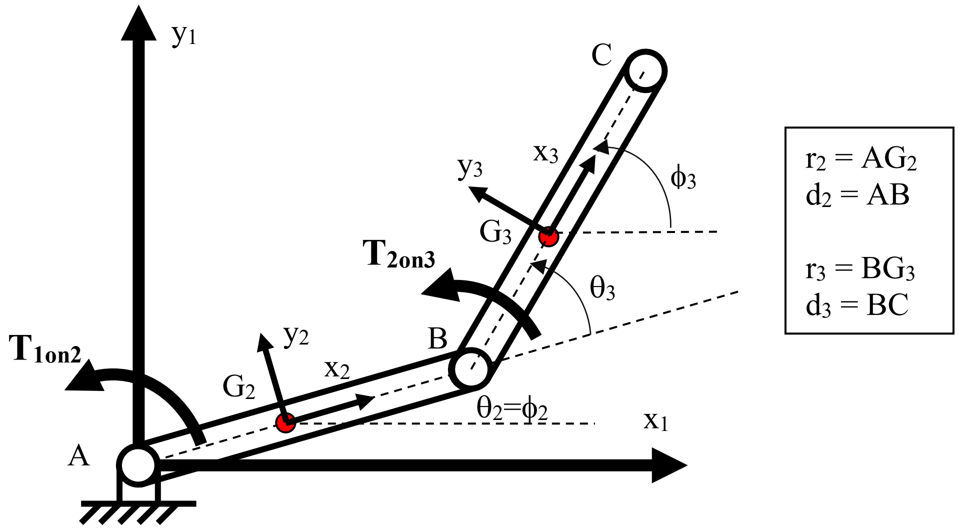
Differential-Algebraic Equations (DAE) for Anthropomorphic Manipulator



Two solid rigid bars with revolute joints A and B
Tool center point (TCP) at C (endpoint)
Centroids G_2 and G_3
Masses m_2 and m_3
Centroidal mass moments of inertia J_{G2} and J_{G3}
T_{1on2} is torque of ground on bar 2 about revolute A measured CCW positive
T_{2on3} is torque of bar 2 on bar 3 about revolute B measured CCW positive
Gravity g acts along negative y axis

nL = 3 nJ1=2 m = 3 (nL-1) – 2 nJ1 = 2

Lagrangian method (from Notes_09_02)

$$\{q\} = \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} \qquad \{\dot{q}\} = \begin{Bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \qquad \{Q\} = \begin{Bmatrix} T_{1\,on\,2} \\ T_{2\,on\,3} \end{Bmatrix}$$

Note: θ_3 measured relative to θ_2

$$J_B = m_3 a_3^2 + J_3$$
$$J_A = J_B + m_2 a_2^2 + m_3 d_2^2 + J_2 + 2 m_3 d_2 a_3 \cos\theta_3$$
$$C = J_B + m_3 d_2 a_3 \cos\theta_3$$
$$D = m_3 d_2 a_3 \sin\theta_3$$
$$G_2 = (m_2 a_2 + m_3 d_2) g \cos\theta_2$$
$$G_3 = m_3 a_3 g \cos(\theta_2 + \theta_3)$$

inverse dynamics

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} + \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix}$$

know driver motion **<u>at any time</u>** t - find $\{q\}$ $\{\dot{q}\}$ $\{\ddot{q}\}$

compute driver torques $T_{1\,on\,2}$ $T_{2\,on\,3}$ from Lagrangian equations

compute joint forces $F_{1\,on\,2}{}^x$ $F_{1\,on\,2}{}^y$ $F_{2\,on\,3}{}^x$ $F_{2\,on\,3}{}^y$ from Newtonian equations

may arbitrarily choose **<u>any other time</u>** t

forward dynamics

$$\begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix}^{-1} \left(\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} - \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} - \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix} \right)$$

know current state $\{q\}$ and $\{\dot{q}\}$ at current t

compute $\{\ddot{q}\}$ from Lagrangian equations

compute $F_{1\,on\,2}{}^x$ $F_{1\,on\,2}{}^y$ $F_{2\,on\,3}{}^x$ $F_{2\,on\,3}{}^y$ from Newtonian equations

must integrate $\{\ddot{q}\}$ to get new $\{q\}$ and $\{\dot{q}\}$ at the **<u>next time step</u>**

Newtonian method (from free body diagrams)

$$
\begin{aligned}
F_{1\,on\,2}^{\;x} - F_{2\,on\,3}^{\;x} &= m_2 \ddot{x}_2 \\
F_{1\,on\,2}^{\;y} - F_{2\,on\,3}^{\;y} - m_2 g &= m_2 \ddot{y}_2 \\
\{s_2\}^A \times \{F_{1\,on\,2}\} - \{s_2\}^B \times \{F_{2\,on\,3}\} + T_{1\,on\,2} - T_{2\,on\,3} &= J_{G2} \ddot{\phi}_2 \\
F_{2\,on\,3}^{\;x} &= m_3 \ddot{x}_3 \\
F_{2\,on\,3}^{\;y} - m_3 g &= m_3 \ddot{y}_3 \\
\{s_3\}^B \times \{F_{2\,on\,3}\} + T_{2\,on\,3} J_{G3} \ddot{\phi}_3
\end{aligned}
$$

inverse dynamics

$$
\begin{bmatrix}
+1 & 0 & 0 & -1 & 0 & 0 \\
0 & +1 & 0 & 0 & -1 & 0 \\
-\left(\{s_2\}^A\right)^Y & +\left(\{s_2\}^A\right)^X & +1 & \left(\{s_2\}^B\right)^Y & -\left(\{s_2\}^B\right)^X & -1 \\
0 & 0 & 0 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & -\left(\{s_3\}^B\right)^Y & +\left(\{s_3\}^B\right)^X & +1
\end{bmatrix}
\begin{Bmatrix}
F_{1\,on\,2}^{\;x} \\ F_{1\,on\,2}^{\;y} \\ T_{1\,on\,2} \\ F_{2\,on\,3}^{\;x} \\ F_{2\,on\,3}^{\;y} \\ T_{2\,on\,3}
\end{Bmatrix}
=
\begin{Bmatrix}
m_2 \ddot{x}_2 \\ m_2 \ddot{y}_2 + m_2 g \\ J_{G2} \ddot{\phi}_2 \\ m_3 \ddot{x}_3 \\ m_3 \ddot{y}_3 + m_3 g \\ J_{G3} \ddot{\phi}_3
\end{Bmatrix}
$$

know driver motion **<u>at any time</u>** t - find positions, velocities and accelerations

compute joint forces $\left\{ F_{1\,on\,2}^{\;x} \quad F_{1\,on\,2}^{\;y} \quad F_{2\,on\,3}^{\;x} \quad F_{2\,on\,3}^{\;x} \quad T_{1\,on\,2} \quad T_{2\,on\,3} \right\}^T$

may arbitrarily choose **<u>any other time</u>** t

forward dynamics

know current positions and velocities at current t

very cumbersome to compute joint forces **<u>and</u>** accelerations **<u>simultaneously</u>**

must integrate accelerations to get new positions and velocities at the **<u>next time step</u>**

DAE dynamics

$$\{q\} = \begin{Bmatrix} x_2 \\ y_2 \\ \phi_2 \\ x_3 \\ y_3 \\ \phi_3 \end{Bmatrix} \qquad \{s_2\}'^A = \begin{Bmatrix} -AG_2 \\ 0 \end{Bmatrix} \qquad \{s_2\}'^B = \begin{Bmatrix} BG_2 \\ 0 \end{Bmatrix} \qquad \{s_3\}'^B = \begin{Bmatrix} -BG_3 \\ 0 \end{Bmatrix} \qquad \{s_3\}'^C = \begin{Bmatrix} CG_3 \\ 0 \end{Bmatrix}$$

$$\{\Phi\}_{KINEMATIC} = \begin{Bmatrix} \{r_2\}^A - \{r_1\}^A \\ \{r_3\}^B - \{r_2\}^B \end{Bmatrix} \qquad \{\Phi\}_{DRIVERS} = \begin{Bmatrix} f_1(\{q\},t) \\ f_2(\{q\},t) \end{Bmatrix}$$

inverse dynamics

$$\underbrace{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}}_{12x12} \underbrace{\begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}}_{12x1} = \underbrace{\begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}}_{12x1} \qquad \{\Phi\} = \begin{Bmatrix} \{r_2\}^A - \{r_1\}^A \\ \{r_3\}^B - \{r_2\}^B \\ \{\Phi\}_{DRIVERS} \end{Bmatrix} \qquad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1\,on\,A2} \\ \{F\}_{B2\,on\,B3} \\ T_{1\,on\,2} \\ T_{2\,on\,3} \end{Bmatrix}$$

know driver motion **<u>at any time</u>** t, find $\{q\}$ $\{\dot{q}\}$ $\{\gamma\}$ $\{Q\}_{EXT}$ $\lfloor\Phi_q\rfloor$

compute $\{\ddot{q}\}$ and $\{\lambda\}$ **<u>simultaneously</u>**

may arbitrarily choose **<u>any other time</u>** t

forward dynamics

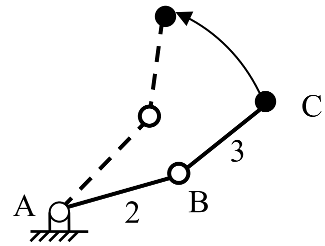
$$\underbrace{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}}_{10x10} \underbrace{\begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}}_{10x1} = \underbrace{\begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}}_{10x1} \qquad \{\Phi\} = \begin{Bmatrix} \{r_2\}^A - \{r_1\}^A \\ \{r_3\}^B - \{r_2\}^B \end{Bmatrix} \qquad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1\,on\,A2} \\ \{F\}_{B2\,on\,B3} \end{Bmatrix}$$

know current state $\{q\}$ and $\{\dot{q}\}$ at current t, find $\{\gamma\}$ $\{Q\}_{EXT}$ $\lfloor\Phi_q\rfloor$

compute $\{\ddot{q}\}$ and $\{\lambda\}$ **<u>simultaneously</u>**

must integrate $\{\ddot{q}\}$ to get new $\{q\}$ and $\{\dot{q}\}$ at the **<u>next time step</u>**

Inverse Dynamics – joint interpolated motion
(independent position controllers on each joint)



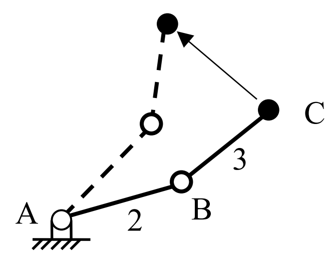
$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV\_A} \\ \{\Phi\}_{REV\_B} \\ \phi_2 - \phi_{2\_START} - \omega_2 t \\ \theta - \theta_{START} - \omega_\theta t \end{Bmatrix}$$

$$\omega_2 = \left(\phi_{2\_END} - \phi_{2\_START}\right) / \Delta t$$
$$\theta = \phi_3 - \phi_2$$
$$\omega_\theta = \left(\theta_{END} - \theta_{START}\right) / \Delta t$$

$$\{q\} = \begin{Bmatrix} x_2 \\ y_2 \\ \phi_2 \\ x_3 \\ y_3 \\ \phi_3 \end{Bmatrix} \qquad \underset{6x6}{\left[\Phi_q\right]} \qquad \{\nu\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \\ \omega_2 \\ \omega_\theta \end{Bmatrix} \qquad \{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV\_A} \\ \{\gamma\}_{REV\_B} \\ 0 \\ 0 \end{Bmatrix} \qquad \{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

$$\underset{12x12}{\begin{bmatrix} [M] & \left[\Phi_q\right]^T \\ \left[\Phi_q\right] & [0] \end{bmatrix}} \underset{12x1}{\begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}} = \underset{12x1}{\begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}} \qquad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1\ on\ A2} \\ \{F\}_{B2\ on\ B3} \\ T_{1\ on\ 2} \\ T_{2\ on\ 3} \end{Bmatrix}$$

Inverse Dynamics – straight-line TCP interpolated motion
(interpolated position controllers on each joint)



$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV\_A} \\ \{\Phi\}_{REV\_B} \\ x_{C3} - x_{C3\_START} - v_{C3\_x}\, t \\ y_{C3} - y_{C3\_START} - v_{C3\_y}\, t \end{Bmatrix}$$

$$v_{C3\_x} = \left(x_{C3\_END} - x_{C3\_START}\right) / \Delta t$$

$$v_{C3\_y} = \left(y_{C3\_END} - y_{C3\_START}\right) / \Delta t$$

$$\{q\} = \begin{Bmatrix} x_2 \\ y_2 \\ \phi_2 \\ x_3 \\ y_3 \\ \phi_3 \end{Bmatrix} \qquad \underset{6x6}{\left[\Phi_q\right]} \qquad \{\nu\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \\ v_{C3\_X} \\ v_{C3\_Y} \end{Bmatrix} \qquad \{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV\_A} \\ \{\gamma\}_{REV\_B} \\ 0 \\ 0 \end{Bmatrix} \qquad \{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{12x12} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}_{12x1} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}_{12x1} \qquad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \\ F_{EFF\_CX} \\ F_{EFF\_Cy} \end{Bmatrix}$$

$$\theta = \phi_3 - \phi_2 \qquad \{r_3\}^C = \begin{Bmatrix} r_2 \cos\phi_2 + r_3 \cos(\phi_2 + \theta) \\ r_2 \sin\phi_2 + r_3 \sin(\phi_2 + \theta) \end{Bmatrix}$$

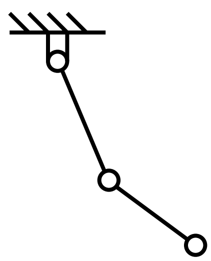
$$\dot{\theta} = \dot{\phi}_3 - \dot{\phi}_2$$

$$\{\dot{r}_3\}^C = \begin{bmatrix} -r_2 \sin\phi_2 - r_3 \sin(\phi_2 + \theta) & -r_3 \sin(\phi_2 + \theta) \\ r_2 \cos\phi_2 r_3 \cos(\phi_2 + \theta) & r_3 \cos(\phi_2 + \theta) \end{bmatrix} \begin{Bmatrix} \dot{\phi}_2 \\ \dot{\theta} \end{Bmatrix}$$

$$\begin{Bmatrix} F_{EFF\_CX} \\ F_{EFF\_Cy} \end{Bmatrix}^T \{\dot{r}_3\}^C + \begin{Bmatrix} T_{1 \text{ on } 2} \\ T_{2 \text{ on } 3} \end{Bmatrix}^T \begin{Bmatrix} \dot{\phi}_2 \\ \dot{\theta} \end{Bmatrix} = 0$$

$$\begin{Bmatrix} T_{1 \text{ on } 2} \\ T_{2 \text{ on } 3} \end{Bmatrix} = -\begin{bmatrix} -r_2 \sin\phi_2 - r_3 \sin(\phi_2 + \theta) & -r_3 \sin(\phi_2 + \theta) \\ r_2 \cos\phi_2 r_3 \cos(\phi_2 + \theta) & r_3 \cos(\phi_2 + \theta) \end{bmatrix}^T \begin{Bmatrix} F_{EFF\_CX} \\ F_{EFF\_Cy} \end{Bmatrix}$$

Forward Dynamics – double pendulum
(no actuators)



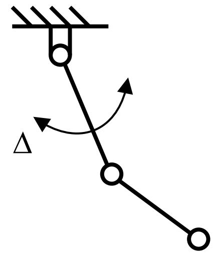
$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV\_A} \\ \{\Phi\}_{REV\_B} \end{Bmatrix}$$

$$\{q\} = \begin{Bmatrix} x_2 \\ y_2 \\ \phi_2 \\ x_3 \\ y_3 \\ \phi_3 \end{Bmatrix} \qquad [\Phi_q]_{4x6} \qquad \{\nu\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \end{Bmatrix} \qquad \{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV\_A} \\ \{\gamma\}_{REV\_B} \end{Bmatrix} \qquad \{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{10x10} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}_{10x1} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}_{10x1} \qquad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \end{Bmatrix}$$

Forward Dynamics – proximal link kinematically driven, distal link pendulum
(position controller **only** on proximal joint)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV\_A} \\ \{\Phi\}_{REV\_B} \\ \phi_2 - \phi_{2\_CENTER} - \Delta \sin(2\pi f t) \end{Bmatrix}$$

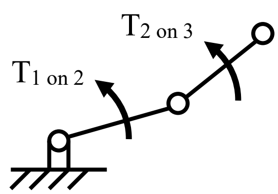


$$\{q\} = \begin{Bmatrix} x_2 \\ y_2 \\ \phi_2 \\ x_3 \\ y_3 \\ \phi_3 \end{Bmatrix} \quad \underset{5x6}{\left[\Phi_q\right]} \quad \{\nu\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \\ 2\pi f\, \Delta \cos(2\pi f t) \end{Bmatrix} \quad \{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV\_A} \\ \{\gamma\}_{REV\_B} \\ -4\pi^2 f^2 \Delta \sin(2\pi f t) \end{Bmatrix} \quad \{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

$$\underset{11x11}{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}} \underset{11x1}{\begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}} = \underset{11x1}{\begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}} \qquad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1\ on\ A2} \\ \{F\}_{B2\ on\ B3} \\ T_{1\ on\ 2} \end{Bmatrix}$$

Forward Dynamics – computed torque control
(torque controllers on each joint)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV\_A} \\ \{\Phi\}_{REV\_B} \end{Bmatrix}$$



$$\{q\} = \begin{Bmatrix} x_2 \\ y_2 \\ \phi_2 \\ x_3 \\ y_3 \\ \phi_3 \end{Bmatrix} \quad \underset{4x6}{\left[\Phi_q\right]} \quad \{\nu\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \end{Bmatrix} \quad \{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV\_A} \\ \{\gamma\}_{REV\_B} \end{Bmatrix} \quad \{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ T_{1\ on\ 2} - T_{2\ on\ 3} \\ 0 \\ -m_3 g \\ T_{2\ on\ 3} \end{Bmatrix}$$

$$\underset{10x10}{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}} \underset{10x1}{\begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}} = \underset{10x1}{\begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}} \qquad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1\ on\ A2} \\ \{F\}_{B2\ on\ B3} \end{Bmatrix}$$