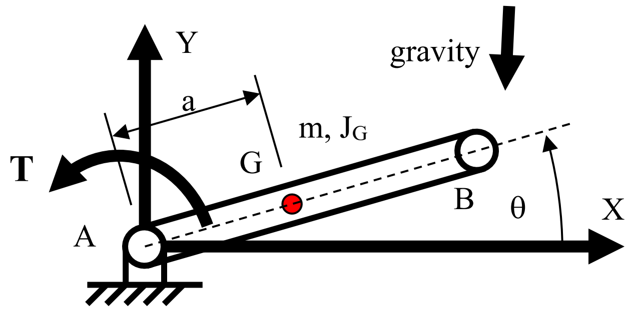
Lagrangian Dynamics for Simple Pendulum



$q=\theta \qquad \dot{q}=\dot{\theta} \qquad \ddot{q}=\ddot{\theta} \qquad Q=T$

$x_G = a\cos\theta \qquad \dot{x} = -a\,\dot{\theta}\sin\theta \qquad \ddot{x} = -a\,\ddot{\theta}\sin\theta - a\,\dot{\theta}^2\cos\theta$

$y_G = a\sin\theta \qquad \dot{y} = a\,\dot{\theta}\cos\theta \qquad \ddot{y} = a\,\ddot{\theta}\cos\theta - a\,\dot{\theta}^2\sin\theta$

$K = \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+\tfrac{1}{2}J_G\,\dot{\theta}^2 = \tfrac{1}{2}m\left(a^2\dot{\theta}^2\sin^2\theta + a^2\dot{\theta}^2\cos^2\theta\right)+\tfrac{1}{2}J_G\,\dot{\theta}^2 = \tfrac{1}{2}\left(J_G+ma^2\right)\dot{\theta}^2$

$P = m\,g\,y = m\,g\,a\sin\theta$

$L = K - P = \tfrac{1}{2}\left(J_G+ma^2\right)\dot{\theta}^2 - m\,g\,a\sin\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=Q \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=T$

$\frac{\partial L}{\partial \dot{\theta}} = \left(J_G+ma^2\right)\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(J_G+ma^2\right)\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = -m\,g\,a\cos\theta$

$\left(J_G+ma^2\right)\ddot{\theta} + m\,g\,a\cos\theta = T$

second order dynamics using Gibbs-Appell

$G = \tfrac{1}{2}m_i\left\{\ddot{r}_i\right\}^T\left\{\ddot{r}_i\right\}+\tfrac{1}{2}\left\{\dot{\omega}_i\right\}^T\left[J_{Gi}\right]\left\{\dot{\omega}_i\right\}+\left\{\dot{\omega}_i\right\}^T\left(\left[\tilde{\omega}_i\right]\left[J_{Gi}\right]\left\{\omega_i\right\}\right)-\left\{F\right\}^T\left\{\ddot{r}_i\right\}-\left\{M\right\}^T\left\{\dot{\omega}_i\right\}$

$G = \tfrac{1}{2}m_i\left\{\ddot{r}_i\right\}^T\left\{\ddot{r}_i\right\}+\tfrac{1}{2}\left\{\dot{\omega}_i\right\}^T\left[J_{Gi}\right]\left\{\dot{\omega}_i\right\}+\left(\left[\dot{\tilde{\omega}}_i\right]\left\{\omega_i\right\}\right)^T\left[J_{Gi}\right]\left\{\omega_i\right\}-\left\{F\right\}^T\left\{\ddot{r}_i\right\}-\left\{M\right\}^T\left\{\dot{\omega}_i\right\}$

$\left\{\ddot{r}\right\} = \begin{Bmatrix}\ddot{x}\\ \ddot{y}\end{Bmatrix} = \begin{Bmatrix}-a\,\ddot{\theta}\sin\theta - a\,\dot{\theta}^2\cos\theta\\ a\,\ddot{\theta}\cos\theta - a\,\dot{\theta}^2\sin\theta\end{Bmatrix} \qquad \left\{F\right\} = \begin{Bmatrix}0\\ -m\,g\end{Bmatrix} \qquad \left\{F\right\}^T\left\{\ddot{r}\right\} = -m\,g\left(a\,\ddot{\theta}\cos\theta - a\,\dot{\theta}^2\sin\theta\right)$

$$G = \tfrac{1}{2} m\left(\ddot{x}^2 + \ddot{y}^2\right) + \tfrac{1}{2} J_G \ddot{\theta}^2 + m\,g\left(a\,\ddot{\theta}\cos\theta - a\,\dot{\theta}^2 \sin\theta\right) - T\ddot{\theta}$$

$$\ddot{x}^2 = a^2\ddot{\theta}^2 \sin^2\theta + 2a^2\dot{\theta}^2\ddot{\theta}\sin\theta\cos\theta + a^2\dot{\theta}^4\cos^2\theta$$
$$\ddot{y}^2 = a^2\ddot{\theta}^2 \cos^2\theta - 2a^2\dot{\theta}^2\ddot{\theta}\sin\theta\cos\theta + a^2\dot{\theta}^4\sin^2\theta$$
$$\ddot{x}^2 + \ddot{y}^2 = a^2\left(\ddot{\theta}^2 + \dot{\theta}^4\right)$$

$$G = \tfrac{1}{2} ma^2\left(\ddot{\theta}^2 + \dot{\theta}^4\right) + \tfrac{1}{2} J_G \ddot{\theta}^2 + m\,g\left(a\,\ddot{\theta}\cos\theta - a\,\dot{\theta}^2 \sin\theta\right) - T\ddot{\theta}$$

$$\frac{\partial G}{\partial \ddot{q}} = 0 \qquad \frac{\partial G}{\partial \ddot{\theta}} = 0$$

$$\frac{\partial G}{\partial \ddot{\theta}} = ma^2\ddot{\theta} + J_G\ddot{\theta} + m\,g\,a\cos\theta - T = 0$$

$$\left(J_G + ma^2\right)\ddot{\theta} + m\,g\,a\cos\theta = T$$

third order dynamics using direct time derivative

$$\left(J_G + ma^2\right)\dddot{\theta} - m\,g\,a\,\dot{\theta}\sin\theta = \dot{T}$$

third-order Lagrange ?????

$$S = \frac{1}{2}\sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i, \quad Q^*_\alpha = \sum_{i=1}^{N} \dot{\boldsymbol{F}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\alpha}\right) - \frac{1}{2}\frac{\partial S}{\partial \dot{q}_\alpha} = Q^*_\alpha, \quad (\alpha = 1, 2, \ldots, n),$$

$$\frac{d}{dt}\left(\frac{\partial S}{\partial \ddot{q}}\right) - \frac{1}{2}\frac{\partial S}{\partial \dot{q}} = Q^* \qquad \frac{d}{dt}\left(\frac{\partial S}{\partial \ddot{\theta}}\right) - \frac{1}{2}\frac{\partial S}{\partial \dot{\theta}} = T^*$$

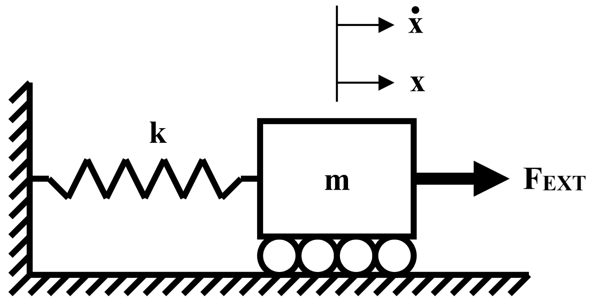
$$\frac{\partial G}{\partial \ddot{\theta}} = ma^2\ddot{\theta} + J_G\ddot{\theta} + m\,g\,a\cos\theta - T$$

$$\frac{d}{dt}\left(\frac{\partial G}{\partial \ddot{\theta}}\right) = \left(J_G + ma^2\right)\dddot{\theta} - m\,g\,a\,\dot{\theta}\sin\theta - \dot{T}$$

$$\frac{\partial G}{\partial \dot{\theta}} = 2ma^2\dot{\theta}^3 - 2m\,g\,a\,\dot{\theta}\sin\theta = 2m\,a\,\dot{\theta}\left(a\,\dot{\theta}^2 - g\sin\theta\right)$$

$$\left(J_G + ma^2\right)\dddot{\theta} - m\,g\,a\,\dot{\theta}\sin\theta - \dot{T} + m\,a\,\dot{\theta}\left(g\sin\theta - a\,\dot{\theta}^2\right) = T^*$$

Lagrangian Dynamics for Spring-Mass



$q = x \qquad \dot{q} = \dot{x} \qquad Q = F_{EXT}$

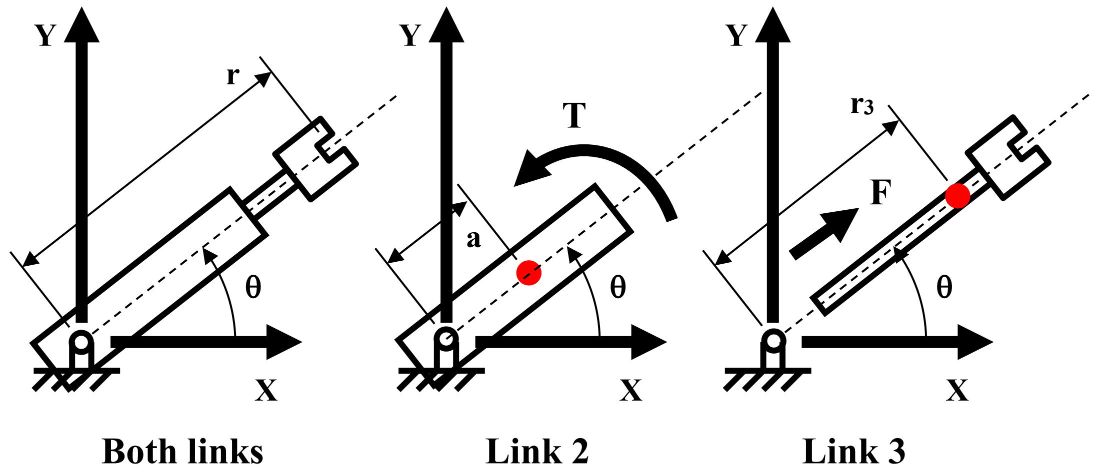
$K = \frac{1}{2} m\dot{x}^2 \qquad P = \frac{1}{2} kx^2 \qquad L = K - P = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F_{EXT}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial L}{\partial x} = -kx$$

$$m\ddot{x} + kx = F_{EXT}$$

Lagrangian Dynamics for Cylindrical Coordinate Manipulator



Main body link 2 - Shaft and end-effector link 3
Mass centers at a and r_3 from waist rotation axis, a=constant, r_3 = variable
Masses m_2 and m_3 - centroidal mass moments of inertia J_2 and J_3
θ CCW from positive x axis – a and r_3 radial from rotation axis
T is rotary actuator torque of ground on body 2 about waist measured CCW positive
F is radial actuator force of body 2 on body 3 measured positive outward
Gravity g acts along negative y axis

$q_2 = \theta \qquad q_3 = r_3 \qquad \dot{q}_2 = \dot{\theta} \qquad \dot{q}_3 = \dot{r}_3 \qquad Q_2 = T \qquad Q_3 = F$

$x_2 = a\cos\theta \qquad \dot{x}_2 = -a\,\dot{\theta}\sin\theta$

$y_2 = a\sin\theta \qquad \dot{y}_2 = a\,\dot{\theta}\cos\theta$

$x_3 = r_3\cos\theta \qquad \dot{x}_3 = \dot{r}_3\cos\theta - r_3\dot{\theta}\sin\theta$

$y_3 = r_3\sin\theta \qquad \dot{y}_3 = \dot{r}_3\sin\theta + r_3\dot{\theta}\cos\theta$

$$K = \tfrac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) + \tfrac{1}{2}m_3\left(\dot{x}_3^2 + \dot{y}_3^2\right) + \tfrac{1}{2}J_2\dot{\theta}^2 + \tfrac{1}{2}J_3\dot{\theta}^2$$

$$K = \tfrac{1}{2}m_2 a^2\dot{\theta}^2 + \tfrac{1}{2}m_3 r_3^2\dot{\theta}^2 + \tfrac{1}{2}m_3\dot{r}_3^2 + \tfrac{1}{2}(J_2 + J_3)\dot{\theta}^2$$

$P = m_2\, y_2\, g + m_3\, y_3\, g \qquad P = m_2\, g\, a\sin\theta + m_3\, g\, r_3\sin\theta$

$L = K - P$

$$L = \tfrac{1}{2}m_2 a^2\dot{\theta}^2 + \tfrac{1}{2}m_3 r_3^2\dot{\theta}^2 + \tfrac{1}{2}m_3\dot{r}_3^2 + \tfrac{1}{2}(J_2 + J_3)\dot{\theta}^2 - g(m_2 a + m_3 r_3)\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = T \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_3}\right) - \frac{\partial L}{\partial r_3} = F$$

$$\frac{\partial L}{\partial \dot{\theta}} = \left(m_2 a^2 + m_3 r_3^2 + J_2 + J_3\right)\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(m_2 a^2 + m_3 r_3^2 + J_2 + J_3\right)\ddot{\theta} + 2 m_3 r_3 \dot{r}_3 \dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -g\left(m_2 a + m_3 r_3\right)\cos\theta$$

$$T = \left(m_2 a^2 + m_3 r_3^2 + J_2 + J_3\right)\ddot{\theta} + 2 m_3 r_3 \dot{r}_3 \dot{\theta} + g\left(m_2 a + m_3 r_3\right)\cos\theta$$

$$\frac{\partial L}{\partial \dot{r}_3} = m_3 \dot{r}_3$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_3}\right) = m_3 \ddot{r}_3$$

$$\frac{\partial L}{\partial r_3} = m_3 r_3 \dot{\theta}^2 - m_3 g \sin\theta$$

$$F = m_3 \ddot{r}_3 - m_3 r_3 \dot{\theta}^2 + m_3 g \sin\theta$$

inverse dynamics

$$T = \left(m_2 a^2 + m_3 r_3^2 + J_2 + J_3\right)\ddot{\theta} + 2 m_3 r_3 \dot{r}_3 \dot{\theta} + g\left(m_2 a + m_3 r_3\right)\cos\theta$$

$$F = m_3 \ddot{r}_3 - m_3 r_3 \dot{\theta}^2 + m_3 g \sin\theta$$

forward dynamics

$$\begin{bmatrix} m_2 a^2 + m_3 r_3^2 + J_2 + J_3 & 0 \\ 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{r}_3 \end{Bmatrix} = \begin{Bmatrix} T - 2 m_3 r_3 \dot{r}_3 \dot{\theta} - g\left(m_2 a + m_3 r_3\right)\cos\theta \\ F + m_3 r_3 \dot{\theta}^2 - m_3 g \sin\theta \end{Bmatrix}$$

third order dynamics using direct time derivatives

$$\dot{T} = \left(m_2 a^2 + m_3 r_3^2 + J_2 + J_3\right)\dddot{\theta} + 2 m_3 \left(\dot{r}_3{}^2 \dot{\theta} + r_3 \ddot{r}_3 \dot{\theta} + r_3 \dot{r}_3 \ddot{\theta}\right) - g\left(m_2 a + m_3 r_3\right)\dot{\theta}\sin\theta + g\, m_3 \dot{r}_3 \cos\theta$$

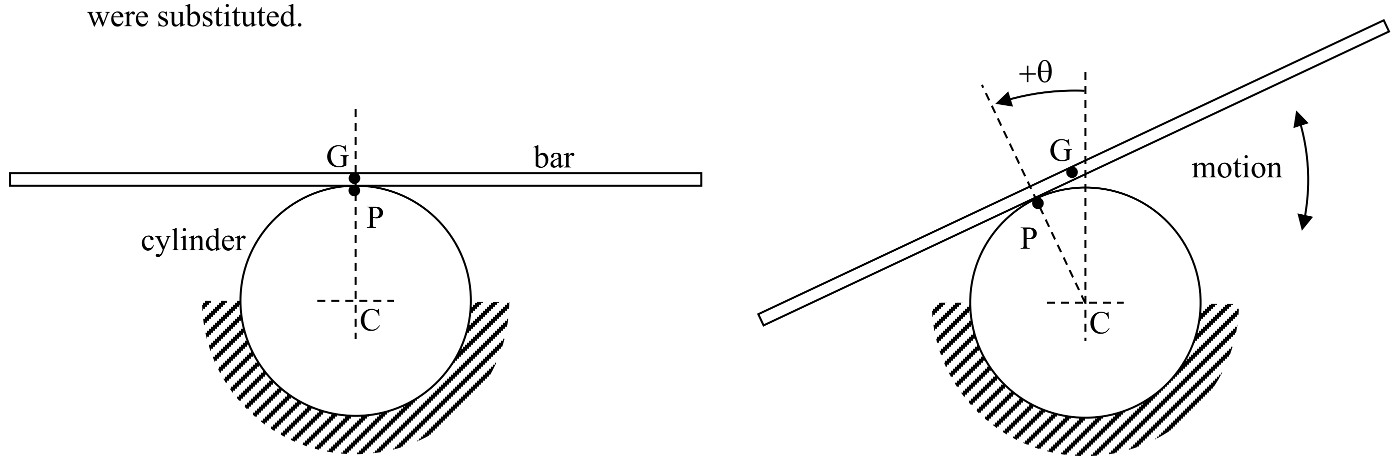
$$\dot{F} = m_3 \dddot{r}_3 - m_3 \dot{r}_3 \dot{\theta}^2 - 2 m_3 r_3 \dot{\theta}\,\ddot{\theta} + m_3 g\, \dot{\theta} \cos\theta$$

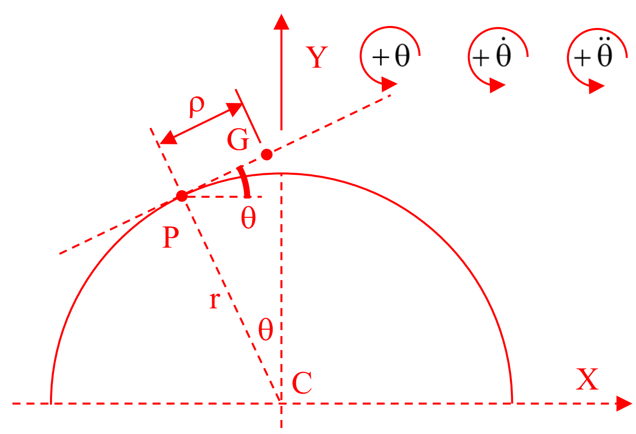
Lagrangian Dynamics for a Bar Balanced on a Cylinder

A steel bar is initially balanced on top of a fixed cylinder as shown below left. When the left end of the bar is depressed and then released, it will oscillate in a rocking fashion as shown below right. The radius of the cylinder is 1.906 inches. The bar is 19.5 inches long and weighs 0.45 pounds. Assume thickness of the bar is negligible and that it rolls without slipping.

a) Determine the natural frequency of oscillation for an initial release angle $\theta_0 = 10^o$.

b) Determine the natural frequency of oscillation if an aluminum bar with identical geometry were substituted.





$$\rho = r\,\theta$$

$$x_G = -r\sin\theta + \rho\cos\theta = -r\sin\theta + r\,\theta\cos\theta$$

$$y_G = +r\cos\theta + \rho\sin\theta = +r\cos\theta + r\,\theta\sin\theta$$

$$\dot{x}_G = -r\,\dot{\theta}\cos\theta + r\,\dot{\theta}\cos\theta - r\,\theta\,\dot{\theta}\sin\theta = -r\,\theta\,\dot{\theta}\sin\theta$$

$$\dot{y}_G = -r\,\dot{\theta}\sin\theta + r\,\dot{\theta}\sin\theta + r\,\theta\,\dot{\theta}\cos\theta = +r\,\theta\,\dot{\theta}\cos\theta$$

$$V_G^{\,2} = r^2\,\theta^2\,\dot{\theta}^2$$

$$K = \tfrac{1}{2} m\,V_G^{\,2} + \tfrac{1}{2} J\,\dot{\theta}^2 = \tfrac{1}{2} m\,r^2\,\theta^2\,\dot{\theta}^2 + \tfrac{1}{2} J\,\dot{\theta}^2$$

$$P = m\,g\,y_G = m\,g\,r\left(\cos\theta + \theta\sin\theta\right)$$

$$L = K - P = \tfrac{1}{2} m\,r^2\,\theta^2\,\dot{\theta}^2 + \tfrac{1}{2} J\,\dot{\theta}^2 - m\,g\,r\left(\cos\theta + \theta\sin\theta\right)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\,r^2\,\theta^2\,\dot{\theta} + J\,\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2\,m\,r^2\,\theta\,\dot{\theta}^2 + m\,r^2\,\theta^2\,\ddot{\theta} + J\,\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = m\,r^2\,\theta\,\dot{\theta}^2 - m\,g\,r\left(-\sin\theta + \sin\theta + \theta\cos\theta\right) = m\,r^2\,\theta\,\dot{\theta}^2 - m\,g\,r\,\theta\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad 2\,m\,r^2\,\theta\,\dot{\theta}^2 + m\,r^2\,\theta^2\,\ddot{\theta} + J\,\ddot{\theta} - \left(m\,r^2\,\theta\,\dot{\theta}^2 - m\,g\,r\,\theta\cos\theta\right) = 0$$

$$\left(J + m\,r^2\,\theta^2\right)\ddot{\theta} + m\,r\left(g\cos\theta + r\,\dot{\theta}^2\right)\theta = 0$$

ASSUME $\quad$ small $\theta \rightarrow \cos\theta \approx 1 \qquad J >> m\,r^2\,\theta^2 \qquad g\cos\theta >> r\,\dot{\theta}^2$

$$J\,\ddot{\theta} + m\,r\,g\,\theta = 0 \qquad \omega_N = \sqrt{mgr/J} \qquad J = m\,L^2/12 \qquad \omega_N = \sqrt{12gr/L^2}$$

using $r = 1.906$ inch, $L = 19.5$ inch, $m = 0.45$ lbm, $g = 386$ ips^2

$$\omega_N = \sqrt{12\left(\frac{386\text{ in}}{\sec^2}\right)\frac{1.906\text{ in}}{(19.5\text{ in})^2}} = 4.818\text{ rad/sec} = 0.767\text{ Hz}$$

use $\quad \theta = \theta_o\cos(\omega_N t) \qquad \dot{\theta} = -\theta_o\omega_N\sin(\omega_N t) \qquad \theta_o = 10° = 0.1745$ rad

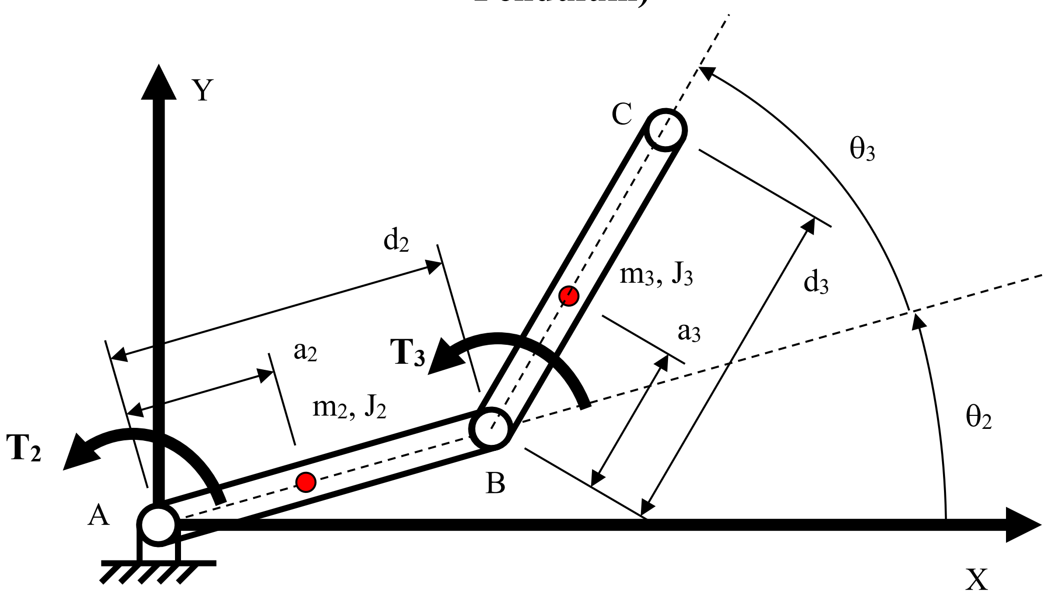
check $\quad \cos\theta_o = 0.9848 \approx 1 \qquad 1.5\% = \text{OK}$

check $\quad J = m\,L^2/12 = 14.26\text{ lbm.in}^2 >> m\,r^2\,\theta_o^{\,2} = 0.0526\text{ lbm.in}^2 \qquad 0.4\% = \text{OK}$

check $\quad g\cos\theta >> r\,\dot{\theta}^2 \qquad g >> r\,\dot{\theta}^2_{MAX} = r\,\theta_o^2\,\omega_N^2 = m\,g\,r^2\,\theta_o^2/J \qquad$ same as $\quad J >> m\,r^2\,\theta_o^2$

Steel and aluminum bars with the same geometry should have the same frequency because mass cancels out of ω_N.

Lagrangian Dynamics for Two Link Anthropomorphic Manipulator (Double Pendulum)



Two solid rigid bars with revolute joints A and B
Lengths d_2 and d_3 - mass centers at a_2 and a_3 from proximal ends
Masses m_2 and m_3 - centroidal mass moments of inertia J_2 and J_3
θ_2 CCW from positive x axis — T_2 is torque of ground on bar 2 about pin A, CCW positive
θ_3 CCW from centerline of bar 2 — T_3 is torque of bar 2 on bar 3 about pin B, CCW positive
Gravity g acts along negative y axis

$$q_2 = \theta_2 \qquad \dot{q}_2 = \dot{\theta}_2 \qquad Q_2 = T_2$$
$$q_3 = \theta_3 \qquad \dot{q}_3 = \dot{\theta}_3 \qquad Q_3 = T_3$$

$$x_2 = a_2 \cos\theta_2 \qquad \dot{x}_2 = -a_2\dot{\theta}_2 \sin\theta_2$$
$$y_2 = a_2 \sin\theta_2 \qquad \dot{y}_2 = a_2\dot{\theta}_2 \cos\theta_2$$

$$x_3 = d_2 \cos\theta_2 + a_3 \cos(\theta_2 + \theta_3) \qquad \dot{x}_3 = -d_2\dot{\theta}_2 \sin\theta_2 - a_3(\dot{\theta}_2 + \dot{\theta}_3)\sin(\theta_2 + \theta_3)$$
$$y_3 = d_2 \sin\theta_2 + a_3 \sin(\theta_2 + \theta_3) \qquad \dot{y}_3 = d_2\dot{\theta}_2 \cos\theta_2 + a_3(\dot{\theta}_2 + \dot{\theta}_3)\cos(\theta_2 + \theta_3)$$

$$K = \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \tfrac{1}{2} m_3 (\dot{x}_3^2 + \dot{y}_3^2) + \tfrac{1}{2} J_2 \dot{\theta}_2^2 + \tfrac{1}{2} J_3 (\dot{\theta}_2 + \dot{\theta}_3)^2$$

$$K = \tfrac{1}{2} m_2 a_2^2 \dot{\theta}_2^2 + \tfrac{1}{2} m_3 d_2^2 \dot{\theta}_2^2 + \tfrac{1}{2} m_3 a_3^2 (\dot{\theta}_2 + \dot{\theta}_3)^2 + m_3 d_2 a_3 \dot{\theta}_2 (\dot{\theta}_2 + \dot{\theta}_3)\cos\theta_3$$
$$+ \tfrac{1}{2} J_2 \dot{\theta}_2^2 + \tfrac{1}{2} J_3 (\dot{\theta}_2 + \dot{\theta}_3)^2$$

$$P = m_2 y_2 g + m_3 y_3 g$$

$$P = (m_2 a_2 + m_3 d_2) g \sin\theta_2 + m_3 a_3 g \sin(\theta_2 + \theta_3)$$

$$L = K - P \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = T_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = (m_2 a_2^2 + m_3 d_2^2 + J_2)\dot{\theta}_2 + (m_3 a_3^2 + J_3)(\dot{\theta}_2 + \dot{\theta}_3) + m_3 d_2 a_3 (2\dot{\theta}_2 + \dot{\theta}_3)\cos\theta_3$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = (m_2 a_2^2 + m_3 d_2^2 + J_2)\ddot{\theta}_2 + (m_3 a_3^2 + J_3)(\ddot{\theta}_2 + \ddot{\theta}_3)$$
$$+ m_3 d_2 a_3 (2\ddot{\theta}_2 + \ddot{\theta}_3)\cos\theta_3 - m_3 d_2 a_3 (2\dot{\theta}_2\dot{\theta}_3 + \dot{\theta}_3^2)\sin\theta_3$$

$$\frac{\partial L}{\partial \theta_2} = -(m_2 a_2 + m_3 d_2)\cos\theta_2 g - m_3 a_3 \cos(\theta_2 + \theta_3) g$$

$$T_2 = (m_2 a_2^2 + m_3 d_2^2 + J_2)\ddot{\theta}_2 + (m_3 a_3^2 + J_3)(\ddot{\theta}_2 + \ddot{\theta}_3)$$
$$+ m_3 d_2 a_3 (2\ddot{\theta}_2 + \ddot{\theta}_3)\cos\theta_3 - m_3 d_2 a_3 (2\dot{\theta}_2\dot{\theta}_3 + \dot{\theta}_3^2)\sin\theta_3$$
$$+ (m_2 a_2 + m_3 d_2)\cos\theta_2 g + m_3 a_3 \cos(\theta_2 + \theta_3) g$$

$$T_2 = (m_2 a_2^2 + m_3 d_2^2 + J_2 + m_3 a_3^2 + J_3 + 2 m_3 d_2 a_3 \cos\theta_3)\ddot{\theta}_2$$
$$+ (m_3 a_3^2 + J_3 + m_3 d_2 a_3 \cos\theta_3)(\ddot{\theta}_3)$$
$$- (2 m_3 d_2 a_3 \sin\theta_3)\dot{\theta}_2\dot{\theta}_3$$
$$- (m_3 d_2 a_3 \sin\theta_3)\dot{\theta}_3^2$$
$$+ (m_2 a_2 + m_3 d_2)\cos\theta_2 g + m_3 a_3 \cos(\theta_2 + \theta_3) g$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) - \frac{\partial L}{\partial \theta_3} = T_3$$

$$\frac{\partial L}{\partial \dot{\theta}_3} = (m_3 a_3^2 + J_3)(\dot{\theta}_2 + \dot{\theta}_3) + m_3 d_2 a_3 \dot{\theta}_2 \cos\theta_3$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) = (m_3 a_3^2 + J_3)(\ddot{\theta}_2 + \ddot{\theta}_3) + m_3 d_2 a_3 \ddot{\theta}_2 \cos\theta_3 - m_3 d_2 a_3 \dot{\theta}_2 \dot{\theta}_3 \sin\theta_3$$

$$\frac{\partial L}{\partial \theta_3} = -m_3 d_2 a_3 \dot{\theta}_2 \left(\dot{\theta}_2 + \dot{\theta}_3\right) \sin\theta_3 - m_3 a_3 \cos\left(\theta_2 + \theta_3\right) g$$

$$\begin{aligned} T_3 &= \left(m_3 a_3^2 + J_3\right)\left(\ddot{\theta}_2 + \ddot{\theta}_3\right) + m_3 d_2 a_3 \ddot{\theta}_2 \cos\theta_3 - m_3 d_2 a_3 \dot{\theta}_2 \dot{\theta}_3 \sin\theta_3 \\ &\quad + m_3 d_2 a_3 \dot{\theta}_2 \left(\dot{\theta}_2 + \dot{\theta}_3\right) \sin\theta_3 + m_3 a_3 \cos\left(\theta_2 + \theta_3\right) g \end{aligned}$$

$$\begin{aligned} T_3 &= \left(m_3 a_3^2 + J_3 + m_3 d_2 a_3 \cos\theta_3\right) \ddot{\theta}_2 \\ &+ \left(m_3 a_3^2 + J_3\right) \ddot{\theta}_3 \\ &+ \left(m_3 d_2 a_3 \sin\theta_3\right) \dot{\theta}_2^2 \\ &+ m_3 a_3 \cos\left(\theta_2 + \theta_3\right) g \end{aligned}$$

+++

$$\begin{aligned} J_B &= m_3 a_3^2 + J_3 \\ J_A &= J_B + m_2 a_2^2 + m_3 d_2^2 + J_2 + 2 m_3 d_2 a_3 \cos\theta_3 \\ C &= J_B + m_3 d_2 a_3 \cos\theta_3 \\ D &= m_3 d_2 a_3 \sin\theta_3 \\ G_2 &= \left(m_2 a_2 + m_3 d_2\right) g \cos\theta_2 \\ G_3 &= m_3 a_3 g \cos\left(\theta_2 + \theta_3\right) \end{aligned}$$

inverse dynamics

$$\begin{aligned} T_2 &= J_A \ddot{\theta}_2 + C\ddot{\theta}_3 - 2D\dot{\theta}_2\dot{\theta}_3 - D\dot{\theta}_3^2 + G_2 + G_3 \\ T_3 &= C\ddot{\theta}_2 + J_B \ddot{\theta}_3 + D\dot{\theta}_2^2 + G_3 \end{aligned}$$

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} + \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix}$$

forward dynamics

$$\begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix}^{-1} \left(\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} - \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} - \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix} \right)$$

third order dynamics using direct time derivatives

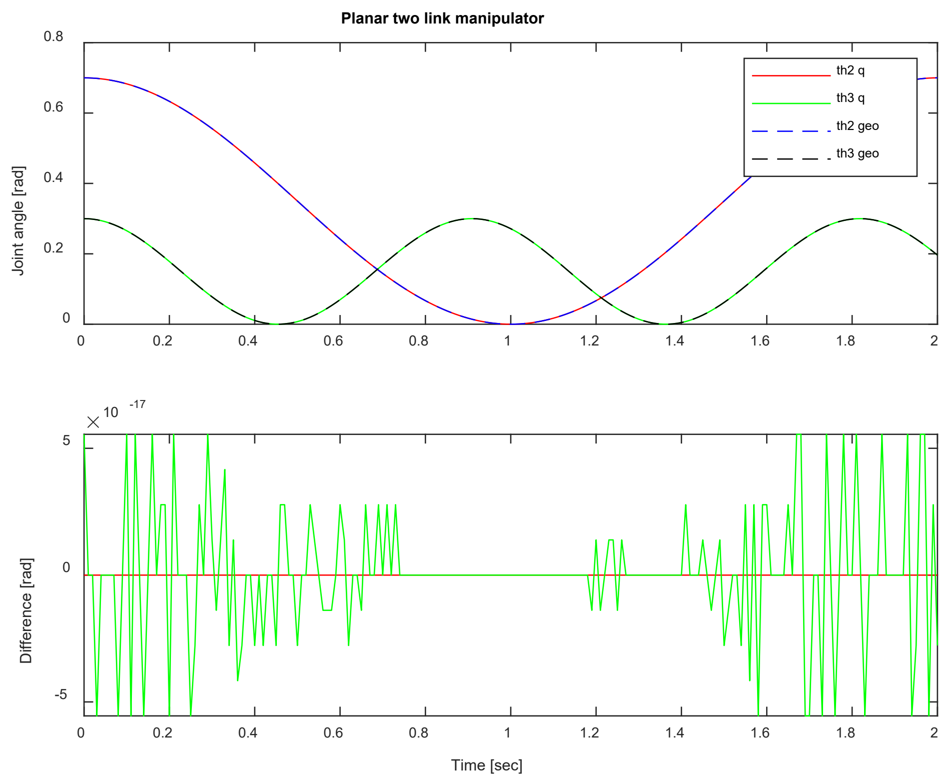
$$\begin{aligned} \dot{T}_2 &= J_A \dddot{\theta}_2 + \dot{J}_A \ddot{\theta}_2 + C\dddot{\theta}_3 + \dot{C}\ddot{\theta}_3 - 2\dot{D}\dot{\theta}_2\dot{\theta}_3 - 2D\ddot{\theta}_2\dot{\theta}_3 - 2D\dot{\theta}_2\ddot{\theta}_3 - \dot{D}\dot{\theta}_3^2 - 2D\dot{\theta}_3\ddot{\theta}_3 + \dot{G}_2 + \dot{G}_3 \\ \dot{T}_3 &= C\dddot{\theta}_2 + \dot{C}\ddot{\theta}_2 + J_B \dddot{\theta}_3 + \dot{D}\dot{\theta}_2^2 + 2D\dot{\theta}_2\ddot{\theta}_2 + \dot{G}_3 \end{aligned}$$

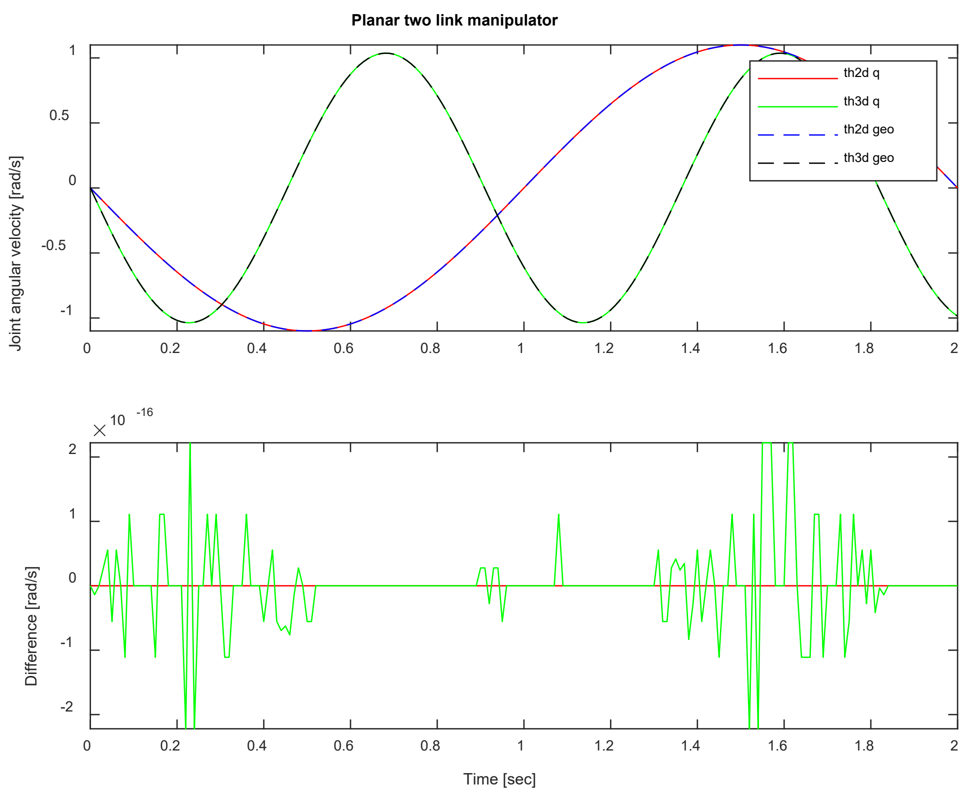
$$\dot{J}_A = -2m_3 d_2 a_3 \dot{\theta}_3 \sin\theta_3$$
$$\dot{C} = -m_3 d_2 a_3 \dot{\theta}_3 \sin\theta_3$$
$$\dot{D} = m_3 d_2 a_3 \dot{\theta}_3 \cos\theta_3$$
$$\dot{G}_2 = -\left(m_2 a_2 + m_3 d_2\right) g\, \dot{\theta}_2 \sin\theta_2$$
$$\dot{G}_3 = -m_3 a_3 g\, (\dot{\theta}_2 + \dot{\theta}_3) \sin(\theta_2 + \theta_3)$$

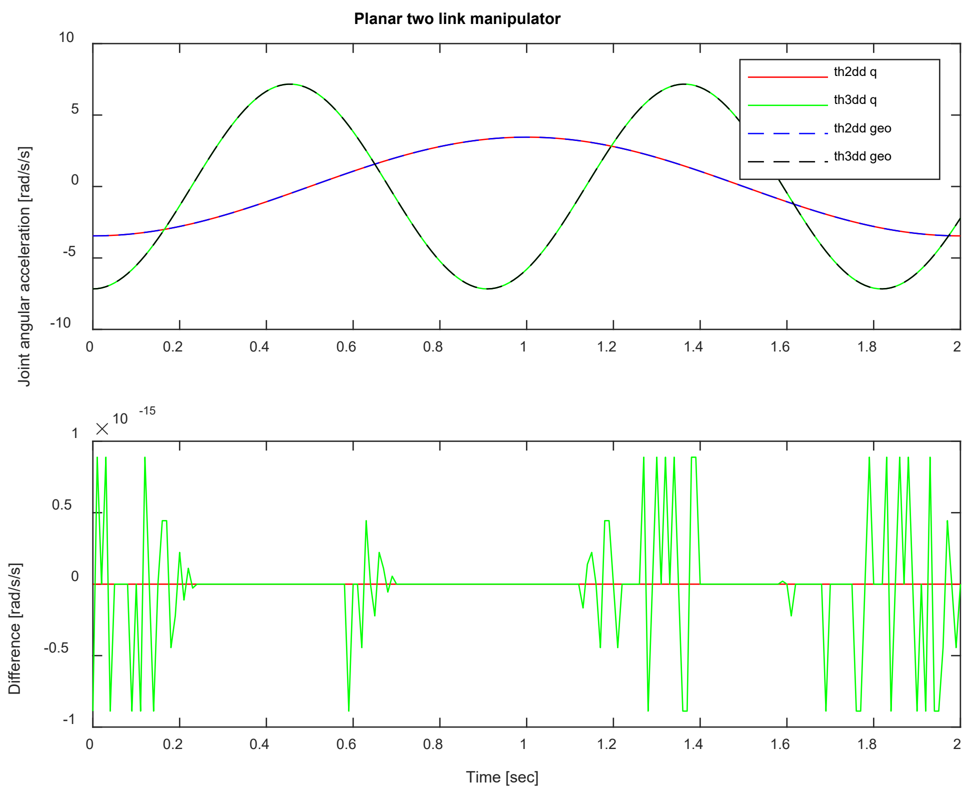
$$\dot{T}_2 = J_A \dddot{\theta}_2 + \dot{J}_A \ddot{\theta}_2 + C\dddot{\theta}_3 + \dot{C}\ddot{\theta}_3 - 2\dot{D}\dot{\theta}_2\dot{\theta}_3 - 2D\ddot{\theta}_2\dot{\theta}_3 - 2D\dot{\theta}_2\ddot{\theta}_3 - \dot{D}\dot{\theta}_3^2 - 2D\dot{\theta}_3\ddot{\theta}_3 + \dot{G}_2 + \dot{G}_3$$
$$\dot{T}_3 = C\dddot{\theta}_2 + \dot{C}\ddot{\theta}_2 + J_B\dddot{\theta}_3 + \dot{D}\dot{\theta}_2^2 + 2D\dot{\theta}_2\ddot{\theta}_2 + \dot{G}_3$$

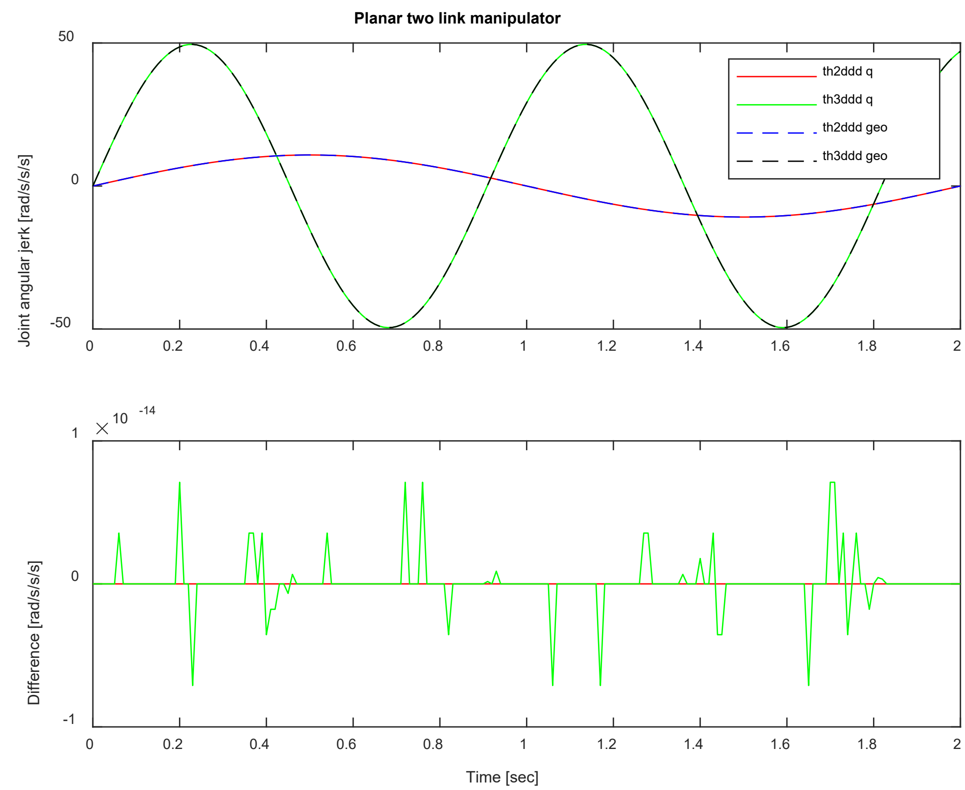
$$\begin{aligned}\dot{T}_2 = {} & J_A \dddot{\theta}_2 - 2m_3 d_2 a_3 \dot{\theta}_3 \sin\theta_3 \ddot{\theta}_2 + C\dddot{\theta}_3 - m_3 d_2 a_3 \dot{\theta}_3 \sin\theta_3 \ddot{\theta}_3 \\ & - 2m_3 d_2 a_3 \dot{\theta}_3 \cos\theta_3 \dot{\theta}_2 \dot{\theta}_3 - 2D\ddot{\theta}_2\dot{\theta}_3 - 2D\dot{\theta}_2\ddot{\theta}_3 - m_3 d_2 a_3 \dot{\theta}_3 \cos\theta_3 \dot{\theta}_3^2 - 2D\dot{\theta}_3\ddot{\theta}_3 \\ & - \left(m_2 a_2 + m_3 d_2\right) g\, \dot{\theta}_2 \sin\theta_2 - m_3 a_3 g\, (\dot{\theta}_2 + \dot{\theta}_3) \sin(\theta_2 + \theta_3)\end{aligned}$$
$$\dot{T}_3 = C\dddot{\theta}_2 - m_3 d_2 a_3 \dot{\theta}_3 \sin\theta_3 \ddot{\theta}_2 + J_B \dddot{\theta}_3 + m_3 d_2 a_3 \dot{\theta}_3 \cos\theta_3 \dot{\theta}_2^2 + 2D\dot{\theta}_2\ddot{\theta}_2 - m_3 a_3 g\, (\dot{\theta}_2 + \dot{\theta}_3) \sin(\theta_2 + \theta_3)$$

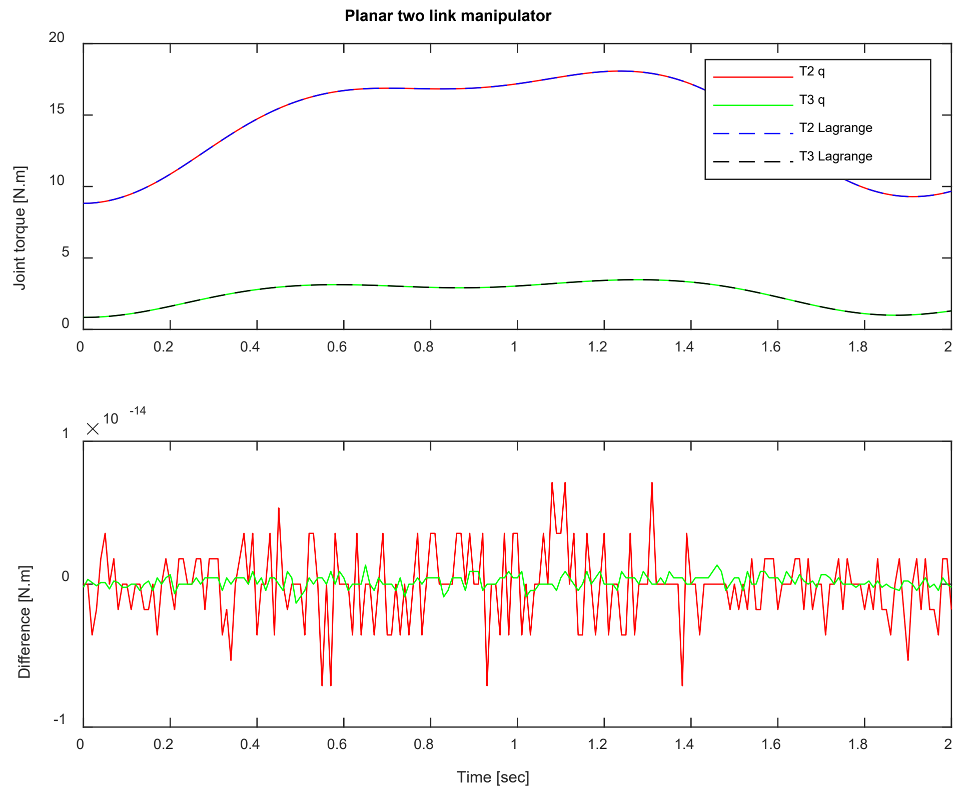
$$\begin{aligned}\dot{T}_2 = {} & J_A \dddot{\theta}_2 + C\dddot{\theta}_3 \\ & - m_3 d_2 a_3 \cos\theta_3 \left(2\dot{\theta}_2 + \dot{\theta}_3\right)\dot{\theta}_3{}^2 - 2\left(D + m_3 d_2 a_3 \sin\theta_3\right)\ddot{\theta}_2\dot{\theta}_3 - 2D\dot{\theta}_2\ddot{\theta}_3 - \left(2D + m_3 d_2 a_3 \sin\theta_3\right)\dot{\theta}_3\ddot{\theta}_3 \\ & - \left(m_2 a_2 + m_3 d_2\right) g\, \dot{\theta}_2 \sin\theta_2 - m_3 a_3 g\, (\dot{\theta}_2 + \dot{\theta}_3) \sin(\theta_2 + \theta_3)\end{aligned}$$
$$\dot{T}_3 = C\dddot{\theta}_2 + J_B \dddot{\theta}_3 + m_3 d_2 a_3 \left(\cos\theta_3 \dot{\theta}_2{}^2 - \sin\theta_3 \ddot{\theta}_2\right)\dot{\theta}_3 + 2D\dot{\theta}_2\ddot{\theta}_2 - m_3 a_3 g\, (\dot{\theta}_2 + \dot{\theta}_3) \sin(\theta_2 + \theta_3)$$

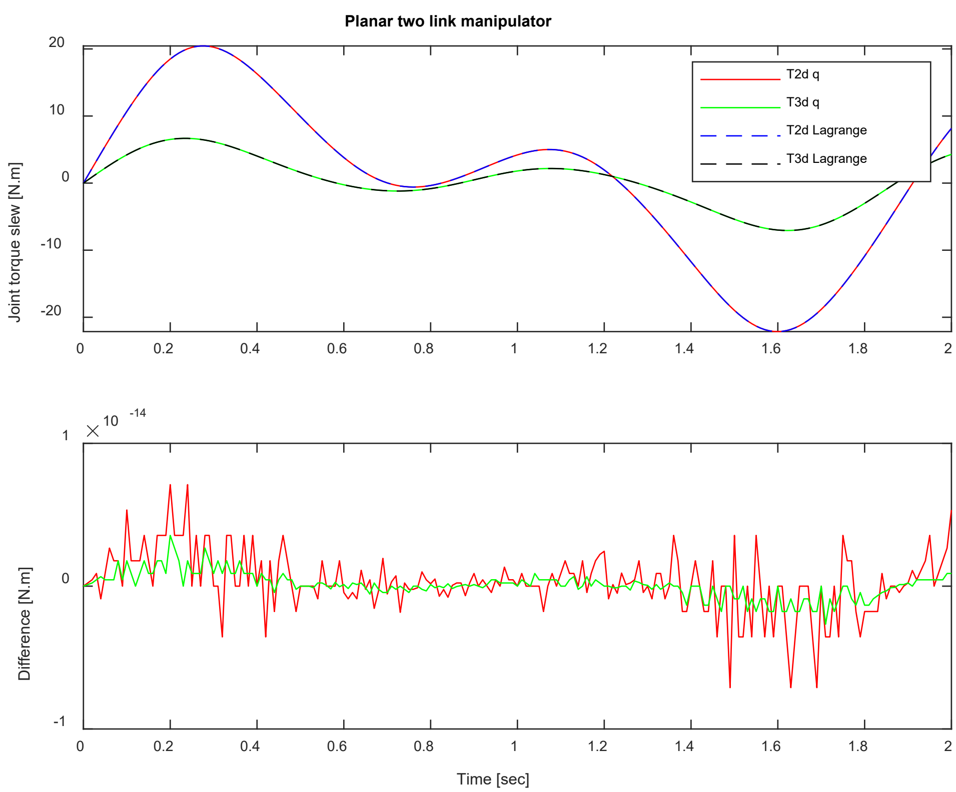

Planar two link manipulator
Joint angle [rad]
th2 q
th3 q
th2 geo
th3 geo
× 10^-17
Difference [rad]
Time [sec]


Planar two link manipulator
Joint angular velocity [rad/s]
th2d q
th3d q
th2d geo
th3d geo
× 10^-16
Difference [rad/s]
Time [sec]

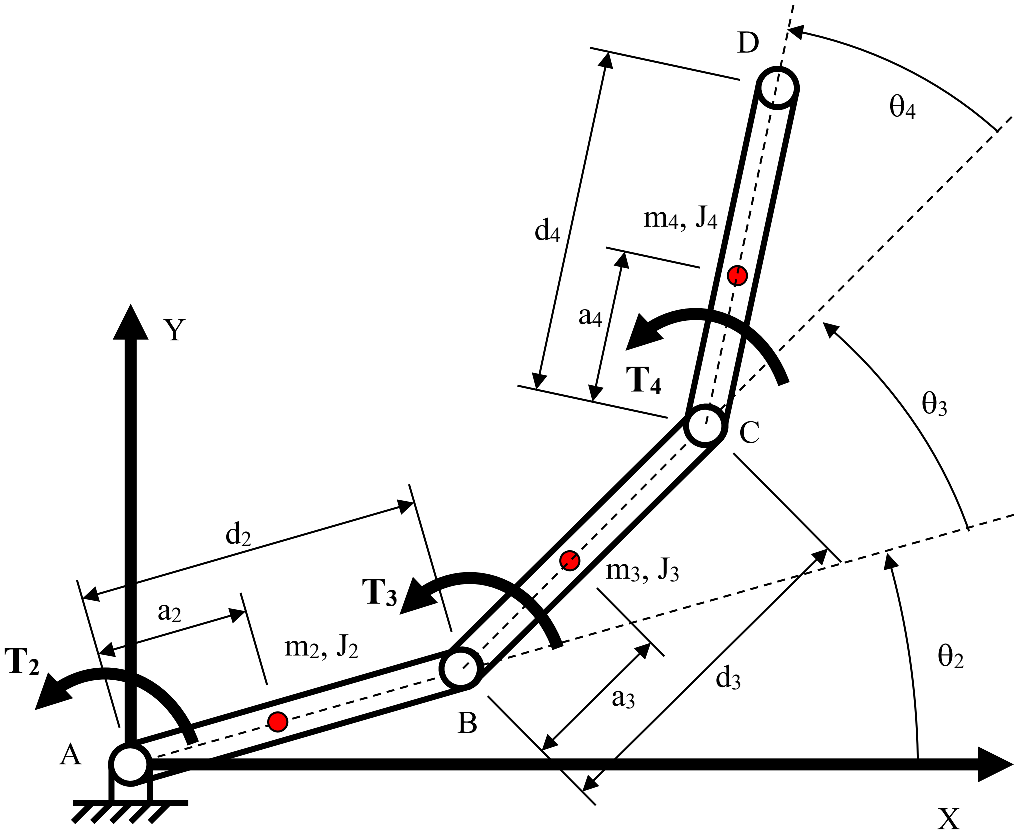

Planar two link manipulator
Joint angular acceleration [rad/s/s]
th2dd q
th3dd q
th2dd geo
th3dd geo
10
5
0
-5
-10
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
×10⁻¹⁵
Difference [rad/s/s]
1
0.5
0
-0.5
-1
Time [sec]


Planar two link manipulator
Joint angular jerk [rad/s/s/s]
th2ddd q
th3ddd q
th2ddd geo
th3ddd geo
50
0
-50
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
×10⁻¹⁴
Difference [rad/s/s/s]
1
0
-1
Time [sec]


Planar two link manipulator
Joint torque [N.m]
T2 q
T3 q
T2 Lagrange
T3 Lagrange
Difference [N.m]
× 10^-14
Time [sec]


Planar two link manipulator
Joint torque slew [N.m]
T2d q
T3d q
T2d Lagrange
T3d Lagrange
Difference [N.m]
× 10^-14
Time [sec]

Lagrangian Dynamics for Three Link Anthropomorphic Manipulator



Three solid rigid bars with revolute joints A, B and C
Lengths d_2 d_3 d_4 - mass centers at a_2 a_3 a_4 from proximal ends
Masses m_2 m_3 m_4 - centroidal mass moments of inertia J_2 J_3 J_4

θ_2 CCW from positive x axis	T_2 is torque of ground on bar 2 about pin A, CCW positive
θ_3 CCW from centerline of bar 2	T_3 is torque of bar 2 on bar 3 about pin B, CCW positive
θ_4 CCW from centerline of bar 3	T_4 is torque of bar 3 on bar 4 about pin C, CCW positive

Gravity g acts along negative y axis

$$q_2 = \theta_2 \qquad \dot{q}_2 = \dot{\theta}_2 \qquad Q_2 = T_2$$
$$q_3 = \theta_3 \qquad \dot{q}_3 = \dot{\theta}_3 \qquad Q_3 = T_3$$
$$q_4 = \theta_4 \qquad \dot{q}_4 = \dot{\theta}_4 \qquad Q_4 = T_4$$

$$x_2 = a_2 \cos\theta_2 \qquad \dot{x}_2 = -a_2\dot{\theta}_2 \sin\theta_2$$
$$y_2 = a_2 \sin\theta_2 \qquad \dot{y}_2 = a_2\dot{\theta}_2 \cos\theta_2$$

$$x_3 = d_2 \cos\theta_2 + a_3 \cos(\theta_2 + \theta_3) \qquad \dot{x}_3 = -d_2\dot{\theta}_2 \sin\theta_2 - a_3(\dot{\theta}_2 + \dot{\theta}_3)\sin(\theta_2 + \theta_3)$$
$$y_3 = d_2 \sin\theta_2 + a_3 \sin(\theta_2 + \theta_3) \qquad \dot{y}_3 = d_2\dot{\theta}_2 \cos\theta_2 + a_3(\dot{\theta}_2 + \dot{\theta}_3)\cos(\theta_2 + \theta_3)$$

$$x_4 = d_2\cos\theta_2 + d_3\cos(\theta_2+\theta_3) + a_4\cos(\theta_2+\theta_3+\theta_4)$$
$$\dot{x}_4 = -d_2\dot\theta_2\sin\theta_2 - d_3(\dot\theta_2+\dot\theta_3)\sin(\theta_2+\theta_3) - a_4(\dot\theta_2+\dot\theta_3+\dot\theta_4)\sin(\theta_2+\theta_3+\theta_4)$$

$$y_4 = d_2\sin\theta_2 + d_3\sin(\theta_2+\theta_3) + a_4\sin(\theta_2+\theta_3+\theta_4)$$
$$\dot{y}_4 = d_2\dot\theta_2\cos\theta_2 + d_3(\dot\theta_2+\dot\theta_3)\cos(\theta_2+\theta_3) + a_4(\dot\theta_2+\dot\theta_3+\dot\theta_4)\cos(\theta_2+\theta_3+\theta_4)$$

$$K = \tfrac{1}{2}m_2(\dot x_2^2+\dot y_2^2) + \tfrac{1}{2}m_3(\dot x_3^2+\dot y_3^2) + \tfrac{1}{2}m_4(\dot x_4^2+\dot y_4^2)$$
$$+\tfrac{1}{2}J_2\dot\theta_2^2 + \tfrac{1}{2}J_3(\dot\theta_2+\dot\theta_3)^2 + \tfrac{1}{2}J_4(\dot\theta_2+\dot\theta_3+\dot\theta_4)^2$$

$$K = \tfrac{1}{2}m_2a_2^2\dot\theta_2^2 + \tfrac{1}{2}m_3d_2^2\dot\theta_2^2 + \tfrac{1}{2}m_3a_3^2(\dot\theta_2+\dot\theta_3)^2 + m_3d_2a_3\dot\theta_2(\dot\theta_2+\dot\theta_3)\cos\theta_3$$
$$+\tfrac{1}{2}m_4d_2^2\dot\theta_2^2 + \tfrac{1}{2}m_4d_3^2(\dot\theta_2+\dot\theta_3)^2 + \tfrac{1}{2}m_4a_4^2(\dot\theta_2+\dot\theta_3+\dot\theta_4)^2$$
$$+m_4d_2d_3\dot\theta_2(\dot\theta_2+\dot\theta_3)\cos\theta_3 + m_4d_2a_4\dot\theta_2(\dot\theta_2+\dot\theta_3+\dot\theta_4)\cos(\theta_3+\theta_4)$$
$$+m_4d_3a_4(\dot\theta_2+\dot\theta_3)(\dot\theta_2+\dot\theta_3+\dot\theta_4)\cos\theta_4$$
$$+\tfrac{1}{2}J_2\dot\theta_2^2 + \tfrac{1}{2}J_3(\dot\theta_2+\dot\theta_3)^2 + \tfrac{1}{2}J_4(\dot\theta_2+\dot\theta_3+\dot\theta_4)^2$$

$$P = m_2y_2g + m_3y_3g + m_4y_4g$$

$$P = (m_2a_2+m_3d_2+m_4d_2)g\sin\theta_2 + (m_3a_3+m_4d_3)g\sin(\theta_2+\theta_3) + m_4a_4g\sin(\theta_2+\theta_3+\theta_4)$$

$$L = K - P \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = T_2$$

$$\frac{\partial L}{\partial\dot\theta_2} = (m_2a_2^2+m_3d_2^2+m_4d_2^2+J_2)\dot\theta_2 + (m_3a_3^2+m_4d_3^2+J_3)(\dot\theta_2+\dot\theta_3) + (m_4a_4^2+J_4)(\dot\theta_2+\dot\theta_3+\dot\theta_4)$$
$$+(m_3d_2a_3+m_4d_2d_3)(2\dot\theta_2+\dot\theta_3)\cos\theta_3 + m_4d_2a_4(2\dot\theta_2+\dot\theta_3+\dot\theta_4)\cos(\theta_3+\theta_4)$$
$$+m_4d_3a_4(2\dot\theta_2+2\dot\theta_3+\dot\theta_4)\cos\theta_4$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_2}\right) = (m_2a_2^2+m_3d_2^2+m_4d_2^2+J_2)\ddot\theta_2 + (m_3a_3^2+m_4d_3^2+J_3)(\ddot\theta_2+\ddot\theta_3) + (m_4a_4^2+J_4)(\ddot\theta_2+\ddot\theta_3+\ddot\theta_4)$$
$$+(m_3d_2a_3+m_4d_2d_3)(2\ddot\theta_2+\ddot\theta_3)\cos\theta_3 - (m_3d_2a_3+m_4d_2d_3)\dot\theta_3(2\dot\theta_2+\dot\theta_3)\sin\theta_3$$
$$+m_4d_2a_4(2\ddot\theta_2+\ddot\theta_3+\ddot\theta_4)\cos(\theta_3+\theta_4) - m_4d_2a_4(\dot\theta_3+\dot\theta_4)(2\dot\theta_2+\dot\theta_3+\dot\theta_4)\sin(\theta_3+\theta_4)$$
$$+m_4d_3a_4(2\ddot\theta_2+2\ddot\theta_3+\ddot\theta_4)\cos\theta_4 - m_4d_3a_4\dot\theta_4(2\dot\theta_2+2\dot\theta_3+\dot\theta_4)\sin\theta_4$$

$$\frac{\partial L}{\partial \theta_2} = -(m_2 a_2 + m_3 d_2 + m_4 d_2) g \cos\theta_2 - (m_3 a_3 + m_4 d_3) g \cos(\theta_2 + \theta_3) - m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)$$

$$\begin{aligned}
T_2 = &\left(m_2 a_2^2 + m_3 d_2^2 + m_4 d_2^2 + J_2\right)\ddot\theta_2 + \left(m_3 a_3^2 + m_4 d_3^2 + J_3\right)\left(\ddot\theta_2 + \ddot\theta_3\right) + \left(m_4 a_4^2 + J_4\right)\left(\ddot\theta_2 + \ddot\theta_3 + \ddot\theta_4\right) \\
&+ (m_3 d_2 a_3 + m_4 d_2 d_3)\left(2\ddot\theta_2 + \ddot\theta_3\right)\cos\theta_3 - (m_3 d_2 a_3 + m_4 d_2 d_3)\dot\theta_3\left(2\dot\theta_2 + \dot\theta_3\right)\sin\theta_3 \\
&+ m_4 d_2 a_4\left(2\ddot\theta_2 + \ddot\theta_3 + \ddot\theta_4\right)\cos(\theta_3 + \theta_4) - m_4 d_2 a_4\left(\dot\theta_3 + \dot\theta_4\right)\left(2\dot\theta_2 + \dot\theta_3 + \dot\theta_4\right)\sin(\theta_3 + \theta_4) \\
&+ m_4 d_3 a_4\left(2\ddot\theta_2 + 2\ddot\theta_3 + \ddot\theta_4\right)\cos\theta_4 - m_4 d_3 a_4 \dot\theta_4\left(2\dot\theta_2 + 2\dot\theta_3 + \dot\theta_4\right)\sin\theta_4 \\
&+ (m_2 a_2 + m_3 d_2 + m_4 d_2) g \cos\theta_2 + (m_3 a_3 + m_4 d_3) g \cos(\theta_2 + \theta_3) + m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)
\end{aligned}$$

$$\begin{aligned}
T_2 = &\begin{pmatrix} m_2 a_2^2 + m_3 d_2^2 + m_3 a_3^2 + m_4 d_2^2 + m_4 d_3^2 + m_4 a_4^2 + J_2 + J_3 + J_4 \\ + 2(m_3 d_2 a_3 + m_4 d_2 d_3)\cos\theta_3 + 2 m_4 d_2 a_4 \cos(\theta_3 + \theta_4) + 2 m_4 d_3 a_4 \cos\theta_4 \end{pmatrix}\ddot\theta_2 \\
&+ \begin{pmatrix} m_3 a_3^2 + m_4 d_3^2 + m_4 a_4^2 + J_4 + J_3 \\ + (m_3 d_2 a_3 + m_4 d_2 d_3)\cos\theta_3 + m_4 d_2 a_4 \cos(\theta_3 + \theta_4) + 2 m_4 d_3 a_4 \cos\theta_4 \end{pmatrix}\ddot\theta_3 \\
&+ \left(m_4 a_4^2 + J_4 + m_4 d_2 a_4 \cos(\theta_3 + \theta_4) + m_4 d_3 a_4 \cos\theta_4\right)\ddot\theta_4 \\
&- \left((m_3 d_2 a_3 + m_4 d_2 d_3)\sin\theta_3 + m_4 d_2 a_4 \sin(\theta_3 + \theta_4)\right)\dot\theta_3^2 \\
&- \left(m_4 d_2 a_4 \sin(\theta_3 + \theta_4) + m_4 d_3 a_4 \sin\theta_4\right)\dot\theta_4^2 \\
&- 2\left((m_3 d_2 a_3 + m_4 d_2 d_3)\sin\theta_3 + m_4 d_2 a_4 \sin(\theta_3 + \theta_4)\right)\dot\theta_2\dot\theta_3 \\
&- 2\left(m_4 d_2 a_4 \sin(\theta_3 + \theta_4) + m_4 d_3 a_4 \sin\theta_4\right)\dot\theta_2\dot\theta_4 \\
&- 2\left(m_4 d_2 a_4 \sin(\theta_3 + \theta_4) + m_4 d_3 a_4 \sin\theta_4\right)\dot\theta_3\dot\theta_4 \\
&+ (m_2 a_2 + m_3 d_2 + m_4 d_2) g \cos\theta_2 + (m_3 a_3 + m_4 d_3) g \cos(\theta_2 + \theta_3) + m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)
\end{aligned}$$

$$\begin{aligned}
T_2 = &J_A \ddot\theta_2 + A\ddot\theta_3 + B\ddot\theta_4 - 2(D + E)\dot\theta_2\dot\theta_3 - 2(E + F)\dot\theta_2\dot\theta_4 - 2(E + F)\dot\theta_3\dot\theta_4 \\
&- (D + E)\dot\theta_3^2 - (E + F)\dot\theta_4^2 + G_2 + G_3 + G_4
\end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_3}\right) - \frac{\partial L}{\partial \theta_3} = T_3$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot\theta_3} = &\left(m_3 a_3^2 + m_4 d_3^2 + J_3\right)\left(\dot\theta_2 + \dot\theta_3\right) + \left(m_4 a_4^2 + J_4\right)\left(\dot\theta_2 + \dot\theta_3 + \dot\theta_4\right) \\
&+ (m_3 d_2 a_3 + m_4 d_2 d_3)\dot\theta_2 \cos\theta_3 + m_4 d_2 a_4 \dot\theta_2 \cos(\theta_3 + \theta_4) + m_4 d_3 a_4\left(2\dot\theta_2 + 2\dot\theta_3 + \dot\theta_4\right)\cos\theta_4
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_3}\right) = &\left(m_3 a_3^2 + m_4 d_3^2 + J_3\right)\left(\ddot\theta_2 + \ddot\theta_3\right) + \left(m_4 a_4^2 + J_4\right)\left(\ddot\theta_2 + \ddot\theta_3 + \ddot\theta_4\right) \\
&+ (m_3 d_2 a_3 + m_4 d_2 d_3)\ddot\theta_2 \cos\theta_3 - (m_3 d_2 a_3 + m_4 d_2 d_3)\dot\theta_2\dot\theta_3 \sin\theta_3 \\
&+ m_4 d_2 a_4 \ddot\theta_2 \cos(\theta_3 + \theta_4) - m_4 d_2 a_4 \dot\theta_2\left(\dot\theta_3 + \dot\theta_4\right)\sin(\theta_3 + \theta_4) \\
&+ m_4 d_3 a_4\left(2\ddot\theta_2 + 2\ddot\theta_3 + \ddot\theta_4\right)\cos\theta_4 - m_4 d_3 a_4\left(2\dot\theta_2 + 2\dot\theta_3 + \dot\theta_4\right)\dot\theta_4 \sin\theta_4
\end{aligned}$$

$$\frac{\partial L}{\partial \theta_3} = -m_3 d_2 a_3 \dot\theta_2 (\dot\theta_2 + \dot\theta_3)\sin\theta_3 - m_4 d_2 d_3 \dot\theta_2 (\dot\theta_2 + \dot\theta_3)\sin\theta_3 - m_4 d_2 a_4 \dot\theta_2 (\dot\theta_2 + \dot\theta_3 + \dot\theta_4)\sin(\theta_3 + \theta_4)$$
$$- (m_3 a_3 + m_4 d_3) g \cos(\theta_2 + \theta_3) - m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)$$

$$T_3 = (m_3 a_3^2 + m_4 d_3^2 + J_3)(\ddot\theta_2 + \ddot\theta_3) + (m_4 a_4^2 + J_4)(\ddot\theta_2 + \ddot\theta_3 + \ddot\theta_4)$$
$$+ (m_3 d_2 a_3 + m_4 d_2 d_3)\ddot\theta_2 \cos\theta_3 - (m_3 d_2 a_3 + m_4 d_2 d_3)\dot\theta_2 \dot\theta_3 \sin\theta_3$$
$$+ m_4 d_2 a_4 \ddot\theta_2 \cos(\theta_3 + \theta_4) - m_4 d_2 a_4 \dot\theta_2 (\dot\theta_3 + \dot\theta_4)\sin(\theta_3 + \theta_4)$$
$$+ m_4 d_3 a_4 (2\ddot\theta_2 + 2\ddot\theta_3 + \ddot\theta_4)\cos\theta_4 - m_4 d_3 a_4 (2\dot\theta_2 + 2\dot\theta_3 + \dot\theta_4)\dot\theta_4 \sin\theta_4$$
$$+ m_3 d_2 a_3 \dot\theta_2 (\dot\theta_2 + \dot\theta_3)\sin\theta_3 + m_4 d_2 d_3 \dot\theta_2 (\dot\theta_2 + \dot\theta_3)\sin\theta_3 + m_4 d_2 a_4 \dot\theta_2 (\dot\theta_2 + \dot\theta_3 + \dot\theta_4)\sin(\theta_3 + \theta_4)$$
$$+ (m_3 a_3 + m_4 d_3) g \cos(\theta_2 + \theta_3) + m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)$$

$$T_3 = \begin{pmatrix} m_3 a_3^2 + m_4 d_3^2 + m_4 a_4^2 + J_3 + J_4 \\ + (m_3 d_2 a_3 + m_4 d_2 d_3)\cos\theta_3 + m_4 d_2 a_4 \cos(\theta_3 + \theta_4) + 2 m_4 d_3 a_4 \cos\theta_4 \end{pmatrix}\ddot\theta_2$$
$$+ \begin{pmatrix} m_3 a_3^2 + m_4 d_3^2 + m_4 a_4^2 + J_3 + J_4 \\ + 2 m_4 d_3 a_4 \cos\theta_4 \end{pmatrix}\ddot\theta_3$$
$$+ (m_4 a_4^2 + J_4 + m_4 d_3 a_4 \cos\theta_4)\ddot\theta_4$$
$$- 2 m_4 d_3 a_4 \sin\theta_4 \dot\theta_2 \dot\theta_4$$
$$- 2 m_4 d_3 a_4 \sin\theta_4 \dot\theta_3 \dot\theta_4$$
$$+ (m_3 d_2 a_3 + m_4 d_2 d_3)\sin\theta_3 \dot\theta_2^2 + m_4 d_2 a_4 \sin(\theta_3 + \theta_4)\dot\theta_2^2$$
$$- m_4 d_3 a_4 \sin\theta_4 \dot\theta_4^2$$
$$+ (m_3 a_3 + m_4 d_3) g \cos(\theta_2 + \theta_3) + m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)$$

$$T_3 = A\ddot\theta_2 + J_B \ddot\theta_3 + C\ddot\theta_4 - 2F\dot\theta_2\dot\theta_4 - 2F\dot\theta_3\dot\theta_4 + (D + E)\dot\theta_2^2 - F\dot\theta_4^2 + G_3 + G_4$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_4}\right) - \frac{\partial L}{\partial \theta_4} = T_4$$

$$\frac{\partial L}{\partial \dot\theta_4} = (m_4 a_4^2 + J_4)(\dot\theta_2 + \dot\theta_3 + \dot\theta_4) + m_4 d_2 a_4 \dot\theta_2 \cos(\theta_3 + \theta_4) + m_4 d_3 a_4 (\dot\theta_2 + \dot\theta_3)\cos\theta_4$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_4}\right) = (m_4 a_4^2 + J_4)(\ddot\theta_2 + \ddot\theta_3 + \ddot\theta_4)$$
$$+ m_4 d_2 a_4 \ddot\theta_2 \cos(\theta_3 + \theta_4) - m_4 d_2 a_4 \dot\theta_2 (\dot\theta_3 + \dot\theta_4)\sin(\theta_3 + \theta_4)$$
$$+ m_4 d_3 a_4 (\ddot\theta_2 + \ddot\theta_3)\cos\theta_4 - m_4 d_3 a_4 \dot\theta_4 (\dot\theta_2 + \dot\theta_3)\sin\theta_4$$

$$\frac{\partial L}{\partial \theta_4} = -m_4 d_2 a_4 \dot{\theta}_2 \left(\dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4\right)\sin(\theta_3 + \theta_4) - m_4 d_3 a_4 \left(\dot{\theta}_2 + \dot{\theta}_3\right)\left(\dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4\right)\sin\theta_4$$
$$- m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)$$

$$T_4 = \left(m_4 a_4^2 + J_4\right)\left(\ddot{\theta}_2 + \ddot{\theta}_3 + \ddot{\theta}_4\right)$$
$$+ m_4 d_2 a_4 \ddot{\theta}_2 \cos(\theta_3 + \theta_4) - m_4 d_2 a_4 \dot{\theta}_2 \left(\dot{\theta}_3 + \dot{\theta}_4\right)\sin(\theta_3 + \theta_4)$$
$$+ m_4 d_3 a_4 \left(\ddot{\theta}_2 + \ddot{\theta}_3\right)\cos\theta_4 - m_4 d_3 a_4 \dot{\theta}_4 \left(\dot{\theta}_2 + \dot{\theta}_3\right)\sin\theta_4$$
$$+ m_4 d_2 a_4 \dot{\theta}_2 \left(\dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4\right)\sin(\theta_3 + \theta_4) + m_4 d_3 a_4 \left(\dot{\theta}_2 + \dot{\theta}_3\right)\left(\dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4\right)\sin\theta_4$$
$$+ m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)$$

$$T_4 = \left(m_4 a_4^2 + J_4 + m_4 d_2 a_4 \cos(\theta_3 + \theta_4) + m_4 d_3 a_4 \cos\theta_4\right)\ddot{\theta}_2$$
$$+ \left(m_4 a_4^2 + J_4 + m_4 d_3 a_4 \cos\theta_4\right)\ddot{\theta}_3$$
$$+ \left(m_4 a_4^2 + J_4\right)\ddot{\theta}_4$$
$$+ 2 m_4 d_3 a_4 \sin\theta_4 \dot{\theta}_2 \dot{\theta}_3$$
$$+ \left(m_4 d_3 a_4 \sin\theta_4 + m_4 d_2 a_4 \sin(\theta_3 + \theta_4)\right)\dot{\theta}_2^2$$
$$+ m_4 d_3 a_4 \sin\theta_4 \dot{\theta}_3^2$$
$$+ m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)$$

$$T_4 = B\ddot{\theta}_2 + + C\ddot{\theta}_3 + J_C \ddot{\theta}_4 + 2F\dot{\theta}_2\dot{\theta}_3 + (E + F)\dot{\theta}_2^2 + F\dot{\theta}_3^2 + G_4$$

++

$$J_C = m_4 a_4^2 + J_4$$
$$J_B = J_C + m_3 a_3^2 + m_4 d_3^2 + J_3 + 2 m_4 d_3 a_4 \cos\theta_4$$
$$J_A = J_B + m_2 a_2^2 + m_3 d_2^2 + m_4 d_2^2 + J_2 + 2(m_3 d_2 a_3 + m_4 d_2 d_3)\cos\theta_3 + 2 m_4 d_2 a_4 \cos(\theta_3 + \theta_4)$$

$$A = J_B + (m_3 d_2 a_3 + m_4 d_2 d_3)\cos\theta_3 + m_4 d_2 a_4 \cos(\theta_3 + \theta_4)$$
$$B = J_C + m_4 d_3 a_4 \cos\theta_4 + m_4 d_2 a_4 \cos(\theta_3 + \theta_4)$$
$$C = J_C + m_4 d_3 a_4 \cos\theta_4$$

$$D = (m_3 d_2 a_3 + m_4 d_2 d_3)\sin\theta_3$$
$$E = m_4 d_2 a_4 \sin(\theta_3 + \theta_4)$$
$$F = m_4 d_3 a_4 \sin\theta_4$$

$$G_4 = m_4 a_4 g \cos(\theta_2 + \theta_3 + \theta_4)$$
$$G_3 = (m_3 a_3 + m_4 d_3) g \cos(\theta_2 + \theta_3)$$
$$G_2 = (m_2 a_2 + m_3 d_2 + m_4 d_2) g \cos\theta_2$$

$$T_2 = J_A\ddot{\theta}_2 + A\,\ddot{\theta}_3 + B\ddot{\theta}_4 - 2(D+E)\dot{\theta}_2\dot{\theta}_3 - 2(E+F)\dot{\theta}_2\dot{\theta}_4 - 2(E+F)\dot{\theta}_3\dot{\theta}_4$$
$$- (D+E)\dot{\theta}_3^2 - (E+F)\dot{\theta}_4^2 + G_2 + G_3 + G_4$$

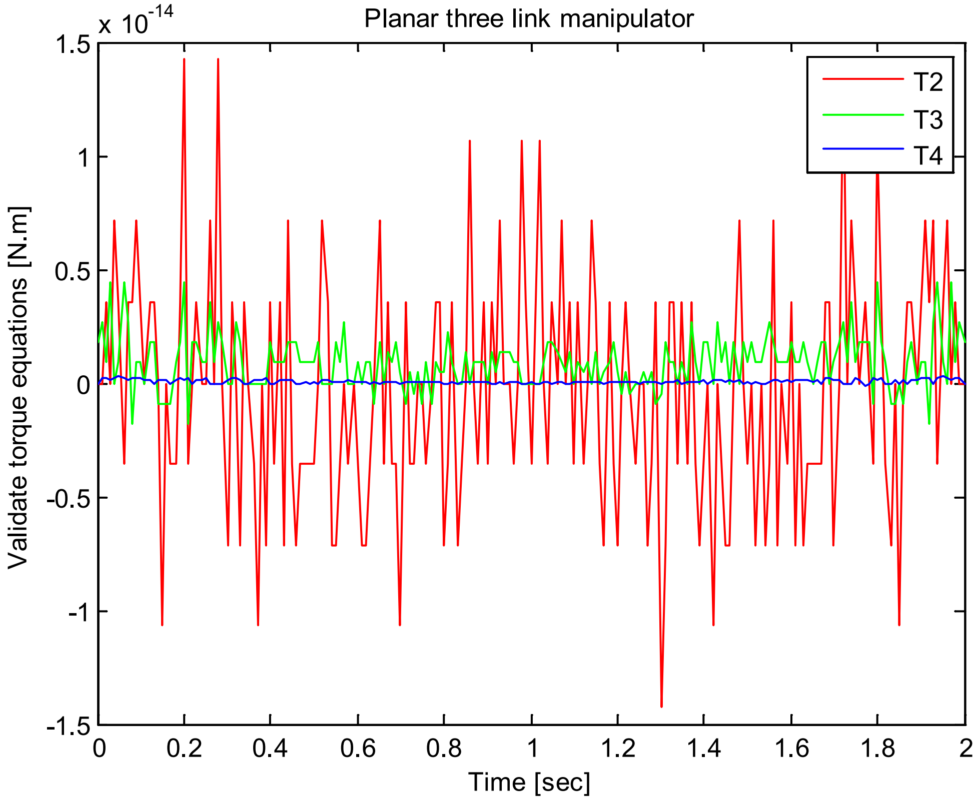
$$T_3 = A\ddot{\theta}_2 + J_B\ddot{\theta}_3 + C\ddot{\theta}_4 - 2F\dot{\theta}_2\dot{\theta}_4 - 2F\dot{\theta}_3\dot{\theta}_4 + (D+E)\dot{\theta}_2^2 - F\dot{\theta}_4^2 + G_3 + G_4$$

$$T_4 = B\ddot{\theta}_2 + +C\ddot{\theta}_3 + J_C\ddot{\theta}_4 + 2F\dot{\theta}_2\dot{\theta}_3 + (E+F)\dot{\theta}_2^2 + F\dot{\theta}_3^2 + G_4$$

$$\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{bmatrix} J_A & A & B \\ A & J_B & C \\ B & C & J_C \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} + \begin{bmatrix} -2(D+E) & -2(E+F) & -2(E+F) \\ 0 & -2F & -2F \\ 2F & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_2\dot{\theta}_3 \\ \dot{\theta}_2\dot{\theta}_4 \\ \dot{\theta}_3\dot{\theta}_4 \end{Bmatrix}$$
$$+ \begin{bmatrix} 0 & -(D+E) & -(E+F) \\ (D+E) & 0 & -F \\ (E+F) & F & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_2^2 \\ \dot{\theta}_3^2 \\ \dot{\theta}_4^2 \end{Bmatrix} + \begin{Bmatrix} G_2 + G_3 + G_4 \\ G_3 + G_4 \\ G_4 \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} = \begin{bmatrix} J_A & A & B \\ A & J_B & C \\ B & C & J_C \end{bmatrix}^{-1} \left\{ \begin{matrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} - \begin{bmatrix} -2(D+E) & -2(E+F) & -2(E+F) \\ 0 & -2F & -2F \\ 2F & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_2\dot{\theta}_3 \\ \dot{\theta}_2\dot{\theta}_4 \\ \dot{\theta}_3\dot{\theta}_4 \end{Bmatrix} \\ - \begin{bmatrix} 0 & -(D+E) & -(E+F) \\ (D+E) & 0 & -F \\ (E+F) & F & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_2^2 \\ \dot{\theta}_3^2 \\ \dot{\theta}_4^2 \end{Bmatrix} - \begin{Bmatrix} G_2 + G_3 + G_4 \\ G_3 + G_4 \\ G_4 \end{Bmatrix} \end{matrix} \right\}$$

Validated with harmonic drivers using Haug's inverse dynamics and then calculate torques per above



Linear State Space Model for Two Link Manipulator

$$\{y\} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} \theta_2 \\ \theta_3 \\ \dot\theta_2 \\ \dot\theta_3 \end{Bmatrix} \qquad \{\dot y\} = \begin{Bmatrix} \dot y_1 \\ \dot y_2 \\ \dot y_3 \\ \dot y_4 \end{Bmatrix} = \begin{Bmatrix} \dot\theta_2 \\ \dot\theta_3 \\ \ddot\theta_2 \\ \ddot\theta_3 \end{Bmatrix} \qquad \{\dot y\} = f(\{y\}) \qquad \{\dot y\} \approx [A_{LINEAR}]\{y\}$$

linearize about nominal values of $\{y\}$

$$\begin{Bmatrix} \ddot\theta_2 \\ \ddot\theta_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix}^{-1} \left(\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} - \begin{Bmatrix} -D\dot\theta_3^2 - 2D\dot\theta_2\dot\theta_3 \\ +D\dot\theta_2^2 \end{Bmatrix} - \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix} \right)$$

$$J_A = J_B + m_2 a_2^2 + m_3 d_2^2 + J_2 + 2 m_3 d_2 a_3 \cos\theta_3$$
$$J_B = m_3 a_3^2 + J_3$$

$$C = J_B + m_3 d_2 a_3 \cos\theta_3$$
$$D = m_3 d_2 a_3 \sin\theta_3$$

$$G_2 = (m_2 a_2 + m_3 d_2) g \cos\theta_2$$
$$G_3 = m_3 a_3 g \cos(\theta_2 + \theta_3)$$

$$\begin{Bmatrix} \ddot\theta_2 \\ \ddot\theta_3 \end{Bmatrix} = \frac{1}{J_A J_B - C^2} \begin{bmatrix} J_B & -C \\ -C & J_A \end{bmatrix} \begin{Bmatrix} G \\ H \end{Bmatrix}$$

$$\begin{Bmatrix} G \\ H \end{Bmatrix} = \begin{Bmatrix} T_2 + D\dot\theta_3^2 + 2D\dot\theta_2\dot\theta_3 - G_2 - G_3 \\ T_3 - D\dot\theta_2^2 - G_3 \end{Bmatrix}$$
$$= \begin{Bmatrix} T_2 + m_3 d_2 a_3 \sin\theta_3 \dot\theta_3^2 + 2 m_3 d_2 a_3 \sin\theta_3 \dot\theta_2 \dot\theta_3 - (m_2 a_2 + m_3 d_2) g \cos\theta_2 - m_3 a_3 g \cos(\theta_2 + \theta_3) \\ T_3 - m_3 d_2 a_3 \sin\theta_3 \dot\theta_2^2 - m_3 a_3 g \cos(\theta_2 + \theta_3) \end{Bmatrix}$$

$$\begin{Bmatrix} \delta\ddot\theta_2 \\ \delta\ddot\theta_3 \end{Bmatrix} = \left(\frac{2C(\partial C/\partial\theta_3) - J_B(\partial J_A/\partial\theta_3)}{(J_A J_B - C^2)^2} \begin{bmatrix} J_B & -C \\ -C & J_A \end{bmatrix} \begin{Bmatrix} G \\ H \end{Bmatrix} \right) \delta\theta_3$$
$$+ \left(\frac{1}{J_A J_B - C^2} \begin{bmatrix} 0 & -\partial C/\partial\theta_3 \\ -\partial C/\partial\theta_3 & \partial J_A/\partial\theta_3 \end{bmatrix} \begin{Bmatrix} G \\ H \end{Bmatrix} \right) \delta\theta_3$$
$$+ \frac{1}{J_A J_B - C^2} \begin{bmatrix} J_B & -C \\ -C & J_A \end{bmatrix} \begin{bmatrix} \partial G/\partial\theta_2 & \partial G/\partial\theta_3 & \partial G/\partial\dot\theta_2 & \partial G/\partial\dot\theta_3 \\ \partial H/\partial\theta_2 & \partial H/\partial\theta_3 & \partial H/\partial\dot\theta_2 & 0 \end{bmatrix} \begin{Bmatrix} \delta\theta_2 \\ \delta\theta_3 \\ \delta\dot\theta_2 \\ \delta\dot\theta_3 \end{Bmatrix}$$

$$\partial G/\partial\theta_2 = (m_2 a_2 + m_3 d_2) g\sin\theta_2 + m_3 a_3 g\sin(\theta_2+\theta_3)$$
$$\partial G/\partial\theta_3 = m_3 d_2 a_3\cos\theta_3\dot\theta_3^2 + 2m_3 d_2 a_3\cos\theta_3\dot\theta_2\dot\theta_3 + m_3 a_3 g\sin(\theta_2+\theta_3)$$
$$\partial G/\partial\dot\theta_2 = 2D\dot\theta_3$$
$$\partial G/\partial\dot\theta_3 = 2D(\dot\theta_2+\dot\theta_3)$$

$$\partial H/\theta_2 = m_3 a_3 g\sin(\theta_2+\theta_3)$$
$$\partial H/\theta_3 = -m_3 d_2 a_3\cos\theta_3\dot\theta_2^2 + m_3 a_3 g\sin(\theta_2+\theta_3)$$
$$\partial H/\dot\theta_2 = -2D\dot\theta_2$$
$$\partial H/\dot\theta_3 = 0$$

$$\partial J_A/\partial\theta_3 = -2D$$
$$\partial C/\partial\theta_3 = -D$$

$$\begin{Bmatrix}\delta\ddot\theta_2\\ \delta\ddot\theta_3\end{Bmatrix} = \frac{D}{(J_AJ_B-C^2)^2}\begin{bmatrix}-2J_B(J_B+C) & J_AJ_B+2J_BC+C^2\\ J_AJ_B+2J_BC+C^2 & -2(2J_AJ_B+J_AC-C^2)\end{bmatrix}\begin{Bmatrix}G\\H\end{Bmatrix}\delta\theta_3$$
$$+\frac{1}{J_AJ_B-C^2}\begin{bmatrix}J_B & -C\\ -C & J_A\end{bmatrix}\begin{bmatrix}\partial G/\partial\theta_2 & \partial G/\partial\theta_3 & \partial G/\partial\dot\theta_2 & \partial G/\partial\dot\theta_3\\ \partial H/\partial\theta_2 & \partial H/\partial\theta_3 & \partial H/\partial\dot\theta_2 & 0\end{bmatrix}\begin{Bmatrix}\delta\theta_2\\ \delta\theta_3\\ \delta\dot\theta_2\\ \delta\dot\theta_3\end{Bmatrix}$$

$$\begin{Bmatrix}P_{23}\\ P_{33}\end{Bmatrix} = \frac{D}{(J_AJ_B-C^2)^2}\begin{bmatrix}-2J_B(J_B+C) & J_AJ_B+2J_BC+C^2\\ J_AJ_B+2J_BC+C^2 & -2(2J_AJ_B+J_AC-C^2)\end{bmatrix}\begin{Bmatrix}G\\H\end{Bmatrix}$$

$$\begin{Bmatrix}\delta\dot\theta_2\\ \delta\dot\theta_3\\ \delta\ddot\theta_2\\ \delta\ddot\theta_3\end{Bmatrix} = \begin{bmatrix}\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}\\ \left(\frac{1}{J_AJ_B-C^2}\begin{bmatrix}J_B & -C\\ -C & J_A\end{bmatrix}\begin{bmatrix}\partial G/\partial\theta_2 & \partial G/\partial\theta_3 & \partial G/\partial\dot\theta_2 & \partial G/\partial\dot\theta_3\\ \partial H/\partial\theta_2 & \partial H/\partial\theta_3 & \partial H/\partial\dot\theta_2 & 0\end{bmatrix}+\begin{bmatrix}0 & P_{23} & 0 & 0\\ 0 & P_{33} & 0 & 0\end{bmatrix}\right)\end{bmatrix}\begin{Bmatrix}\delta\theta_2\\ \delta\theta_3\\ \delta\dot\theta_2\\ \delta\dot\theta_3\end{Bmatrix}$$

$$[A_{LINEAR}] \approx \begin{bmatrix}\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}\\ \left(\frac{1}{J_AJ_B-C^2}\begin{bmatrix}J_B & -C\\ -C & J_A\end{bmatrix}\begin{bmatrix}\partial G/\partial\theta_2 & \partial G/\partial\theta_3 & \partial G/\partial\dot\theta_2 & \partial G/\partial\dot\theta_3\\ \partial H/\partial\theta_2 & \partial H/\partial\theta_3 & \partial H/\partial\dot\theta_2 & 0\end{bmatrix}+\begin{bmatrix}0 & P_{23} & 0 & 0\\ 0 & P_{33} & 0 & 0\end{bmatrix}\right)\end{bmatrix}$$