Two-Dimensional Forward Dynamics

<u>Dynamically driven</u>

nc < nq

$\lfloor \Phi_q \rfloor$ does not have full row rank

cannot solve kinematics only $\{\ddot{q}\} = [\Phi_q]^{-1}\{\gamma\}$

given $\{q\}$ $\{\dot{q}\}$ $\{Q\}_{APPLIED}$ $[M]$

compute $\lfloor \Phi_q \rfloor$ $\{\gamma\}$

$$\begin{bmatrix} \underset{nq\ x\ nq}{[M]} & \underset{nq\ x\ nc}{[\Phi_q]^T} \\ \underset{nc\ x\ nq}{[\Phi_q]} & \underset{nc\ x\ nc}{[0]} \end{bmatrix} \begin{Bmatrix} \underset{nq\ x\ 1}{\{\ddot{q}\}} \\ \underset{nc\ x\ 1}{\{\lambda\}} \end{Bmatrix} = \begin{Bmatrix} \underset{nq\ x\ 1}{\{Q\}_{APPLIED}} \\ \underset{nc\ x\ 1}{\{\gamma\}} \end{Bmatrix} \qquad [EOM] = \underset{(nc+nq)\ x\ (nc+nq)}{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}}$$

$$\begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}^{-1} \begin{Bmatrix} \{Q\}_{APPLIED} \\ \{\gamma\} \end{Bmatrix}$$

must use forward time integration of $\{\ddot{q}\}$ to get $\{q\}$ $\{\dot{q}\}$ at next time step

Differential-Algebraic Equations (DAE)

differential equations for dynamics $\begin{bmatrix}[M] & [\Phi_q]^T\end{bmatrix}\begin{Bmatrix}\{\ddot{q}\} \\ \{\lambda\}\end{Bmatrix} = \{Q\}_{APPLIED}$

algebraic equations for kinematics $\begin{bmatrix}[\Phi_q] & [0]\end{bmatrix}\begin{Bmatrix}\{\ddot{q}\} \\ \{\lambda\}\end{Bmatrix} = \{\gamma\}$

Kinematics

generalized coordinates $\{q\} = \begin{Bmatrix}\{q_2\} \\ \{q_3\} \\ \{q_4\}\end{Bmatrix}$ $\{q_i\} = \begin{Bmatrix}x_i \\ y_i \\ \phi_i\end{Bmatrix}$ size nq x 1, nq = 3 (nL-1)

constraints $\{\Phi\} = \begin{Bmatrix}\{\Phi\}_{KINEMATIC} \\ \{\Phi\}_{DRIVER}\end{Bmatrix}$ size nc x 1 nc = nk + nd

Jacobian $[\Phi_q]$ element i, j $= \dfrac{\partial \Phi_i}{\partial q_j}$ size nc x nq

Inverse dynamics

nc = nq for kinematically driven problem

know driver motion at **<u>any time</u>** t, find $\{q\}$ $\{\dot{q}\}$ $\{\ddot{q}\}$

position solution $\{q\}_{NEW} = \{q\}_{OLD} + [\Phi_q]^{-1}\{\Phi\}$

velocity solution $\{\dot{q}\} = [\Phi_q]^{-1}\{\nu\}$ $\{\nu\} = \begin{Bmatrix}\{\nu\}_{KINEMATIC} \\ \{\nu\}_{DRIVER}\end{Bmatrix}$ size nc x 1

$\{\nu\}_{KINEMATIC} = \{0_{nkx1}\}$ $\{\nu\}_{DRIVER} = -\{\Phi_t\}_{DRIVER}$

acceleration solution $\{\ddot{q}\} = [\Phi_q]^{-1}\{\gamma\}$ $\{\gamma\} = \begin{Bmatrix}\{\gamma\}_{KINEMATIC} \\ \{\gamma\}_{DRIVER}\end{Bmatrix}$ size nc x 1

$\{\gamma\}_{KINEMATIC}$ based on $\{\Phi\}_{KINEMATIC}$ $\{\gamma\}_D = -\{\Phi_{tt}\}_{DRIVER}$

constraint forces $\{\lambda\} = \left([\Phi_q]^T\right)^{-1}\left(\{Q\}_{APPLIED} - [M]\{\ddot{q}\}\right)$

Forward dynamics

use initial conditions for $\{q\}$ $\{\dot{q}\}$ and integrate forward in time using small increments

solve kinematics and dynamics simultaneously $\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{Q\}_{APPLIED} \\ \{\gamma\} \end{Bmatrix}$

insufficient constraints to fully control mobility $\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{KINEMATIC} \\ \{\Phi\}_{DRIVER} \end{Bmatrix}$

Jacobian not full row rank $\underset{nc \times nq}{\lfloor \Phi_q \rfloor}$ nc < nq

EOM matrix has full row rank $[EOM] = \underset{(nc+nq) \times (nc+nq)}{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}}$

example four bar mechanism with M=1 and no kinematic driver defined

nq = 9 but nc = 8

same size $\underset{9 \times 1}{\{q\}}$ $\underset{9 \times 1}{\{\dot{q}\}}$ $\underset{9 \times 1}{\{\ddot{q}\}}$ $\underset{9 \times 9}{[M]}$ $\underset{9 \times 1}{\{Q\}_{APPLIED}}$

smaller $\underset{8 \times 1}{\{\Phi\}_{KINEMATIC}}$ $\underset{8 \times 1}{\{\nu\}_{KINEMATIC}} = \{0_{8x1}\}$ $\underset{8 \times 1}{\{\gamma\}_{KINEMATIC}}$ $\underset{8 \times 1}{\{\lambda\}_{KINEMATIC}}$ $\underset{8 \times 9}{\lfloor \Phi_q \rfloor_{KINEMATIC}}$

$$[EOM] = \underset{17 \times 17}{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}}$$

must know $\{q\}$ $\{\dot{q}\}$ $[M]$ $\{Q\}_{APPLIED}$ at current time step

can compute $\{\gamma\}$ $\lfloor \Phi_q \rfloor$ at current time step

use DAE to find $\{\ddot{q}\}$ and $\{\lambda\}$ $\begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}^{-1} \begin{Bmatrix} \{Q\}_{APPLIED} \\ \{\gamma\} \end{Bmatrix}$

no drivers $\{\Phi\}_{DRIVER}$ no driver forces $\{\lambda\}_{DRIVER}$

integrate $\{\ddot{q}\}$ to predict $\{q\}$ $\{\dot{q}\}$ at next time step

example five bar mechanism with M=2 and no kinematic drivers defined

nq = 12 but nc = 10

$$\underset{12\times 1}{\{q\}} \quad \underset{12\times 1}{\{\dot{q}\}} \quad \underset{12\times 1}{\{\ddot{q}\}} \quad \underset{12\times 12}{[M]} \quad \underset{12\times 1}{\{Q\}_{APPLIED}}$$

$$\underset{10\times 1}{\{\Phi\}_{KINEMATIC}} \quad \underset{10\times 1}{\{\nu\}_{KINEMATIC}} = \{0_{10\times 1}\} \quad \underset{10\times 1}{\{\gamma\}_{KINEMATIC}} \quad \underset{10\times 1}{\{\lambda\}_{KINEMATIC}} \quad \underset{10\times 12}{[\Phi_q]_{KINEMATIC}}$$

$$[EOM] = \underset{22\times 22}{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}}$$

no drivers $\{\Phi\}_{DRIVER}$ no driver forces $\{\lambda\}_{DRIVER}$

example five bar mechanism with M=2 and one kinematic driver defined

nq = 12 but nc = 11

same size $\underset{12\times 1}{\{q\}} \quad \underset{12\times 1}{\{\dot{q}\}} \quad \underset{12\times 1}{\{\ddot{q}\}} \quad \underset{12\times 12}{[M]} \quad \underset{12\times 1}{\{Q\}_{APPLIED}}$

$$\underset{11\times 1}{\{\Phi\}} = \begin{Bmatrix} \{\Phi\}_{KINEMATIC} \\ \Phi_{DRIVER} \end{Bmatrix} \quad \underset{11\times 1}{\{\nu\}} = \begin{Bmatrix} \{0_{10\times 1}\} \\ \nu_{DRIVER} \end{Bmatrix} \quad \underset{11\times 1}{\{\gamma\}} = \begin{Bmatrix} \{\gamma\}_{KINEMATIC} \\ \gamma_{DRIVER} \end{Bmatrix}$$

$$\underset{11\times 1}{\{\lambda\}} = \begin{Bmatrix} \{\lambda\}_{KINEMATIC} \\ \lambda_{DRIVER} \end{Bmatrix} \quad \underset{11\times 12}{[\Phi_q]} = \begin{bmatrix} [\Phi_q]_{KINEMATIC} \\ [\Phi_q]_{DRIVER} \end{bmatrix}$$

$$[EOM] = \underset{23\times 23}{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}}$$

one driver Φ_{DRIVER} one driver force λ_{DRIVER}