$ $[A]=[\Phi_q]$

$([M]\{\ddot{q}\}-\{Q\}_{APPLIED})^T\{\dot{q}\}+\{\lambda\}^T[\Phi_q]\{\dot{q}\}=0$ for arbitrary size but kinematically consistent $\{\dot{q}\}$

$([M]\{\ddot{q}\}-\{Q\}_{APPLIED}+[\Phi_q]^T\{\lambda\})^T\{\dot{q}\}=0$

$[M]\{\ddot{q}\}+[\Phi_q]^T\{\lambda\}=\{Q\}_{APPLIED}$

$[\Phi_q]^T\{\lambda\}$ each row in $\{\lambda\}$ is multiplied times corresponding column in $[\Phi_q]^T$

each row in $\{\lambda\}$ corresponds to matching row in $[\Phi_q]$ and $\{\Phi\}$

Lagrange multipliers

$[M]\{\ddot{q}\}+[\Phi_q]^T\{\lambda\}=\{Q\}_{APPLIED}$

$[M]\{\ddot{q}\}=\{Q\}_{APPLIED}-[\Phi_q]^T\{\lambda\}$

$[M]\{\ddot{q}\}=\{Q\}_{ALL}=\{Q\}_{APPLIED}+\{Q\}_{CONSTRAINTS}$

$\{Q\}_{CONSTRAINTS}=-[\Phi_q]^T\{\lambda\}$

$$\{Q\}_{ALL}=\begin{Bmatrix}\sum\left(\{Q_{on\,2}\}_{ALL}\right)\\ \sum\left(\{Q_{on\,3}\}_{ALL}\right)\\ \sum\left(\{Q_{on\,4}\}_{ALL}\right)\end{Bmatrix} \qquad \{Q\}_{CONSTRAINTS}=\begin{Bmatrix}\sum\left(\{Q_{on\,2}\}_{CONSTRAINTS}\right)\\ \sum\left(\{Q_{on\,3}\}_{CONSTRAINTS}\right)\\ \sum\left(\{Q_{on\,4}\}_{CONSTRAINTS}\right)\end{Bmatrix}$$

Equations of motion (EOM)

$$[M]\{\ddot{q}\} + [\Phi_q]^T\{\lambda\} = \{Q\}_{APPLIED}$$

$$[\Phi_q]\{\ddot{q}\} = \{\gamma\}$$

$$\underset{nq\ x\ 1}{\{q\}} \quad \underset{nq\ x\ 1}{\{\ddot{q}\}} \quad \underset{nq\ x\ nq}{[M]} \quad \underset{nq\ x\ 1}{\{Q\}_{APPLIED}} \quad \underset{nc\ x\ 1}{\{\Phi\}} \quad \underset{nc\ x\ 1}{\{\gamma\}} \quad \underset{nc\ x\ 1}{\{\lambda\}} \quad \underset{nc\ x\ nq}{[\Phi_q]}$$

nq = number of generalized coordinates
nk = number of kinematic constraints
nd = number of driver constraints
nc = total number of constraints (nc = nk + nd)

Inverse dynamics – kinematically driven

solve kinematics $\{\ddot{q}\} = [\Phi_q]^{-1}\{\gamma\}$ $[\Phi_q]$ must have full rank nc = nq

compute constraint forces $\{\lambda\} = \left([\Phi_q]^T\right)^{-1}\left(\{Q\}_{APPLIED} - [M]\{\ddot{q}\}\right)$

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{KINEMATIC} \\ \{\Phi\}_{DRIVER} \end{Bmatrix} \qquad \{\lambda\} = \begin{Bmatrix} \{\lambda\}_{KINEMATIC} \\ \{\lambda\}_{DRIVER} \end{Bmatrix}$$

Inverse dynamics – simultaneous EOM matrix

$[M]\{\ddot{q}\} + [\Phi_q]^T\{\lambda\} = \{Q\}_{APPLIED}$ and $[\Phi_q]\{\ddot{q}\} = \{\gamma\}$

$$\begin{bmatrix} \underset{nq\ x\ nq}{[M]} & \underset{nq\ x\ nc}{[\Phi_q]^T} \\ \underset{nc\ x\ nq}{[\Phi_q]} & \underset{nc\ x\ nc}{[0]} \end{bmatrix} \begin{Bmatrix} \underset{nq\ x\ 1}{\{\ddot{q}\}} \\ \underset{nc\ x\ 1}{\{\lambda\}} \end{Bmatrix} = \begin{Bmatrix} \underset{nq\ x\ 1}{\{Q\}_{APPLIED}} \\ \underset{nc\ x\ 1}{\{\gamma\}} \end{Bmatrix} \qquad [EOM] = \underset{(nc+nq)\ x\ (nc+nq)}{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}}$$

$$\begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}^{-1} \begin{Bmatrix} \{Q\}_{APPLIED} \\ \{\gamma\} \end{Bmatrix}$$

Statics

$\{\ddot{q}\} = \{0\}$

$[M]\{\ddot{q}\} + [\Phi_q]^T\{\lambda\} = \{Q\}_{APPLIED}$

$\{\lambda\} = \left([\Phi_q]^T\right)^{-1}\{Q\}_{APPLIED}$

Lagrange multipliers for specific constraints

$\{F_{on\ i}\}_{CONSTRAINT} = -[\Phi_{r\ i}]^T_{CONSTRAINT}\{\lambda\}_{CONSTRAINT}$

$(T_{on\ i})_{CONSTRAINT} = \left(\left([B_i]\{s_i\}'^P\right)^T[\Phi_{r\ i}]^T_{CONSTRAINT} - [\Phi_{\phi\ i}]^T_{CONSTRAINT}\right)\{\lambda\}_{CONSTRAINT}$

$\{F_{on\ j}\}_{CONSTRAINT} = -[\Phi_{r\ j}]^T_{CONSTRAINT}\{\lambda\}_{CONSTRAINT}$

$(T_{on\ j})_{CONSTRAINT} = \left(\left([B_j]\{s_j\}'^P\right)^T[\Phi_{r\ j}]^T_{CONSTRAINT} - [\Phi_{\phi\ j}]^T_{CONSTRAINT}\right)\{\lambda\}_{CONSTRAINT}$

Revolute

$\{\Phi\}_{REV} = \{r_j\}^P - \{r_i\}^P = \{0_{2x1}\}$ **Note:** Haug uses $\{r_i\}^P - \{r_j\}^P$

$\{F_{on\ i}\}_{REV} = \{\lambda\}_{REV}$ at $\{r_i\}^P$

$(T_{on\ i})_{REV} = 0$

$\{F_{on\ j}\}_{REV} = -\{\lambda\}_{REV}$ at $\{r_j\}^P$

$(T_{on\ j})_{REV} = 0$

check body i $[\Phi_{r\ i}]^P_{REV} = -[I_2]$ $[\Phi_{\phi\ i}]^P_{REV} = -[B_i]\{s_i\}'^P$ **OK**

check body j $[\Phi_{r\ j}]^P_{REV} = [I_2]$ $[\Phi_{\phi\ j}]^P_{REV} = [B_j]\{s_j\}'^P$ **OK**

Double revolute

$$\Phi_{REV\_REV} = \{d_{ij}\}^T \{d_{ij}\} - L^2 = 0 \qquad L = \text{constant length}$$
$$\text{for} \quad \{d_{ij}\} = \{r_j\}^P - \{r_i\}^P$$
$$\text{and} \quad \{\dot{d}_{ij}\} = \{\dot{r}_j\}^P - \{\dot{r}_i\}^P$$
$$\text{and} \quad \{\ddot{d}_{ij}\} = \{\ddot{r}_j\}^P - \{\ddot{r}_i\}^P$$
$$\text{and} \quad \{\dddot{d}_{ij}\} = \{\dddot{r}_j\}^P - \{\dddot{r}_i\}^P$$

$$\{F_{on\ i}\}_{REV\_REV} = 2\{d_{ij}\}\lambda_{REV\_REV} \qquad \text{at } \{r_i\}^P$$

$$(T_{on\ i})_{CONSTRAINT} = 0$$

$$\{F_{on\ j}\}_{REV\_REV} = -2\{d_{ij}\}\lambda_{REV\_REV} \qquad \text{at } \{r_j\}^P$$

$$(T_{on\ j})_{CONSTRAINT} = 0$$

Parallel vectors (planar parallel-1)

$\{a_i\}$ parallel to $\{a_j\}$

$$\Phi_{PARALLEL} = \{a_i\}^T [R]^T \{a_j\} = 0$$
$$\text{for} \quad \{a_i\} = \{r_i\}^Q - \{r_i\}^P \quad \text{and} \quad \{a_j\} = \{r_j\}^Q - \{r_j\}^P$$

$$\{F_{on\ i}\}_{PARALLEL} = \{0_{2x1}\}$$

$$(T_{on\ i})_{PARALLEL} = (\text{norm}\{a_i\})(\text{norm}\{a_j\})\,\lambda_{PARALLEL}$$

$$\{F_{on\ j}\}_{PARALLEL} = \{0_{2x1}\}$$

$$(T_{on\ j})_{PARALLEL} = -(\text{norm}\{a_i\})(\text{norm}\{a_j\})\,\lambda_{PARALLEL}$$

Pin-in-slot (planar parallel-2)

$\{a_i\}$ parallel to $\{d_{ij}\}$

$$\Phi_{PIN\_SLOT} = \{a_i\}^T [R]^T \{d_{ij}\} = 0$$

for $\{d_{ij}\} = \{r_j\}^P - \{r_i\}^P$ and $\{a_i\} = \{r_i\}^Q - \{r_i\}^P$

and $\{\dot{d}_{ij}\}$ $\{\ddot{d}_{ij}\}$ $\{\dddot{d}_{ij}\}$ from above

$$\{F_{on\ i}\}_{PIN\_SLOT} = [R]\{a_i\}\lambda_{PIN\_SLOT} \qquad \text{at } \{r_j\}^P$$

$$(T_{on\ i})_{CONSTRAINT} = (\text{norm}\{a_i\})(\text{norm}\{d_{ij}\})\lambda_{PIN\_SLOT}$$

$$\{F_{on\ j}\}_{PIN\_SLOT} = -[R]\{a_i\}\lambda_{PIN\_SLOT} \qquad \text{at } \{r_j\}^P$$

$$(T_{on\ j})_{PIN\_SLOT} = 0$$

Relative angle driver

$$\Phi_{ANGLE} = \phi_j - \phi_i - C - f(t) = 0 \qquad C = \text{constant}$$

$$\{F_{on\ i}\}_{ANGLE} = \{0_{2x1}\}$$

$$(T_{on\ i})_{ANGLE} = \lambda_{ANGLE}$$

$$\{F_{on\ j}\}_{ANGLE} = \{0_{2x1}\}$$

$$(T_{on\ j})_{ANGLE} = -\lambda_{ANGLE}$$

Gear pair (chain/sprockets, belt/pulleys)

$$\Phi_{GEAR} = \phi_j - K\phi_i - C = 0 \qquad K = \text{constant}, C = \text{constant}$$

external gears $K = -\rho_i / \rho_j$, internal gears $K = +\rho_i / \rho_j$

$$\{F_{on\ i}\}_{GEAR} = \{0_{2x1}\}$$

$$(T_{on\ i})_{CONSTRAINT} = K\,\lambda_{GEAR}$$

$$\{F_{on\ j}\}_{GEAR} = \{0_{2x1}\}$$

$$\left(T_{on\ j}\right)_{CONSTRAINT} = -\lambda_{GEAR}$$

<u>Gear pair on rotating link k</u>

$$\Phi_{GEAR\_ON\_K} = \left(\phi_j - \phi_k\right) - K\left(\phi_i - \phi_k\right) - C = 0 \qquad K = \text{constant},\ C = \text{constant} \quad \text{from above}$$

$$\left\{F_{on\ i}\right\}_{GEAR} = \left\{0_{2x1}\right\}$$

$$\left(T_{on\ i}\right)_{CONSTRAINT} = K\ \lambda_{GEAR}$$

$$\left\{F_{on\ j}\right\}_{GEAR} = \left\{0_{2x1}\right\}$$

$$\left(T_{on\ j}\right)_{CONSTRAINT} = -\lambda_{GEAR}$$

$$\left\{F_{on\ k}\right\}_{GEAR} = \left\{0_{2x1}\right\}$$

$$\left(T_{on\ k}\right)_{CONSTRAINT} = \left(1 - K\right)\lambda_{GEAR}$$