Three-Dimensional Coordinate Transformations

$$\{r_i\}^P = \{r_i\} + [A_i]\{s_i\}'^P \qquad \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}^P = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \begin{bmatrix} a_{i\,11} & a_{i\,12} & a_{i\,13} \\ a_{i\,21} & a_{i\,22} & a_{i\,23} \\ a_{i\,31} & a_{i\,32} & a_{i\,33} \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}'^P$$

$$\{s_i\}^P = [A_i]\{s_i\}'^P \qquad \{s_i\}'^P = [A_i]^T\{s_i\}^P$$

[A] matrices are orthonormal $[A]^{-1} = [A]^T$

- all columns are unit vectors
- all columns are mutually orthogonal
- all rows are unit vectors
- all rows are mutually orthogonal
- det [A] = +1

Single rotations

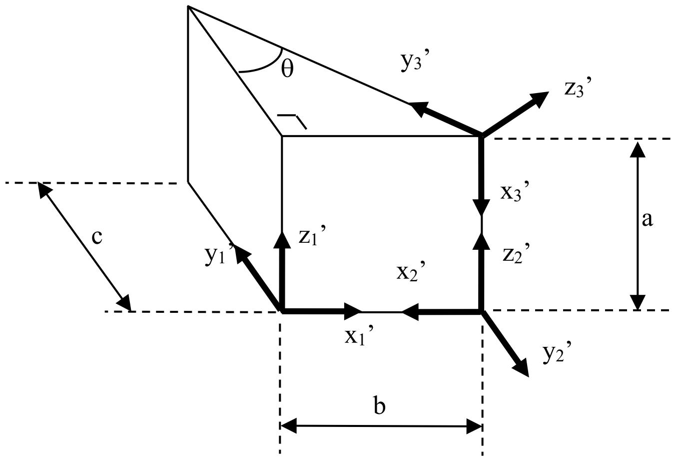
$$[A_{\theta X}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta_X & -S\theta_X \\ 0 & S\theta_X & C\theta_X \end{bmatrix}$$

$$[A_{\theta Y}] = \begin{bmatrix} C\theta_Y & 0 & S\theta_Y \\ 0 & 1 & 0 \\ -S\theta_Y & 0 & C\theta_Y \end{bmatrix}$$

$$[A_{\theta Z}] = \begin{bmatrix} C\theta_z & -S\theta_Z & 0 \\ S\theta_Z & C\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

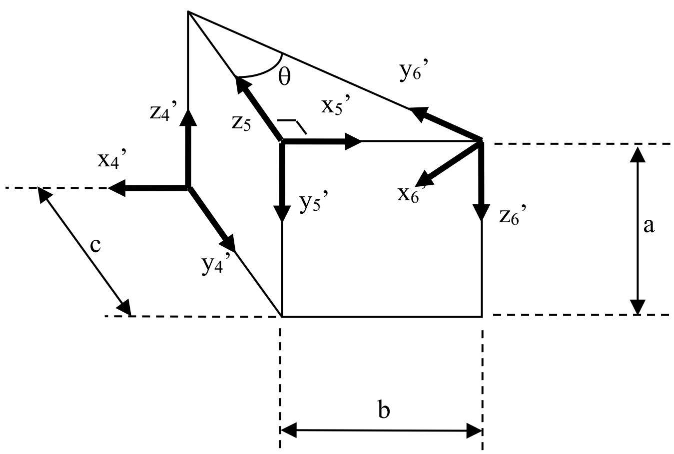
Coordinate Frames at the Corners of a Wedge

Adapted from Introduction to Robotics, J.J. Craig, Addison-Wesley, 1989



$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{Bmatrix} b \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix}'$$

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{Bmatrix} b \\ 0 \\ a \end{Bmatrix} + \begin{bmatrix} 0 & -S\theta & C\theta \\ 0 & C\theta & S\theta \\ -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix}'$$



$$\begin{Bmatrix} x_4 \\ y_4 \\ z_4 \end{Bmatrix}' = \begin{Bmatrix} 0 \\ c \\ a \end{Bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} x_5 \\ y_5 \\ z_5 \end{Bmatrix}'$$

$$\begin{Bmatrix} x_4 \\ y_4 \\ z_4 \end{Bmatrix}' = \begin{Bmatrix} -b \\ c \\ a \end{Bmatrix} + \begin{bmatrix} C\theta & S\theta & 0 \\ S\theta & -C\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} x_6 \\ y_6 \\ z_6 \end{Bmatrix}'$$

Euler Angles

Euler sequences – ZXZ (original), ZYZ, YXY, YZY, XYX, XZX

ZXZ sequence (θ_1 about global z - θ_2 about intermediate x - θ_3 about local z)

$$[A_{\theta Z}][A_{\theta X}][A_{\theta Z}] = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta_2 & -S\theta_2 \\ 0 & S\theta_2 & C\theta_2 \end{bmatrix} \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 \\ S\theta_3 & C\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A_{\theta Z}][A_{\theta X}][A_{\theta Z}] = \begin{bmatrix} -S\theta_1 C\theta_2 S\theta_3 + C\theta_1 C\theta_3 & -S\theta_1 C\theta_2 C\theta_3 - C\theta_1 S\theta_3 & S\theta_1 S\theta_2 \\ C\theta_1 C\theta_2 S\theta_3 + S\theta_1 C\theta_3 & C\theta_1 C\theta_2 C\theta_3 - S\theta_1 S\theta_3 & -C\theta_1 S\theta_2 \\ S\theta_2 S\theta_3 & S\theta_2 C\theta_3 & C\theta_2 \end{bmatrix}$$

$\theta_2 = a\cos(a_{33})$ $\theta_1 = a\tan 2(-a_{13} / a_{23})$ $\theta_3 = a\tan 2(a_{31} / a_{32})$ fails for $\theta_2 = 0$ and π

ZYZ sequence

YXY sequence

YZY sequence

XYX sequence

XZX sequence

<u>Cardan-Bryant-Tait sequences</u> – XYZ, XZY, YXZ, YZX, ZXY, ZYX

<u>XYZ sequence</u> (θ_x about global x - θ_y about intermediate y - θ_z about local z)

$$[A_{\theta X}][A_{\theta Y}][A_{\theta Z}] = \begin{bmatrix} C\theta_Y C\theta_Z & -C\theta_Y S\theta_Z & S\theta_Y \\ S\theta_X S\theta_Y C\theta_Z + C\theta_X S\theta_Z & -S\theta_X S\theta_Y S\theta_Z + C\theta_X C\theta_Z & -S\theta_X C\theta_Y \\ -C\theta_X S\theta_Y C\theta_Z + S\theta_X S\theta_Z & C\theta_X S\theta_Y S\theta_Z + S\theta_X C\theta_Z & C\theta_X C\theta_Y \end{bmatrix}$$

$\theta_Y = a\sin(a_{13})$ $\theta_Z = a\tan 2(-a_{12}/a_{11})$ $\theta_X = a\tan 2(-a_{23}/a_{33})$ fails for $\theta_Y = \pi/2$ and $3\pi/2$

<u>ZYX sequence</u> (θ_z about global z - θ_y about intermediate y - θ_x about local x)

$$[A_{\theta Z}][A_{\theta Y}][A_{\theta X}] = \begin{bmatrix} C\theta_Y C\theta_Z & S\theta_X S\theta_Y C\theta_Z - C\theta_X S\theta_Z & C\theta_X S\theta_Y C\theta_Z + S\theta_X S\theta_Z \\ C\theta_Y S\theta_Z & S\theta_X S\theta_Y S\theta_Z + C\theta_X C\theta_Z & C\theta_X S\theta_Y S\theta_Z - S\theta_X C\theta_Z \\ -S\theta_Y & S\theta_X C\theta_Y & C\theta_X C\theta_Y \end{bmatrix}$$

$\theta_Y = a\sin(-a_{31})$ $\theta_X = a\tan 2(a_{32}/a_{33})$ $\theta_Z = a\tan 2(a_{21}/a_{11})$ fails for $\theta_Y = \pi/2$ and $3\pi/2$

<u>XZY sequence</u> (θ_x about global x - θ_z about intermediate z - θ_y about local y)

$$[A_{\theta X}][A_{\theta Z}][A_{\theta Y}] = \begin{bmatrix} C\theta_Y C\theta_Z & -S\theta_Z & S\theta_Y C\theta_Z \\ C\theta_X C\theta_Y S\theta_Z + S\theta_X S\theta_Y & C\theta_X C\theta_Z & C\theta_X S\theta_Y S\theta_Z - S\theta_X C\theta_Y \\ S\theta_X C\theta_Y S\theta_Z - C\theta_X S\theta_Y & S\theta_X C\theta_Z & S\theta_X S\theta_Y S\theta_Z + C\theta_X C\theta_Y \end{bmatrix}$$

$\theta_Z = a\sin(-a_{12})$ $\theta_X = a\tan 2(a_{32}/a_{22})$ $\theta_Y = a\tan 2(a_{13}/a_{11})$ fails for $\theta_Z = \pi/2$ and $3\pi/2$

<u>YZX sequence</u>

<u>YXZ sequence</u>

<u>ZXY sequence</u>

Derivatives of Cardan-Bryant-Tait angles – ZYX

ZYX sequence (θ_z about global z - θ_y about intermediate y - θ_x about local x)

$$[A_{\theta Z}][A_{\theta Y}][A_{\theta X}] = \begin{bmatrix} C\theta_Y C\theta_Z & C\theta_Z S\theta_Y S\theta_X - S\theta_Z C\theta_X & C\theta_X S\theta_Y C\theta_Z + S\theta_X S\theta_Z \\ C\theta_Y S\theta_Z & S\theta_X S\theta_Y S\theta_Z + C\theta_X C\theta_Z & C\theta_X S\theta_Y S\theta_Z - C\theta_Z S\theta_X \\ -S\theta_Y & S\theta_X C\theta_Y & C\theta_X C\theta_Y \end{bmatrix}$$

$$\begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot\theta_z \end{Bmatrix} + [A_{\theta Z}]\begin{Bmatrix} 0 \\ \dot\theta_Y \\ 0 \end{Bmatrix} + [A_{\theta Z}][A_{\theta Y}]\begin{Bmatrix} \dot\theta_X \\ 0 \\ 0 \end{Bmatrix}$$

$$[A_{\theta Z}]\begin{Bmatrix} 0 \\ \dot\theta_Y \\ 0 \end{Bmatrix} = \begin{bmatrix} C\theta_z & -S\theta_Z & 0 \\ S\theta_Z & C\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \dot\theta_Y \\ 0 \end{Bmatrix}$$

$$[A_{\theta Z}][A_{\theta Y}]\begin{Bmatrix} \dot\theta_X \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} C\theta_z & -S\theta_Z & 0 \\ S\theta_Z & C\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C\theta_Y & 0 & S\theta_Y \\ 0 & 1 & 0 \\ -S\theta_Y & 0 & C\theta_Y \end{bmatrix}\begin{Bmatrix} \dot\theta_X \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} C\theta_Z C\theta_Y & -S\theta_Z & C\theta_Z S\theta_Y \\ S\theta_Z C\theta_Y & C\theta_Z & S\theta_Z S\theta_Y \\ -S\theta_Y & 0 & C\theta_Y \end{bmatrix}\begin{Bmatrix} \dot\theta_X \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} = \begin{bmatrix} C\theta_Z C\theta_Y & -S\theta_Z & 0 \\ S\theta_Z C\theta_Y & C\theta_Z & 0 \\ -S\theta_Y & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot\theta_X \\ \dot\theta_Y \\ \dot\theta_z \end{Bmatrix}$$

$$\begin{Bmatrix} \dot\theta_X \\ \dot\theta_Y \\ \dot\theta_z \end{Bmatrix} = \frac{1}{C\theta_Y}\begin{bmatrix} C\theta_Z & S\theta_Z & 0 \\ -C\theta_Y S\theta_Z & C\theta_Y C\theta_Z & 0 \\ S\theta_Y C\theta_Z & S\theta_Y S\theta_Z & C\theta_Y \end{bmatrix}\begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} \qquad \text{fails for } \theta_Y = \pi/2 \text{ and } 3\pi/2$$

$$\{\omega\} = [A]\{\omega\}'$$

$$\{\omega\}' = [A]^T\{\omega\} = [A_{\theta X}]^T[A_{\theta Y}]^T[A_{\theta Z}]^T\{\omega\}$$

$$\begin{Bmatrix} \omega_X' \\ \omega_Y' \\ \omega_Z' \end{Bmatrix} = \begin{Bmatrix} \dot\theta_X \\ 0 \\ 0 \end{Bmatrix} + [A_{\theta x}]^T\begin{Bmatrix} 0 \\ \dot\theta_Y \\ 0 \end{Bmatrix} + [A_{\theta X}]^T[A_{\theta Y}]^T\begin{Bmatrix} 0 \\ 0 \\ \dot\theta_z \end{Bmatrix}$$

$$[A_{\theta x}]^T \begin{Bmatrix} 0 \\ \dot{\theta}_Y \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta_X & S\theta_X \\ 0 & -S\theta_X & C\theta_X \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\theta}_Y \\ 0 \end{Bmatrix}$$

$$[A_{\theta X}]^T [A_{\theta Y}]^T \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta_X & S\theta_X \\ 0 & -S\theta_X & C\theta_X \end{bmatrix} \begin{bmatrix} C\theta_Y & 0 & -S\theta_Y \\ 0 & 1 & 0 \\ S\theta_Y & 0 & C\theta_Y \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{Bmatrix} = \begin{bmatrix} C\theta_Y & 0 & -S\theta_Y \\ S\theta_X S\theta_Y & C\theta_X & S\theta_X C\theta_Y \\ C\theta_X S\theta_Y & -S\theta_X & C\theta_X C\theta_Y \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{Bmatrix}$$

$$\begin{Bmatrix} \omega_X' \\ \omega_Y' \\ \omega_Z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -S\theta_Y \\ 0 & C\theta_X & S\theta_X C\theta_Y \\ 0 & -S\theta_X & C\theta_X C\theta_Y \end{bmatrix} \begin{Bmatrix} \dot{\theta}_X \\ \dot{\theta}_Y \\ \dot{\theta}_Z \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\theta}_X \\ \dot{\theta}_Y \\ \dot{\theta}_Z \end{Bmatrix} = \frac{1}{C\theta_Y} \begin{bmatrix} C\theta_Y & S\theta_X S\theta_Y & C\theta_X S\theta_Y \\ 0 & C\theta_X C\theta_Y & -S\theta_X C\theta_Y \\ 0 & S\theta_X & C\theta_X \end{bmatrix} \begin{Bmatrix} \omega_X' \\ \omega_Y' \\ \omega_Z' \end{Bmatrix} \qquad \text{fails for } \theta_Y = \pi/2 \text{ and } 3\pi/2$$

Chasles' Angle and Euler Parameters

Rotation χ about unit direction $\{\hat{u}\}$

$$[A]=\begin{bmatrix} u^2V\chi + C\chi & uvV\chi - wS\chi & uwV\chi + vS\chi \\ uvV\chi + wS\chi & v^2V\chi + C\chi & vwV\chi - uS\chi \\ uwV\chi - vS\chi & vwV\chi + uS\chi & w^2V\chi + C\chi \end{bmatrix}$$

$$C\chi = (\mathrm{tr}[A]-1)/2 \qquad \{\hat{u}\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \frac{1}{2S\chi}\begin{Bmatrix} a_{32}-a_{23} \\ a_{13}-a_{31} \\ a_{21}-a_{12} \end{Bmatrix}$$

fails for $\chi = 0$ and π

Euler parameters (unit quarternion)

$$\{p\} = \begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} C\frac{\chi}{2} \\ uS\frac{\chi}{2} \\ vS\frac{\chi}{2} \\ wS\frac{\chi}{2} \end{Bmatrix} \qquad \{p\}^T\{p\} = e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

$$[A] = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

$$e_0^{\,2} = (\mathrm{tr}[A]+1)/4 \qquad \{e\} = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \frac{1}{4e_0}\begin{Bmatrix} a_{32}-a_{23} \\ a_{13}-a_{31} \\ a_{21}-a_{12} \end{Bmatrix}$$

fails for $e_0 = 0$ at $\chi = \pi$

$$[E] = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix} \qquad [G] = \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix}$$

$$[A] = [E][G]^T = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_1 e_2 + e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix}$$

$$[E][E]^T = [G][G]^T = [I_3]$$

$$[E]^T[E] = [G]^T[G] = [I_4] - \{p\}\{p\}^T$$

$$[E]\{p\} = [G]\{p\} = \{0\}$$

Velocity

$$\{\omega\} = [A]\{\omega\}'$$

$$\{\omega\} = 2[E]\{\dot{p}\} = -2[\dot{E}]\{p\}$$

$$\{\omega\}' = [A]^T\{\omega\}$$

$$\{\omega\}' = 2[G]\{\dot{p}\} = -2[\dot{G}]\{p\}$$

$$\{\dot{p}\} = \tfrac{1}{2}[E]^T\{\omega\} = \tfrac{1}{2}[G]^T\{\omega\}'$$

$$\{p\}^T\{\dot{p}\} = \{\dot{p}\}^T\{p\} = 0$$

$$[\tilde{\omega}] = [\dot{A}][A]^T = -2[E][\dot{E}]^T = 2[\dot{E}][E]^T$$

$$[\tilde{\omega}]' = [A]^T[\dot{A}] = 2[G][\dot{G}]^T = -2[\dot{G}][G]^T$$

$$[\dot{A}] = [\tilde{\omega}][A] = [A][\tilde{\omega}]' = 2[E][\dot{G}]^T = 2[\dot{E}][G]^T$$

Acceleration

$$\{\dot{\omega}\} = [A]\{\dot{\omega}\}'$$

$$\{\dot{\omega}\} = 2[E]\{\ddot{p}\}$$

$$\{\dot{\omega}\}' = [A]^T\{\dot{\omega}\}$$

$$\{\dot{\omega}\}' = 2[G]\{\ddot{p}\}$$

$$\{\ddot{p}\} = \tfrac{1}{2}[E]^T\{\dot{\omega}\} + \tfrac{1}{2}[\dot{E}]^T\{\omega\} = \tfrac{1}{2}[E]^T\{\dot{\omega}\} - \tfrac{1}{4}\{p\}\left(\{\omega\}^T\{\omega\}\right)$$

$$\{\ddot{p}\} = \tfrac{1}{2}[G]^T\{\dot{\omega}\}' + \tfrac{1}{2}[\dot{G}]^T\{\omega\}' = \tfrac{1}{2}[G]^T\{\dot{\omega}\}' - \tfrac{1}{4}\{p\}\left(\{\omega\}'^T\{\omega\}'\right)$$

$$\{\dot{p}\}^T\{\dot{p}\} = \tfrac{1}{4}\{\omega\}^T\{\omega\} = \tfrac{1}{4}\{\omega\}'^T\{\omega\}' = \tfrac{1}{4}\omega^2$$

$$[\dot{E}]\{\dot{p}\} = [\dot{G}]\{\dot{p}\} = 0$$

$$[\ddot{A}] = 2[\dot{E}][\dot{G}]^T + 2[E][\ddot{G}]^T = 2[\ddot{E}][G]^T + 2[\dot{E}][\dot{G}]^T$$

$$[E][\ddot{G}]^T = [\ddot{E}][G]^T$$

Jerk

$$\{\dddot{\omega}\} = [A]\{\dddot{\omega}\}' + [A][\tilde{\omega}]'\{\ddot{\omega}\}'$$

$$\{\ddot{\omega}\} = 2[E]\{\dddot{p}\} + 2[\dot{E}]\{\ddot{p}\} = 2[E]\{\dddot{p}\} + \tfrac{1}{2}[\tilde{\omega}]\{\dot{\omega}\} + \tfrac{1}{4}\{\omega\}\{\omega\}^T\{\omega\}$$

$$\{\ddot{\omega}\}' = [A]^T\{\ddot{\omega}\} - [A]^T[\tilde{\omega}]\{\dot{\omega}\}$$

$$\{\ddot{\omega}\}' = 2[G]\{\dddot{p}\} + 2[\dot{G}]\{\ddot{p}\} = 2[G]\{\dddot{p}\} - \tfrac{1}{2}[\tilde{\omega}]'\{\dot{\omega}\}' + \tfrac{1}{4}\{\omega\}'\{\omega\}'^T\{\omega\}'$$

$$\begin{aligned}\{\dddot{p}\} &= \tfrac{1}{2}[E]^T\{\ddot{\omega}\} + [\dot{E}]^T\{\dot{\omega}\} + \tfrac{1}{2}[\ddot{E}]^T\{\omega\}\\ &= \tfrac{1}{2}[E]^T\left(\{\ddot{\omega}\} - \tfrac{1}{2}[\tilde{\omega}]\{\dot{\omega}\} - \tfrac{1}{2}\{\omega\}\{\omega\}^T\{\omega\}\right) - \tfrac{3}{4}\{p\}\{\omega\}^T\{\dot{\omega}\}\end{aligned}$$

$$\begin{aligned}\{\dddot{p}\} &= \tfrac{1}{2}[G]^T\{\ddot{\omega}\}' + [\dot{G}]^T\{\dot{\omega}\}' + \tfrac{1}{2}[\ddot{G}]^T\{\omega\}'\\ &= \tfrac{1}{2}[G]^T\left(\{\ddot{\omega}\}' + \tfrac{1}{2}[\tilde{\omega}]'\{\dot{\omega}\}' - \tfrac{1}{4}\{\omega\}'\{\omega\}'^T\{\omega\}'\right) - \tfrac{3}{4}\{p\}\{\omega\}'^T\{\dot{\omega}\}'\end{aligned}$$

$$[\dot{G}][G]^T \neq 0 \qquad [\dot{G}][\dot{G}]^T \neq 0 \qquad [\dot{G}]\{\ddot{p}\} \neq 0 \qquad [G]\{\ddot{p}\} \neq 0 \qquad [\dot{G}]\{\ddot{p}\} + [\ddot{G}]\{\dot{p}\} = 0$$

$$[\dddot{A}] = 2[E][\dddot{G}]^T + 4[\dot{E}][\ddot{G}]^T + 2[\ddot{E}][\dot{G}]^T = 2[\dddot{E}][G]^T + 4[\ddot{E}][\dot{G}]^T + 2[\dot{E}][\ddot{G}]^T$$

$$[E][\dddot{G}]^T + [\dot{E}][\ddot{G}]^T = [\dddot{E}][G]^T + [\ddot{E}][\dot{G}]^T$$

Snap

$$\{\ddot{\omega}\}' = 2[G]\{\dddot{p}\} + 2[\dot{G}]\{\ddot{p}\} = 2[G]\{\dddot{p}\} - \tfrac{1}{2}[\tilde{\omega}]'\{\dot{\omega}\}' + \tfrac{1}{4}\{\omega\}'\{\omega\}'^{T}\{\omega\}'$$

$$\{\dddot{\omega}\}' = 2[G]\{\ddddot{p}\} + 4[\dot{G}]\{\dddot{p}\} + 2[\ddot{G}]\{\ddot{p}\}$$

$$\{\dddot{p}\} = \tfrac{1}{2}[G]^{T}\{\ddot{\omega}\}' + [\dot{G}]^{T}\{\dot{\omega}\}' + \tfrac{1}{2}[\ddot{G}]^{T}\{\omega\}'$$

$$\{\ddddot{p}\} = \tfrac{1}{2}[G]^{T}\{\dddot{\omega}\}' + \tfrac{3}{2}[\dot{G}]^{T}\{\ddot{\omega}\}' + \tfrac{3}{2}[\ddot{G}]^{T}\{\dot{\omega}\}' + \tfrac{1}{2}[\dddot{G}]^{T}\{\omega\}'$$

Numerical partial derivatives of rotation matrices with respect to Euler parameters can produce different results

$$[A] = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\ e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\ e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix} \qquad \text{numerical } \frac{\partial a_{11}}{\partial e_3} = 0$$

$$[A] = [E][G]^T = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_1 e_2 + e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \qquad \text{numerical } \frac{\partial a_{11}}{\partial e_3} = -2e_3$$

Rodriguez Parameters

$$\{R\} = \tan\left(\frac{\chi}{2}\right)\{\hat{u}\}$$