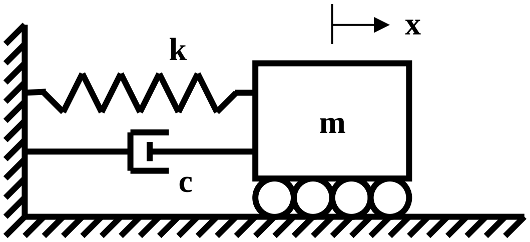
State Space Model For Spring-Mass-Damper



$m\ddot{x} + c\dot{x} + kx = 0 \qquad \ddot{x} = -\frac{k}{m}x - \frac{c}{m}\dot{x}$

$\{q\} = x \qquad \{\dot{q}\} = \dot{x} \qquad \{y\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$

use states $\{y\}$ to compute derivative of states $\{\dot{y}\} = f(\{y\})$

$$\{\dot{y}\} = \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \end{Bmatrix} = \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

$\{\dot{y}\} = [A]\{y\} \qquad \{\dot{y}\} = f(\{y\})$

assumed solution $\{y\} = \begin{Bmatrix} a_1 e^{\lambda t} \\ a_2 e^{\lambda t} \end{Bmatrix} \qquad \{\dot{y}\} = \begin{Bmatrix} \lambda a_1 e^{\lambda t} \\ \lambda a_2 e^{\lambda t} \end{Bmatrix} = \lambda\{y\}$

$\{\dot{y}\} = [A]\{y\} \qquad \lambda\{y\} = [A]\{y\} \qquad [A]\{y\} - \lambda\{y\} = \{0\}$

$([A] - \lambda[I_2])\{y\} = \{0\} \qquad \det([A] - \lambda[I_2]) = 0$ λ is an eigenvalue, NOT $\{\lambda\}$ from EOM

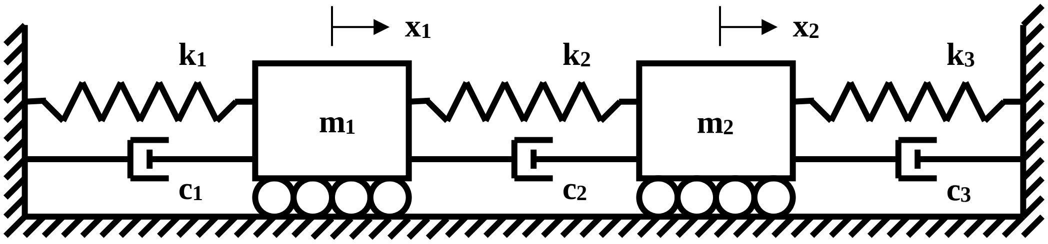
$$\det\begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & (-\frac{c}{m} - \lambda) \end{bmatrix} = 0 \qquad \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

for c = 0, $\lambda = \sqrt{-\frac{k}{m}} = i\sqrt{\frac{k}{m}} = i\,\omega_n \qquad \omega_n = \sqrt{\frac{k}{m}}$

critical damping $\left(\frac{c_{CR}}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_{CR} = 2\,m\,\varpi_n \qquad \zeta = \frac{c}{c_{CR}} = \frac{c}{2\,m\,\omega_n}$

underdamped $\zeta < 1 \qquad \lambda = -\zeta\,\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$

State Space Model For Double Spring-Mass-Damper



$$m_1\ddot{x}_1 = -k_1x_1 - c_1\dot{x}_1 - k_2(x_1 - x_2) - c_2(\dot{x}_1 - \dot{x}_2)$$

$$m_2\ddot{x}_2 = -k_3x_2 - c_3\dot{x}_2 - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1)$$

$$\{q\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad \{\dot{q}\} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \qquad \{y\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}$$

use states $\{y\}$ to compute derivative of states $\{\dot{y}\} = f(\{y\})$

$$\{\dot{y}\} = \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \end{Bmatrix} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k_1 + k_2}{m_1}\right) & +\frac{k_2}{m_1} & -\left(\frac{c_1 + c_2}{m_1}\right) & +\frac{c_2}{m_1} \\ +\frac{k_2}{m_2} & -\left(\frac{k_2 + k_3}{m_2}\right) & +\frac{c_2}{m_2} & -\left(\frac{c_2 + c_3}{m_2}\right) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = [A]\{y\}$$

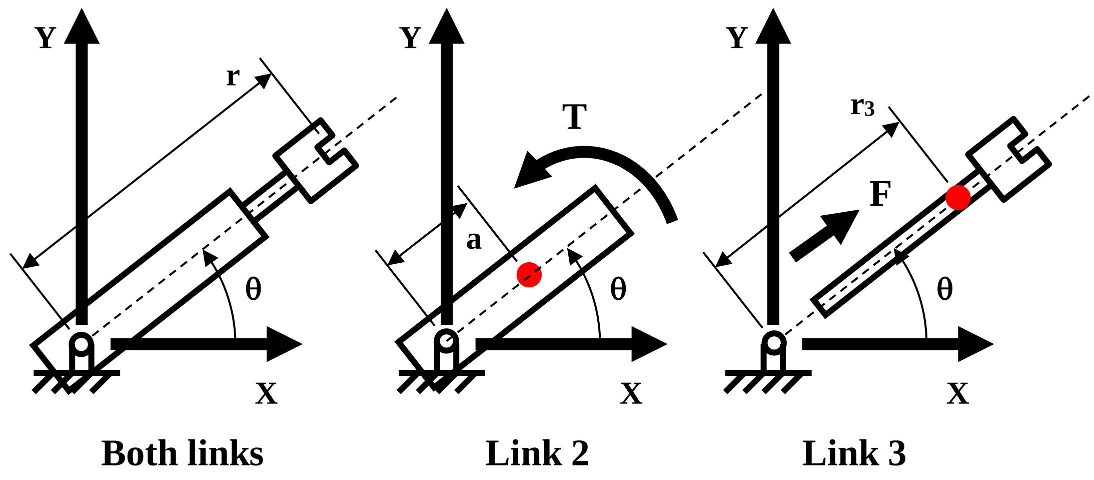
$$\{\dot{y}\} = [A]\{y\} \qquad \{\dot{y}\} = f(\{y\})$$

assumed solution $\{y\} = \{a_i e^{\lambda t}\}$ $\quad \{\dot{y}\} = \lambda\{y\}$

for $c_1 = c_2 = c_3 = 0$,
$a = (m_1k_2 + m_1k_3 + m_2k_1 + m_2k_2) / (m_1m_2)$, $b = (k_1k_2 + k_1k_3 + k_2k_3) / (m_1m_2)$

$$\lambda = i\sqrt{\frac{a}{2}\left(1 \pm \sqrt{1 - \frac{4b}{a^2}}\right)}$$

State Space Model for Cylindrical Coordinate Manipulator



Main body link 2 - Shaft and end-effector link 3
Mass centers at a and r_3 from waist rotation axis, a=constant, r_3 = variable
Masses m_2 and m_3 - centroidal mass moments of inertia J_2 and J_3
θ CCW from positive x axis – a and r_3 radial from rotation axis
T is rotary actuator torque of ground on body 2 about waist measured CCW positive
F is radial actuator force of body 2 on body 3 measured positive outward
Gravity g acts along negative y axis

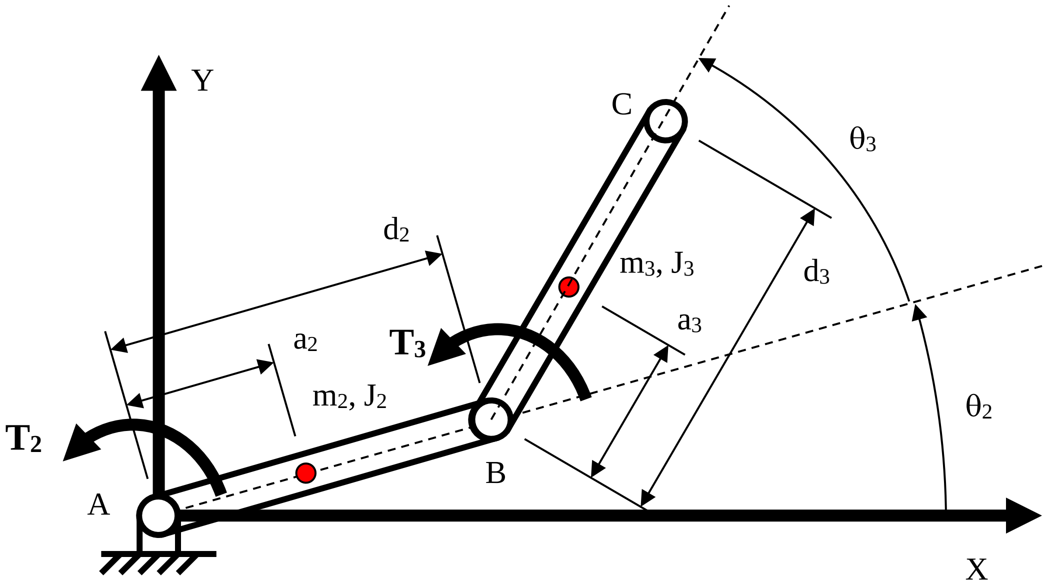
$$\begin{bmatrix} m_2 a^2 + m_3 r_3^2 + J_2 + J_3 & 0 \\ 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{r}_3 \end{Bmatrix} = \begin{Bmatrix} T - 2 m_3 r_3 \dot{r}_3 \dot{\theta} - g(m_2 a + m_3 r_3)\cos\theta \\ F + m_3 r_3 \dot{\theta}^2 - m_3 g \sin\theta \end{Bmatrix}$$

$$\{q\} = \begin{Bmatrix} \theta \\ r_3 \end{Bmatrix} \qquad \{\dot{q}\} = \begin{Bmatrix} \dot{\theta} \\ \dot{r}_3 \end{Bmatrix} \qquad \{y\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix} = \begin{Bmatrix} \theta \\ r_3 \\ \dot{\theta} \\ \dot{r}_3 \end{Bmatrix}$$

use states $\{y\}$ to compute derivative of states $\{\dot{y}\} = f(\{y\})$

$$\{\dot{y}\} = \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \end{Bmatrix} = \begin{Bmatrix} \dot{\theta} \\ \dot{r}_3 \\ \ddot{\theta} \\ \ddot{r}_3 \end{Bmatrix} = \begin{Bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta \\ r_3 \\ \dot{\theta} \\ \dot{r}_3 \end{Bmatrix} \\ \begin{bmatrix} m_2 a^2 + m_3 r_3^2 + J_2 + J_3 & 0 \\ 0 & m_3 \end{bmatrix}^{-1} \begin{Bmatrix} T - 2 m_3 r_3 \dot{r}_3 \dot{\theta} - g\left(m_2 a + m_3 r_3\right)\cos\theta \\ F + m_3 r_3 \dot{\theta}^2 - m_3 g \sin\theta \end{Bmatrix} \end{Bmatrix}$$

State Space Model for Two Link Anthropomorphic Manipulator (Double Pendulum)



Two solid rigid bars with revolute joints A and B
Lengths d_2 and d_3 - mass centers at a_2 and a_3 from proximal ends
Masses m_2 and m_3 - centroidal mass moments of inertia J_2 and J_3
θ_2 CCW from positive x axis — T_2 is torque of ground on bar 2 about pin A, CCW positive
θ_3 CCW from centerline of bar 2 — T_3 is torque of bar 2 on bar 3 about pin B, CCW positive
Gravity g acts along negative y axis

$$J_B = m_3 a_3^2 + J_3$$
$$J_A = J_B + m_2 a_2^2 + m_3 d_2^2 + J_2 + 2 m_3 d_2 a_3 \cos\theta_3$$

$$C = J_B + m_3 d_2 a_3 \cos\theta_3$$
$$D = m_3 d_2 a_3 \sin\theta_3$$

$$G_2 = (m_2 a_2 + m_3 d_2) g \cos\theta_2$$
$$G_3 = m_3 a_3 g \cos(\theta_2 + \theta_3)$$

$$\begin{Bmatrix}\ddot{\theta}_2\\ \ddot{\theta}_3\end{Bmatrix} = \begin{bmatrix}J_A & C\\ C & J_B\end{bmatrix}^{-1}\left(\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} - \begin{Bmatrix}-D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3\\ +D\dot{\theta}_2^2\end{Bmatrix} - \begin{Bmatrix}G_2 + G_3\\ G_3\end{Bmatrix}\right)$$

$$\{q\} = \begin{Bmatrix}\theta_2\\ \theta_3\end{Bmatrix} \qquad \{\dot{q}\} = \begin{Bmatrix}\dot{\theta}_2\\ \dot{\theta}_3\end{Bmatrix} \qquad \{y\} = \begin{Bmatrix}\{q\}\\ \{\dot{q}\}\end{Bmatrix} = \begin{Bmatrix}\theta_2\\ \theta_3\\ \dot{\theta}_2\\ \dot{\theta}_3\end{Bmatrix}$$

use states $\{y\}$ to compute derivative of states $\{\dot{y}\} = f(\{y\})$

$$\{\dot{y}\} = \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \\ \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix}^{-1} \left(\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} - \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} - \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix} \right) \end{Bmatrix}$$

Numerically Evaluate Linear State Matrix

assume $\{\dot{y}\} \approx [A_{LINEAR}]\{y\}$ using ns number of states and n number of time samples

$$\begin{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{Bmatrix}_{t1} & \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{Bmatrix}_{t2} & \cdots & \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{Bmatrix}_{tn} \end{bmatrix} \approx [A_{LINEAR}] \begin{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{Bmatrix}_{t1} & \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{Bmatrix}_{t2} & \cdots & \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{Bmatrix}_{tn} \end{bmatrix}$$

$$\underset{ns \times n}{[\dot{Y}]} \approx \underset{ns \times ns}{[A_{LINEAR}]} \underset{ns \times n}{[Y]}$$

Note that samples need not be computed using fixed h, and do not need to be continuous.

<u>for n = ns</u>

$$[A_{LINEAR}] \approx [\dot{Y}][Y]^{-1} = \begin{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{Bmatrix}_{t1} & \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{Bmatrix}_{t2} & \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{Bmatrix}_{t3} & \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{ns} \end{Bmatrix}_{t4} \end{bmatrix} \begin{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{Bmatrix}_{t1} & \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{Bmatrix}_{t2} & \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{Bmatrix}_{t3} & \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{ns} \end{Bmatrix}_{t4} \end{bmatrix}^{-1}$$

<u>for n ≥ ns</u>

$$[A_{LINEAR}] \approx [\dot{Y}][Y]^T \left([Y][Y]^T\right)^{-1}$$

extract $\{\lambda\}$ from $[A_{LINEAR}]$ $\qquad \{f_n\} = \dfrac{abs\{\lambda\}}{2\pi}$ $\qquad h < \dfrac{1}{2 \max\{f_n\}}$ $\qquad$ prefer $\qquad h < \dfrac{1}{10 \max\{f_n\}}$